RENORMALIZATION OF THE SELF FIELD QED

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ABSTRACT

A new form of the regularization of self energy term is derived in the QED based on the self field formalism. In this new form of regularization final result is finite and renormalized.

INTRODUCTION

The electrodynamics is the interaction of charged particles with the radiation fields. In the classical approach to the problem the interaction of the charged particles with the radiation fields is considered nonperturbatively. In standard formulation of the quantumelectrodynamics there are seperately quantized electron and photon fields and their interaction can be added as a second step and perturbatively. In order to understand the theory better other approachs are proposed. The self field approach is one of them.[1]

The self field approach is similiar to the classical electrodynamics. In this approach the electron is interacting with the external field and its self field non-perturbatively. In order to formulate the problem the interacting electron is quantized by the first quantization and the photon field is quantized by its source (electron).

In the self field electrodynamics, we consider the interaction of the electron with external field (classical or quantum) and its self radiation field and formulate the radiation reaction. Then, there appears an important question. Does the free particle have radiation reaction? The physical answer have to be no. But we know from the classical electrodynamics that Lorentz-Dirac equation does not satisfy this condition and this is one of the reason for the existence of runaway solutions. In a physical theory the radiation reaction must go to zero when external field goes to zero.

In standart quantumelectrodynamics all the radiative processes are formulated in terms of free quantized electron and photon fields or Green's functions. In this formulation it is not easy to answer the question mentioned above. In the self field quantumelectrodynamics we can choose our physical quantities such that they go to zero when the external field becomes zero.

This approach is very similiar to the scattering theory. In quantum mechanics we have the scattering solutions. These solutions include the infinite plane wave solution. In order to obtain the physical quantities such as the scattering amplitude , we substract the plane wave or free particle solution from the scattering solution.

The main radiative processes are the self energy of the electron, anomalous magnetic moment and spontaneous decay in the free space or in the cavity for different external fields.(2,3,4,5) The contribution of the vacuum polarization term to the Lamb shift of the bound state electron have been investigated.(6,7) This is the most divergent term in the standart formulation of the QED. This new calculation gives the standart result by using a new regularization mechanism. In the first order iteration self-field QED gives exactly the same result as the standart QED calculation.(8) In the formulation of the self energy problem we have also formally divergent integrals. The source of these terms is the sum all over the intermediate quantum states and it includes intermediate bound states and continuous scattering states. The contributions of the bound states to this sum goes to zero when external field vanishes. But the contributions of the scattering states do not satisfy this criteria, because they include infinite plane wave or free particle solutions. In the next section we develope a new method how to regularize these integrals and obtain a finite result for the self energy of the electron.

SELF ENERGY TERM

Self energy is apart of the general energy shift ΔE_n of a quantum level n of system due to the radiative self energy effects. It is given by

$$\Delta E_n^{S.E.} = \frac{e^2}{2} \int\!\!\int_s d^3x \bar{\psi}_n(\mathbf{x})\, \gamma_\mu \psi_s(\mathbf{x}) \int d^3y \bar{\psi}_s(\mathbf{y})\, \gamma^\mu \psi_n(\mathbf{y}) \times \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{2k} P\left(\frac{2k}{(E_s - E_n)^2 - k^2} \right) \tag{1}$$

Where ψ_n is a fixed level and we sum on the over all levels ψ_s, discrete and continuous.

We summarize first the spin algebra and the angular integrations.The relativistic Coulomb functions are written as

$$\psi_n(\mathbf{r}) = \begin{pmatrix} f_n \Omega_n \\ i g_n \Omega_{n'} \end{pmatrix}, \quad \begin{matrix} n = (j_n, l_n, m_n) \\ n' = (j_n, l'_n, m_n); \quad l'_n = l_n + 1 \end{matrix} \tag{2}$$

where f_n and g_n are the “large” the “small” components respectively. The product of two currents is

$$\bar{\psi}_n \gamma_\mu \psi_s \bar{\psi}_s \gamma^\mu \psi_n = \psi^+{}_n(\mathbf{r}) \psi_s(\mathbf{r}) \psi^+{}_s(\mathbf{r}') \psi_n(\mathbf{r}') - \psi^+{}_n(\mathbf{r}) \alpha \psi_s(\mathbf{r}) \cdot \psi^+{}_s(\mathbf{r}') \alpha \psi_n(\mathbf{r}') \tag{3}$$

After the angular integrations we obtain

$$\Delta E_n^{S.E.} = \frac{e^2}{2} \sum_s \int k dk \int dr \int dr' j_l(kr) j_l(kr') \times P\left[\frac{2k}{(E_s - E_n)^2 - k^2 + i\varepsilon} \right]$$

$$\{ [w_{1n}^*(r) W_{ns}^{lm} w_{1s}(r) + w_{2n}^*(r) W_{n's'}^{lm} w_{2s}(r)] \cdot \left[w_{1s}^*(r') (W_{ns}^{lm})^* w_{1n}(r') + w_{2s}^*(r') (W_{n's'}^{lm}) w_{2n}(r') \right] -$$

$$[w_{1n}^*(r) \mathbf{K}_{ns'}^{lm} w_{2s}(r) + w_{2n}^*(r) \mathbf{K}_{n'}^{lm} w_{1s}(r)] \cdot \left[w_{2s}^*(r') (\mathbf{K}_{ns'}^{lm})^* w_{1n}(r') + w_{1s}^*(r') (\mathbf{K}_{n's}^{lm}) w_{2n}(r') \right] \}$$

(4)

where

$$W_{\substack{ns \\ n's'}}^{lm} = \int d\hat{\mathbf{r}} \left[\frac{k_n k_s}{|k_n||k_s|} \sqrt{\frac{k_n + m_n \mp \frac{1}{2}}{2k_n \mp 1} \frac{k_s + m_s \mp \frac{1}{2}}{2k_s \mp 1}} Y^{*m_n - \frac{1}{2}}_{|k_n \mp \frac{1}{2}| - \frac{1}{2}} Y_l^m Y^{m_s - \frac{1}{2}}_{|k_s \mp \frac{1}{2}| - \frac{1}{2}} \right.$$

$$\left. + \frac{k_n k_s}{|k_n||k_s|} \sqrt{\frac{k_n - m_n \mp \frac{1}{2}}{2k_n \mp 1} \frac{k_s - m_s \mp \frac{1}{2}}{2k_s \mp 1}} Y^{*m_n + \frac{1}{2}}_{|k_n \mp \frac{1}{2}| - \frac{1}{2}} Y_l^m Y^{m_s + \frac{1}{2}}_{|k_s \mp \frac{1}{2}| - \frac{1}{2}} \right]$$

$$\mathbf{K}_{ns'}^{lm} = \int d\hat{\mathbf{r}} \left(\frac{ik_n}{|k_n|} \sqrt{\frac{k_n + m_n - \frac{1}{2}}{2k_n - 1}} Y^{*m_n - \frac{1}{2}}_{|k_n - \frac{1}{2}| - \frac{1}{2}} \;,\; i\sqrt{\frac{k_n - m_n - \frac{1}{2}}{2k_n - 1}} Y^{*m_n + \frac{1}{2}}_{|k_n - \frac{1}{2}| - \frac{1}{2}} \right) \cdot$$

$$\sigma \cdot Y_l^m \cdot \begin{pmatrix} -\frac{k_s}{|k|} \sqrt{\frac{k_n + m_n - \frac{1}{2}}{2k_n - 1}} Y^{m_n - \frac{1}{2}}_{|k_n - \frac{1}{2}| - \frac{1}{2}} \\ \sqrt{\frac{k_n + m_n - \frac{1}{2}}{2k_n - 1}} Y^{m_n - \frac{1}{2}}_{|k_n - \frac{1}{2}| - \frac{1}{2}} \end{pmatrix} \qquad (5)$$

We can extend the sum over the intermediate ψ_s states also to the negative energy solutions in order to introduce the energy dependent radial Green's functions $G(r,r';z)$ of the relativistic Coulomb problem, because the negative-energy solutions are equivalent to positive-energy solutions with
$-e$. Then $\Delta E_n^{S.E}$ becomes

$$\Delta E_n^{S.E.} = -4\alpha \sum_s \int \frac{dz}{2\pi i} \int dr \int dr' \int k dk\, j_l(kr) j_l(kr') P\left[\frac{2k}{(z - E_n)^2 - k^2 + i\varepsilon} \right] \times R$$

where R is

$$\begin{aligned} R = \Big[& w_{1n}^*(r) G_{11}(r,r';z) w_{1n}(r') |W_{ns}^{lm}|^2 + w_{2n}^*(r) G_{22}(r,r';z) w_{2n}(r') |W_{n's'}^{lm}|^2 + \\ & w_{1n}^*(r) G_{12}(r,r';z) w_{2n}(r') W_{ns}^{lm} W^{*lm}_{n's'} + w_{2n}^*(r) G_{21}(r,r';z) w_{1n}(r') W_{n's'}^{lm} W_{ns}^{lm} - \\ & w_{1n}^*(r) G_{22}(r,r';z) w_{1n}(r') \mathbf{K}_{ns'}^{lm} \cdot \mathbf{K}^{*lm}_{ns'} - w_{1n}^*(r) G_{21}(r,r';z) w_{2n}(r') \mathbf{K}_{ns'}^{lm} \cdot \mathbf{K}^{*lm}_{n's'} - \\ & w_{2n}^*(r) G_{12}(r,r';z) w_{1n}(r') \mathbf{K}_{n's}^{lm} \mathbf{K}^{*lm}_{ns'} - w_{2n}^*(r) G_{11}(r,r';z) w_{2n}(r') \mathbf{K}_{n's}^{lm} \cdot \mathbf{K}^{*lm}_{n's} \Big] \end{aligned} \qquad (6)$$

Where $G(r,r';z)$ are the matrix elements of the Green's function of the relativistic Coulomb problem and the contour of z- integration is shown in Figure 1. The Green's function $G(r,r';z)$ has the poles corresponding to the bound states, plus the cuts beginning at $\pm m$ corresponding to positive and negative continuous spectra. The other cuts come from the photon Green's function.

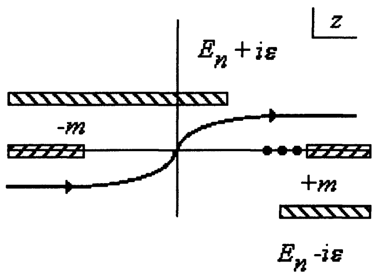

Figure 1. Contour of the *z*- integration.

The Green's function $G(r,r';z)$ can be constructed in terms of the solutions of the radial problem. It is in the following form:

$$G(r,r';z) = \frac{1}{K(z)} \begin{pmatrix} w_1^{(2)}(r_>;z) \\ w_2^{(2)}(r_>;z) \end{pmatrix} \left(w_1^{(1)}(r_<;z), w_2^{(1)}(r_<;z) \right) \tag{7}$$

Where $w^{(1)}(r_<;z)$ and $w^{(2)}(r_>;z)$ are the regular solutions of the radial problem at the origin and at the infinity respectively.[6] They are given by Wichmann and Kroll in terms of confluent hypergeometric functions as

$$w_{\substack{1\\2}}^{(1)}(r_<;z) = \left[2r_<(z^2-1)^{1/2}\right]^{\gamma} \left[\frac{i\sqrt{z+1}}{\sqrt{z-1}}\right] \left[\left(\kappa - \frac{iZ\alpha}{(z^2-1)^{1/2}}\right)\phi\left(\gamma - i\nu, 2\gamma+1, -2i(z^2-1)^{1/2} r_<\right) \pm (\gamma - i\nu)\phi\left(\gamma - i\nu, 2\gamma+1, -2i(z^2-1)^{1/2} r_<\right)\right]$$

$$w_{\substack{1\\2}}^{(2)}(r_>;z) = \left[2r_>(z^2-1)^{1/2}\right]^{\gamma} \left[\frac{i\sqrt{z+1}}{\sqrt{z-1}}\right] \left[\left(\kappa - \frac{iZ\alpha}{(z^2-1)^{1/2}}\right)\chi\left(\gamma - i\nu, 2\gamma+1, -2i(z^2-1)^{1/2} r_>\right) \pm (\gamma - i\nu)\chi\left(\gamma - i\nu, 2\gamma+1, -2i(z^2-1)^{1/2} r_>\right)\right] \tag{8}$$

where *K(z)* is

$$K(z) = -2(z^2-1)^{1/2}\left[\kappa - iZ\alpha/(z^2-1)^{1/2}\right] \frac{\Gamma(-\gamma - i\nu)\Gamma(2\gamma+1)}{\Gamma(\gamma - i\nu)\Gamma(-2\gamma)} \exp\left[\frac{i\pi}{2}(2\gamma+1)\right] \tag{9}$$

and the regular solutions of the confluent hypergeometric equation at the origin and the infinity are given by

$$\phi(\alpha,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 dt e^{zt} t^{\alpha-1}(1-t)^{\gamma-\alpha-1} \tag{10}$$

and

$$\chi(\alpha,\gamma;z)=\frac{\Gamma(\alpha+1-\gamma)}{\Gamma(\alpha)\Gamma(1-\gamma)}\int_0^\infty dt e^{-zt}t^{\alpha-1}(1+t)^{\gamma-\alpha-1} \tag{11}$$

respectively. After the k-integration Eq (6) becomes

$$\Delta E_n^{S.E.}=-4\alpha\sum_s\int\frac{dz}{2\pi i}\int dr\int dr'\left[-\frac{i\pi}{2}\omega j_l(\omega r_\langle)h_l^{(2)}(\omega r_\rangle)\right]R \tag{12}$$

Then by deforming the contour of energy integration we can separate $\Delta E_n^{S.E}$ into low energy and high energy parts:

$$\Delta E_n^{S.E.}(\text{low energy})=-4\alpha\sum_s\int_0^{E_n}\frac{dz}{2\pi i}\int dr\int dr'\left[-\frac{i\pi}{2}\omega j_l(\omega r_\langle)j_l(\omega r_\rangle)\right]R \tag{13}$$

$$\Delta E_n^{S.E.}(\text{high energy})=4\alpha\sum_s\int_{-i\infty}^{+i\infty}\frac{dz}{2\pi i}\int dr\int dr'\left[\frac{i\pi}{2}\omega j_l(\omega r_\langle)h_l^{(2)}(\omega r_\rangle)\right]R \tag{14}$$

In order to do r and r' integrations we represent $G(r,r';z)$ as a double Mellin-Barnes type complex integral. Mellin- Barnes type integral representation of the regular solutions are

$$\phi(\alpha,\gamma,z)=\int_{C_t-i\infty}^{C_t+i\infty}\frac{dt}{2\pi i}\Gamma(-t)\frac{\Gamma(\alpha+t)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma+t)}(-z)^t \tag{15}$$

and

$$\chi(\alpha,\gamma;z)=\int_{C_s-i\infty}^{C_s+i\infty}\frac{ds}{2\pi i}\Gamma(-s)\frac{\Gamma(\alpha+s)\Gamma(1-\gamma-s)}{\Gamma(\alpha)\Gamma(1-\gamma)}z^s \tag{16}$$

where C_t is choosen such that the poles of $\Gamma(-t)$ and the poles of $\Gamma(\alpha+t)$ and $\Gamma(\gamma+t)$ are seperated. Similiarly C_s is choosen such that the poles of $\Gamma(\alpha+s)$ and poles of $\Gamma(-s)$ and $\Gamma(1-\gamma-s)$ are seperated. For the Coulomb problem these conditions are satisfied except the free particle limit. When $Z\alpha$ goes to zero these two sets of poles are not seperated. We discuss this limit in the Appendix A.

Radial Green's function of the photon is

$$j_l(\omega r_\langle)h_l^{(2)}(\omega r_\rangle) \tag{17}$$

Mellin-Barnes type integral representations of j_l is

$$j_l(\omega r_\langle)=-\frac{1}{2}\sqrt{\frac{\pi}{2\omega r_\langle}}\int_{C_s-i\infty}^{C_s+i\infty}\frac{ds}{2\pi i}\frac{\Gamma\left(\frac{l+\frac{1}{2}+s}{2}\right)}{\Gamma\left(1+\frac{l+\frac{1}{2}-s}{2}\right)}(\omega r_\langle)^{-s} \tag{18}$$

This representation gives the Taylor expansion of j_l . Mellin-Barnes type integral representation of $h_l^{(2)}$ is

$$h_l^{(2)}(\omega r_\rangle) = \frac{i}{\pi} e^{i(l+\frac{1}{2})\frac{\pi}{2}} \sqrt{\frac{\pi}{2\omega r_\rangle}} \int_{C_s - i\infty}^{C_s + i\infty} \frac{ds}{2\pi i} \Gamma(-s)\Gamma(-l - \tfrac{1}{2} - s)(i\omega r_\rangle)^{l+\frac{1}{2}+2s} \tag{19}$$

This representation gives the asymptotic expansion of $h_l^{(2)}$.

Finally, we represent the bound state solutions as

$$\begin{pmatrix} f_n(r) \\ g_n(r) \end{pmatrix} = U_n(r) \begin{pmatrix} \sqrt{1+\varepsilon_n} \sum_{n_1} [A_{n_1} + B_{n_1}] \\ \sqrt{1-\varepsilon_n} \sum_{n_1} [A_{n_1} - B_{n_1}] \end{pmatrix} (2p_N r)^{n_1} \tag{20}$$

where

$$\begin{pmatrix} A_{n_1} \\ B_{n_1} \end{pmatrix} = \frac{1}{(2\gamma_n + 1)_{n_1} n_1 !} \begin{pmatrix} n_r (1 - n_r)_{n_1} \\ (N_n - \kappa_n)(-n_r)_{n_1} \end{pmatrix} \tag{21}$$

and $_n(r)$ is given by

$$_n(r) = \left[\frac{\Gamma(2\gamma_n + n_r + 1)}{4N_n(N_n - \kappa_n) n_r !} \right]^{\frac{1}{2}} \frac{(2p_N)^{\frac{3}{2}} (2p_N r)^{\gamma_n - 1}}{\Gamma(2\gamma_n + 1)} \tag{22}$$

with

$$\begin{aligned} & p_N = \frac{Z\alpha}{N_n}, \quad \varepsilon_n = \frac{E_n}{m}, \quad N = [n^2 - 2n_r(|\kappa_n| - \gamma_n)]^{\frac{1}{2}}, \\ & n_r = n_r - |\kappa_n|, \quad \kappa_n = \pm(j_n + \tfrac{1}{2}), \quad \gamma_n = [\kappa_n^2 - (Z\alpha)^2]^{\frac{1}{2}} \end{aligned} \tag{23}$$

Then we can represent the $\Delta E_n^{S.E}$ as a four dimensional complex integral and energy integral:

$$\begin{aligned} & \Delta E_n^{\text{S.E.}}(\text{high energy}) = 4\alpha \int \frac{dz}{2\pi i} \sum_s \{ \\ & T_{\alpha\alpha'}^{11} R_{\alpha\alpha'} [A_{n_1} A_{n_2} |W_{ns}^{lm}|^2 (1+\varepsilon_n) - B_{n_1} B_{n_2} \mathbf{K}_{n's}^{lm} \cdot \mathbf{K}^{*lm}_{n's} (1-\varepsilon_n)] \\ & + T_{\alpha\alpha'}^{22} R_{\alpha\alpha'} [B_{n_1} B_{n_2} |W_{n's'}^{lm}|^2 (1-\varepsilon_n) - A_{n_1} A_{n_2} \mathbf{K}_{ns'}^{lm} \cdot \mathbf{K}^{*lm}_{ns'} (1+\varepsilon_n)] \\ & + \left[A_{n_1} B_{n_2} W_{ns}^{lm} W^{*lm}_{n's'} (1-\varepsilon_n^2)^{\frac{1}{2}} - B_{n_1} A_{n_2} \mathbf{K}_{n's}^{lm} \cdot \mathbf{K}^{*lm}_{ns'} (1-\varepsilon_n^2)^{\frac{1}{2}} \right] T_{\alpha\alpha'}^{12} R_{\alpha\alpha'} \\ & + \left[B_{n_1} A_{n_2} W_{n's'}^{lm} W^{*lm}_{ns} (1-\varepsilon_n^2)^{\frac{1}{2}} - A_{n_1} B_{n_2} \mathbf{K}_{ns'}^{lm} \cdot \mathbf{K}^{*lm}_{n's} (1-\varepsilon_n^2)^{\frac{1}{2}} \right] T_{\alpha\alpha'}^{21} R_{\alpha\alpha'} \Big\} \end{aligned} \tag{24}$$

where $R_{\alpha\alpha'}$, $T^{mn}{}_{\alpha\alpha'}$, b and c are defined by

$$R_{\alpha\alpha'} = -\sum_{n_1 n_2} a_{n_1 n_2} \frac{\pi}{2^3} e^{i\frac{\pi}{2}(l+\frac{1}{2})} \times$$

$$\frac{\Gamma(l+\frac{1}{2}+v_1)}{\Gamma\left(1+\frac{l+\frac{1}{2}-v_1}{2}\right)}\Gamma(-v_2)\Gamma(-l-\tfrac{1}{2}-v_2)\frac{\Gamma(-s_1)\Gamma(\alpha+s_1)\Gamma(2\gamma+1_1)}{\Gamma(\alpha_1)\Gamma(2\gamma+1+s_1)} \tag{25}$$

$$\times\frac{\Gamma(-s_2)\Gamma(\alpha'+s_2)\Gamma(-2\gamma-s_2)}{\Gamma(\alpha')\Gamma(-2\gamma)}k(z)\left(\frac{2p_N}{2ip}\right)^{2\gamma_n+n_1+n_2+1}\frac{\Gamma(b)\ {}_2F_1\left(1,b,c+1;\frac{1}{2}\right)}{c\left(-1+\frac{p_N}{ip}\right)^{b}}$$

$$T^{mn}{}_{\alpha\alpha'}=\left[\left(\kappa-\frac{iZ\alpha}{p}\right)\delta_{\alpha,\gamma-i\nu}-\theta\left(2m-\frac{3}{2}\right)\delta_{\alpha,\gamma+1-i\nu}\right]$$
$$\times\left[\left(\kappa-\frac{iZ\alpha}{p}\right)\delta_{\alpha',\gamma-i\nu}-\theta\left(2n-\frac{3}{2}\right)\delta_{\alpha',\gamma+1-i\nu}\right] \tag{26}$$

$$b=2\gamma_n+n_1+n_2+1+2\gamma+s_1+s_2-s_3+2s_4+l+\tfrac{1}{2} \tag{27}$$

$$c=\gamma_n+n_1+s_2+2s_4+l \tag{28}$$

Regularization of $R_{\alpha\alpha'}$:

In the electron Green's function when $Z\alpha$ goes to zero $G(r,r';z)$ becomes the free particle Green's function. But when we used $G(r,r';z)$ in the calculation of self energy we must be carefull. Because the self energy of the free particle is included *in the definition of the particle itself.* In order to get rid of this problem, we must substract the free particle contribution from *G*. This is a new kind of *renormalization.*

We also have this ambiguity when we examined the Mellin-Barnes type integral representation of the Green's function. In $R_{\alpha\alpha'}$ we have s_1, s_2, v_1 and v_2 integrals. In v_1 and v_2 integrals, the contours are well defined. As it has been pointed out in the above in s_1 and s_2 integrals, the contours are not well defined. They are well defined when $Z\alpha \neq 0$. But they are not well defined when $Z\alpha = 0$, because two sets of the poles are not seperated and in this limit they coincide .That means the free particle limit of the transition amplitudes or the matrix elements of Green's function of the relativistic Coulomb problem are not well defined. If we use the direct product of the contours of s_1 and s_2 integrals we get formally divergent series.

Generally, when there is a double complex integral we cannot define the integration contour as a direct product of two separate contours of the one dimensional complex integrals.[9] In order to understand the physical meaning of this formal divergence and in order to regularize these integrals we examine the poles of the in the complex s_1 and s_2 planes.

In the Appendix A we discuss the regularization of the scattering solutions. We know from the scattering theory that the scattering solutions of the Coulomb problem always include plane wave or free particle solutions. Scattering probabilities or cross sections are physically measurable quantities. In order to calculate the physically measurable quantities we change the boundary conditions of the scattering solutions. The scattering amplitutes or cross sections are defined by substracting the plane waves from the scattering solutions. Then the final results are finite.

Here we also have the same problem. We are using the transition amplitudes and they also include the plane wave solutions. In order to use the transition amplitudes in the calculation of physical measurable self -energy we regularize them in the same way. That is equivalent to find an integration contour such that it seperates the two sets of poles or zeros in the s_1 and s_2 planes.

The poles of the $R_{\alpha\alpha'}(s_1,s_2)$ in the complex s_1 and s_2 planes are shown in Fig. 2 and 3. When $Z\alpha = 0$ the poles of $\Gamma(\gamma - i\nu + s_2)$ and $\Gamma(-2\gamma - s_2)$ are coincide. In order to separate the poles we regularize the integrals as follows:

$$\frac{\Gamma(\gamma - i\nu)\Gamma(\alpha + s_1)}{\Gamma(\alpha)} =$$

$$\lim_{\varepsilon \to 0}\left\{\frac{\Gamma(\gamma - i\nu)\Gamma(\alpha + s_1)}{\Gamma(\alpha)} - \frac{\Gamma(\gamma - i\varepsilon)\Gamma(\alpha + s_1 - i\varepsilon + i\nu)}{\Gamma(\alpha + i\nu - i\varepsilon)} + \frac{\Gamma(\gamma + i\varepsilon)\Gamma(\alpha + s_1 + i\varepsilon + i\nu)}{\Gamma(\alpha + i\nu + i\varepsilon)}\right\}$$

(29)

and

$$\frac{\Gamma(\gamma - i\nu)\Gamma(\alpha' + s_2)}{\Gamma(\alpha')\Gamma(-\gamma - i\nu)} =$$

$$\lim_{\varepsilon \to 0}\left\{\frac{\Gamma(\gamma - i\nu)\Gamma(\alpha' + s_2)}{\Gamma(\alpha')\Gamma(-\gamma - i\nu)} - \frac{\Gamma(\gamma - i\varepsilon)\Gamma(\alpha' + s_2 - i\varepsilon + i\nu)}{\Gamma(\alpha' + i\nu - i\varepsilon)\Gamma(-\gamma - i\varepsilon)} + \frac{\Gamma(\gamma + i\varepsilon)\Gamma(\alpha' + s_2 + i\varepsilon + i\nu)}{\Gamma(\alpha' + i\nu + i\varepsilon)\Gamma(-\gamma - i\varepsilon)}\right\}$$

(30)

Then we choose the contours such that the poles of $\Gamma(-s_1)$ and the poles of $\Gamma(\alpha + s_1)$ and the zeros of $(\Gamma(2\gamma + 1 + s_1))^{-1}$ are seperated. In the same way the poles of $\Gamma(-2\gamma - s_2)$ and $\Gamma(-s_2)$ and the poles of $\Gamma(\gamma - i\nu + s_2)$ are seperated.

We substitute R_{reg} into $\Delta E_n^{S.E}$ and calculate the complex integrals. We calculate the integral of $R_{\alpha\alpha'}$ as a sum of the residues at the poles $s_1 = -\alpha - p_1$ and $s_2 = -\alpha' - p_2$ where p_1 and p_2 range over *0,1,2,...* . In the similiar way we calculate ν_1 and ν_2 integrals also as residue integrals. They can be written as the sum over the residues at $\nu_1 = -l - \frac{1}{2} - q_1$ and $\nu_2 = q_2$ and $\nu_2 = -l - \frac{1}{2} + q_2$ where q_1 and q_2 range over *0,1,2,....* By using these expressions we do *z*-integration. Thus the self energy contribution to the Lamb shift becomes a finite expression. All of the series in this expression are convergent. In order to compare this result with the experiment we need numerical sum of these series.

APPENDIX A

We discuss the relation between scattering solutions and scattering amplitudes of the Dirac Coulomb problem. In order to obtain a relation between them we examine the regular solutions of the Dirac-Coulomb problem. For the Dirac-Coulomb problem we have a regular solution around the origin and the asymptotic forms of this solution are function of sine or cosine. Here we use the regular solutions at the infinity. The asymptotic form of this solutions give exponantial waves.

Regular solutions at the infinity were developed by Wichmann and Kroll. They are

$$\begin{pmatrix} rf \\ rg \end{pmatrix} = N\begin{pmatrix} i\sqrt{z-1} \\ \sqrt{z+1} \end{pmatrix} e^{ipr}$$
$$\times\left[\left(\kappa - \tfrac{iZ\alpha}{p}\right)\chi(\gamma - i\nu, 2\gamma + 1; -2ipr) \mp (\gamma - i\nu)\chi(\gamma + 1 - i\nu, 2\gamma + 1; -2ipr)\right] \qquad \text{(A-1)}$$

where $\chi(a,c;x)$ is the regular solution of the confluent hypergeometric differential equation at the infinity, and it is the linear combination of the regular and irregular solutions of the confluent hypergeometric differential equation at the origin: Asymptotic form of the regular solution at the infinity gives spherical waves which includes the free particle solutions or unscattered plane waves. The difference between the scattering solutions and the free particle solution is the scattering amplitude and it goes to zero when $Z\alpha$ goes to zero.

$\chi(a,c;x)$ function can be represented as Mellin-Barnes type integral:

$$\chi(\gamma - i\nu, 2\gamma + 1; -2ipr) = \int_{C_s - i\infty}^{C_s + i\infty} \frac{ds}{2\pi i}\Gamma(-s)\frac{\Gamma(\gamma - i\nu + s)\Gamma(-2\gamma - s)}{\Gamma(\gamma - i\nu)\Gamma(-2\gamma)} z^s$$
$$\equiv \int_{C_s - i\infty}^{C_s + i\infty} \frac{ds}{2\pi i}\Gamma(-s)M(\gamma, \nu; s)z^s \qquad \text{(A-2)}$$

where C_s is the integration contour which can be choosen such that the set of the poles $\Gamma(-2\gamma - s)$ and $\Gamma(-s)$ and the poles of $\Gamma(\gamma - i\nu + s)$ are seperated. However, Mellin-Barnes type integral representation of $\chi(\gamma - i\nu, 2\gamma + 1; -2ipr)$ is not well defined when $i\nu$ goes to zero. In this limit some of the poles of $\Gamma(\gamma - i\nu + s)$ and $\Gamma(-2\gamma - s)$ coincide and the seperation of two sets of poles is not clear. In order to solve this ambiguity we regularize the $M(\gamma, \nu; s)$ in the following way:

$$M(\gamma, \nu; s) = \lim_{\varepsilon \to 0}\left[M(\gamma, \nu; s) - M(\gamma, \varepsilon; s) + M(\gamma, -\varepsilon; s)\right] \qquad \text{(A-3)}$$

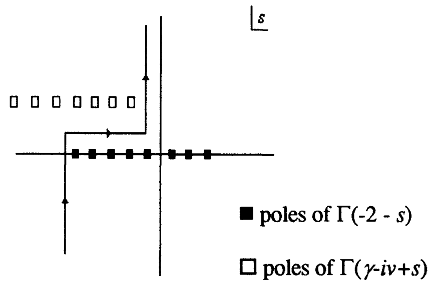

Figure 2. Poles of $M(\gamma, \nu, s)$

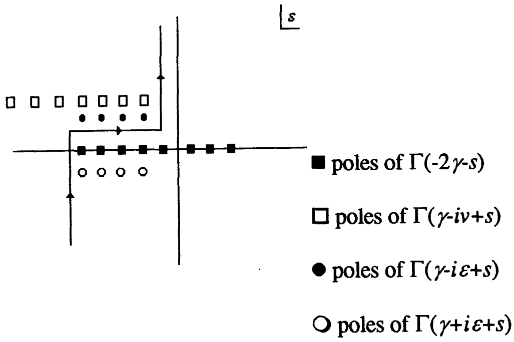

Figure 3. Poles of $M_{reg}(\gamma, \nu, s)$.

The new set of poles are in Fig.3. Where we have choosen the integration contour such that one of the regularizing terms gives contribution to the integral. We close the integration contour from left hand side. Then the asymptotic form of this solution becomes

$$\chi_{reg}(\gamma - i\nu, 2\gamma + 1; -2ipr) \cong (-2ipr)^{-\gamma + i\nu} \frac{\Gamma(-\gamma - i\nu)}{\Gamma(-2\gamma)} - (-2ipr)^{-\gamma + i\varepsilon} \frac{\Gamma(-\gamma - i\varepsilon)}{\Gamma(-2\gamma)} \tag{A-4}$$

The asymptotic solution of the Dirac Coulomb problem is

$$\begin{pmatrix} rf \\ rg \end{pmatrix} = N \begin{pmatrix} i\sqrt{z-1} \\ \sqrt{z+1} \end{pmatrix} \frac{e^{ipr}}{\Gamma(-2\gamma)} \left[e^{\frac{\pi\nu}{2} + i \arg \Gamma(-\gamma - i\nu) + i\delta_\nu} \left| \Gamma(-\gamma - i\nu) \right| - \left| \Gamma(-\gamma) \right| \right] \tag{A-5}$$

where

$$\delta_\nu = \nu \ln 2pr \tag{A-6}$$

This solution represents a spherical wave and and when ν goes to zero it becomes zero. The second term corresponds to the spherical wave expansion of $e^{ik\cdot r}$. By this regularization we obtain a transformation from the scattering solution to the scattering amplitudes.

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