# Light-front two-dimensional QED: Self-field approach

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The self-field approach to quantum electrodynamics (QED) is used to study the bound state problem in light-front two-dimensional QED with massive matter fields. A composite matter field describing bound states is introduced and the relativistic bound state equation for the composite field including a self-potential is obtained. The Hamiltonian form of the bound state equation in terms of the invariant mass squared operator is given. The eigenvalue problem of this operator is solved for a fixed value of the self-potential, the corresponding eigenfunctions and the mass spectrum are found. In the case of massless matter fields, there are no self-field terms in the bound state equation, and the invariant mass spectrum can be evaluated explicitly. Possible ways of deriving more complete information about the bound state spectrum are briefly discussed.
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## I. INTRODUCTION

Bound state problems in relativistic field theory, particularly quantum electrodynamics (QED), are basic ones for particle physics. It is well known that in perturbative QED the bound state problems cannot be treated starting from first principles. Instead one begins from a Schrödinger or Dirac-like equation obtained from some approximation to the Bethe-Salpeter relations and then calculates the perturbation diagrams to the bound state solutions of these equations.

A nonperturbative treatment of two and many body systems in closed form is possible in the self-field formulation of QED [1,2]. Here one starts from two fermion fields $\psi_1$ and $\psi_2$ coupled by the usual electromagnetic minimal coupling. One then eliminates the electromagnetic field, introduces a composite field $\Phi=\psi_1\otimes\psi_2$ and derives a two-body wave equation for this composite field including the radiative corrections. This relativistic bound state equation is a genuine and exact one obtained directly from the action without using any approximation. For two-dimensional QED, the two-body wave equation was constructed and analyzed in Ref. [3].

In the present paper we aim to apply the self-field approach to light-front two-dimensional QED ($QED_2$) defined on the circle. Light-front quantization [4] has a number of advantages. These include kinematical Lorentz boosts and a simpler vacuum structure. By quantizing at equal time on the light front a gauge theory can be reduced to an eigenvalue problem for the invariant mass squared operator. The relativistic spectrum emerges as the set of eigenvalues of this operator.

In the self-field approach the quantum theory is first quantized. The electromagnetic field has no separate degrees of freedom, they are determined by the matter degrees of freedom, but then one must include nonlinear self-field terms. We consider the standard $QED_2$ with one matter field $\psi$ and describe bound states by the composite field $\Phi=\psi\otimes\psi$. We use the advantages of the light-front formulation in order to get exact expressions for the bound state wave functions and spectrums for both massless and massive fermions.

Our paper is organized as follows. In Sec. II we give the light-front formulation of $QED_2$. In Sec. III we introduce the composite matter field and rewrite the action of the model entirely in terms of this field. We then vary the action with respect to the composite field and derive a bound state equation. In Sec. IV we give the Hamiltonian form of the bound state equation in terms of the invariant mass squared operator $M^2$. We solve exactly the eigenvalue problem for $M^2$ for a fixed value of the self-potential and find explicitly the corresponding eigenfunctions and the spectrum. We consider also the massless case when the mass of the matter fields is zero. In Sec. V we conclude with a discussion.

## II. LIGHT-FRONT TWO-DIMENSIONAL QED

Two-dimensional quantum electrodynamics describes matter fields interacting with an electromagnetic field in two-dimensional space-time. The Lagrangian density of $QED_2$ is

$$\mathcal{L}=i\bar{\psi}\gamma^\mu\partial_\mu\psi-m\bar{\psi}\psi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-e\bar{\psi}\gamma^\mu\psi A_\mu, \quad (2.1)$$

where $(\mu,\nu)=\overline{0,1,}$ $\gamma^\mu$ are Dirac matrices, $F^{\mu\nu}\equiv\partial^\mu A^\nu-\partial^\nu A^\mu$ is the electromagnetic field strength. The matter field $\psi$ is a two-component Dirac spinor, and $\bar{\psi}=\psi^\star\gamma^0$.

We choose the light-front coordinates $x^\pm=x^0\pm x^1$, $x^+$ and $x^-$ playing the role of time and space coordinates, respectively. The metric tensor $g^{\mu\nu}$ for the light-front coordinates has the form

$$g^{++}=g^{--}=0, \quad g^{+-}=g^{-+}=2.$$

We must distinguish upper and lower indices; for example, time and space derivatives are

$$\partial^-=2\partial_+\equiv 2\partial/\partial x^+,$$

$$\partial^+=2\partial_-\equiv 2\partial/\partial x^-.$$

The algebra of Dirac matrices $\gamma^\pm=\gamma^0\pm\gamma^1$ is

$$\gamma^+\gamma^+=\gamma^-\gamma^-=0,$$

$$\gamma^+\gamma^-+\gamma^-\gamma^+=4.$$

Using this algebra, we can define projection operators $\Lambda^{(\pm)} \equiv \frac{1}{4}\gamma^{\mp}\gamma^{\pm}$ and projected spinors $\psi_{\pm} \equiv \Lambda^{(\pm)}\psi$.

In terms of the light-front coordinates the Lagrangian density (2.1) is rewritten as

$$\mathcal{L} = (\psi_+^\star i\partial^- \psi_+ + \psi_-^\star i\partial^+ \psi_-) - \frac{1}{2}m(\psi_-^\star \gamma^+ \psi_+ + \psi_+^\star \gamma^- \psi_-)$$

$$+ \frac{1}{2}(\partial_- A_+ - \partial_+ A_-)^2 - \frac{1}{2}e(j^+ A_+ + j^- A_-), \tag{2.2}$$

where $A_{\pm} = A_0 \pm A_1$, while $j^{\pm} \equiv 2\psi_{\pm}^\star \psi_{\pm}$ are light-front matter currents.

In what follows we work in the light-front gauge $A_- = 0$. We suppose that light-front space is a circle of length $L$, $0 \leqslant x^- < L$, and impose the boundary conditions

$$A_+(x^+,0) = A_+(x^+,L), \tag{2.3a}$$

$$\psi_{\pm}(x^+,L) = \exp\{i2\pi\kappa_{\pm}\}\psi_{\pm}(x^+,0), \tag{2.3b}$$

$\kappa_{\pm}$ being arbitrary numbers.

The action of the light-front $QED_2$ is

$$W = \int_{-\infty}^{\infty} dx^+ \int_0^L dx^- \mathcal{L}(x^+,x^-).$$

The Lagrange-Euler equations deduced from it are

$$\partial^+ E = -ej^+, \tag{2.4a}$$

$$\partial^- E = ej^-, \tag{2.4b}$$

and

$$i\partial^+ \psi_- = \frac{1}{2}m\gamma^+ \psi_+, \tag{2.5a}$$

$$i\partial^- \psi_+ = \frac{1}{2}m\gamma^- \psi_- + eA_+ \psi_+, \tag{2.5b}$$

where $E \equiv \partial_- A_+$ is the electric field strength. From Eqs. (2.4a), (2.4b) we have

$$\partial_- j^- + \partial_+ j^+ = 0,$$

i.e., the matter current is conserved.

Equation (2.4a) gives us the Gauss' law and the boundary conditions for $E(x^-,x^+)$:

$$E(L,x^+) - E(0,x^+) = -e\frac{1}{2}\int_0^L dx^- j^+ \equiv -Q^+,$$

i.e., the electric field is not single-valued if the light-front charge $Q^+$ is nonzero (see below).

The equation for $\psi_+$ involves the light-front time derivative, so $\psi_+$ is a dynamical degree of freedom. On the other hand, the equation for $\psi_-$ involves only spatial derivative, so $\psi_-$ is a constrained degree of freedom that should be eliminated in favor of $\psi_+$:

$$\psi_-(x^-,x^+) = -\frac{i}{8}m\gamma^+ \int_0^L \epsilon(x^- - y^-)\psi_+(y^-,x^+)dy^-. \tag{2.6}$$

The solution (2.6) fulfills antiperiodic boundary conditions, $\psi_-(L,x^+) = -\psi_-(0,x^+)$, so that $\kappa_- = \pm\frac{1}{2}$.

The action of the electromagnetic field can be reexpressed by a partial integration, using Eqs. (2.4a), (2.4b) and the boundary conditions, as

$$\frac{1}{2}\int_{-\infty}^{\infty} dx^+ \int_0^L dx^- (\partial_- A_+)^2$$

$$= \frac{e}{4}\int_{-\infty}^{\infty} dx^+ \int_0^L dx^- A_+ j^+ - \frac{1}{2}\int_{-\infty}^{\infty} dx^+ A_+(0,x^+)Q^+.$$

With Eq. (2.6), the total action becomes

$$W[\psi,A] = \int_{-\infty}^{\infty} dx^+ \int_0^L dx^- \left( \psi_+^\star i\partial^- \psi_+ - \frac{e}{4}j^+ A_+ \right)$$

$$- \frac{1}{2}\int_{-\infty}^{\infty} dx^+ A_+(0,x^+)Q^+$$

$$- \frac{i}{2}m\int_{-\infty}^{\infty} dx^+ \int_0^L dx^- \int_0^L dy^- \psi_+^\star(x^-,x^+)$$

$$\times \epsilon(x^- - y^-)\psi_+(y^-,x^+). \tag{2.7}$$

If we solve Eqs. (2.4a), (2.4b), express $A_+$ in terms of $j^+$ and insert the expression obtained into Eq. (2.7), then we get an action written only in terms of the matter fields.

The electromagnetic field equations can be rewritten in the form

$$\partial_-^2 A_+ = -\frac{1}{2}ej^+, \tag{2.8a}$$

$$\partial_- \partial_+ A_+ = \frac{1}{2}ej^-. \tag{2.8b}$$

Equations (2.8a), (2.8b) with the periodic boundary conditions for the gauge field are solved by

$$A_+(x^-,x^+) = -\frac{1}{2}e\int_0^L dy^- D(x^-,y^-|L)j^+(x^+,y^-), \tag{2.9}$$

where the Green's function is

$$D(x^-,y^-|L) \equiv \frac{1}{2}|x^- - y^-| + \frac{x^- y^-}{L} - \frac{1}{2}x^-,$$

i.e., $A_+(x^-,x^+)$ is completely determined by $j^+(x^-,x^+)$.

The Green's function $D(x^-,y^-|L)$ is not symmetric in $x^-$ and $y^-$. The reason for that is nonzero $Q^+$. We can easily see this if for a moment use

$$D^{\rm sym}(x^-,y^-|L)\equiv\frac{1}{2}|x^--y^-|+\frac{x^-y^-}{L},$$

$$D^{\rm sym}(x^-,y^-|L)=D^{\rm sym}(y^-,x^-|L),$$

instead of $D(x^-,y^-|L)$ in Eq. (2.9). The boundary conditions for the field $A_+$ become

$$A_+(L,x^+)=A_+(0,x^+)+\frac{1}{2}Q^+,$$

i.e., $A_+$ is periodic only if $Q^+=0$. So $D^{\rm sym}(x^-,y^-|L)$ is a right choice only for the vanishing charge $Q^+$.

However, only the symmetric part of the Green's function contributes to the action. If we insert Eq. (2.9) into Eq. (2.7), we obtain the action in the light-front gauge as

$$\begin{aligned}W[\psi]=&\int_{-\infty}^{\infty}dx^+\int_0^L dx^-\psi_+^\star i\partial^-\psi_+\\&-\frac{i}{2}m\int_{-\infty}^{\infty}dx^+\int_0^L dx^-\int_0^L dy^-\psi_+^\star\\&\times(x^-,x^+)\epsilon(x^--y^-)\psi_+(y^-,x^+)\\&+\frac{e^2}{8}\int_{-\infty}^{\infty}dx^+\int_0^L dx^-\int_0^L dy^-\\&\times j^+(x^-,x^+)D^{\rm sym}(x^-,y^-|L)j^+(y^-,x^+),\end{aligned}\tag{2.10}$$

the last term representing the current-current interaction.

## III. BOUND STATE EQUATION

Let us define a composite field $\Phi$ by

$$\Phi(x_1^-,x^+|x_2^-,x^+)\equiv\psi_+(x_1^-,x^+)\otimes\psi_+(x_2^-,x^+).\tag{3.1}$$

This is a four-component spinor field, $\Phi^{\alpha\beta}=\psi_+^\alpha\psi_+^\beta$, $(\alpha,\beta)=\overline{1,2}$. However, only one component of the composite field is nonvanishing:

$$\Phi^{11}=\psi_+^1\psi_+^1=\Phi^{++},$$

$$\Phi^{12}=\Phi^{21}=\Phi^{22}=0.$$

The configuration space $(x_1^-,x_2^-)$ is a square of side $L$ ($0\leqslant x_1^-<L$, $0\leqslant x_2^-<L$) with the opposite sides being identified, i.e., a torus.

We can rewrite the action (2.10) entirely in terms of the composite field $\Phi$. In order to do this we multiply the action with the normalization factor (which is constant of motion)

$$\int_0^L dx^-\psi_+^\star(x^-,x^+)\psi_+(x^-,x^+)=1.\tag{3.2}$$

The resultant action in terms of the composite field is

$$\begin{aligned}W[\Phi]=&\frac{1}{2}\int_{-\infty}^{\infty}dx^+\int_0^L dx_1^-\int_0^L dx_2^-\Phi^{\star,++}\\&\times(x_1^-,x^+|x_2^-,x^+)(\pi_{(1)}^-+\pi_{(2)}^-\\&-m\phi^{\rm self})\Phi^{++}(x_1^-,x^+|x_2^-,x^+).\end{aligned}\tag{3.3}$$

Here the index (1) refers to the coordinates of the first field $\psi_+(x^-,x^+)$ in the ansatz (3.1), the index (2) refers to the second field $\psi_+(x_2^-,x^+)$.

The generalized (kinetic) momenta $\pi_{(i)}^\pm$ are given by

$$\pi_{(i)}^\pm=p_{(i)}^\pm+eA_{(i),{\rm self}}^\pm\tag{3.4}$$

with

$$p_{(i)}^\pm\equiv i\partial_{(i)}^\pm$$

and

$$\begin{aligned}A_{(1),{\rm self}}^-(x^-,x^+)\equiv&\phi_{(1)}^{self}(x^-,x^+)\\=&\frac{e}{2}\int_0^L dy^-\int_0^L dz^-\Phi^{\star,++}(y^-,x^+|z^-,x^+)\\&\times D^{\rm sym}(x^-,y^-|L)\Phi^{++}(y^-,x^+|z^-,x^+),\\A_{(2),{\rm self}}^-(x^-,x^+)\equiv&\phi_{(2)}^{\rm self}(x^-,x^+)\\=&\frac{e}{2}\int_0^L dy^-\int_0^L dz^-\Phi^{\star,++}(y^-,x^+|z^-,x^+)\\&\times D^{\rm sym}(x^-,z^-|L)\Phi^{++}(y^-,x^+|z^-,x^+),\\A_{(i),{\rm self}}^+(x^-,x^+)=&0,\end{aligned}$$

the self-potentials $\phi_{(i)}^{\rm self}$ being nonlinear integral expressions.

The self-potential in the mass term

$$\begin{aligned}\phi^{\rm self}(x^+)=&\int_0^L dy^-\int_0^L dz^-\int_0^L d\eta^-\Phi^{\star,++}\\&\times(z^-,x^+|y^-,x^+)\\&\times i\epsilon(y^--\eta^-)\Phi^{++}(\eta^-,x^+|z^-,x^+)\end{aligned}\tag{3.5}$$

does not depend on the light-front space coordinate and does not contribute to the action in the massless case.

We can write the action (3.3) in another, equivalent form. If $\phi_{(i)}^{\rm self}$ are not included into the generalized momenta and are considered separately, then these self-potentials can be shown to reduce to the potential $e^2D^{\rm sym}$. The action becomes

$$\begin{aligned}W[\Phi]=&\frac{1}{2}\int_{-\infty}^{\infty}dx^+\int_0^L dx_1^-\int_0^L dx_2^-\Phi^{\star,++}\\&\times(x_1^-,x^+|x_2^-,x^+)\cdot(p_{(1)}^-+p_{(2)}^-+e^2D^{\rm sym}\\&-m\phi^{\rm self})\Phi^{++}(x_1^-,x^+|x_2^-,x^+).\end{aligned}\tag{3.6}$$

Since there is only one self-potential in the action (3.6), it is much simpler to work with this action rather than with the action (3.3).

In the self-field approach the $\psi$ currents are actual material charge currents, and not just probability currents. The corresponding charges are actual matter charges. With the normalization factor (3.2), we get

$$Q^+ = e,$$

i.e., $Q^+$ is actual charge of the positive chirality matter. For our model with a single charge on the circle, the electric field is not therefore periodic. We must consider at least two matter fields with charges equal in magnitude and opposite in sign in order to get vanishing total charge and single-valued total electric field.

Now we require the action (3.6) to be stationary not with respect to the variation of the original matter field $\psi_+(x^-,x^+)$ but with respect to the total composite field only. This is a weaker condition and leads to the following equation for $\Phi^{++}(x_1^-,x^+|x_2^-,x^+)$ in configuration space:

$$(p_{(1)}^- + p_{(2)}^- + e^2 D^{\rm sym} - m\phi^{\rm self})\Phi^{++}(x_1^-,x^+|x_2^-,x^+) = 0. \tag{3.7}$$

We next introduce center of mass and relative coordinates according to

$$P^\pm = p_{(1)}^\pm + p_{(2)}^\pm, \quad p^\pm = p_{(1)}^\pm - p_{(2)}^\pm,$$

$$R = x_{(1)}^- + x_{(2)}^-, \quad r = x_{(1)}^- - x_{(2)}^-.$$

The configuration space $(r,R)$ is again a torus, but with the circle length $2L$ $(-L \leqslant r < L,\ 0 \leqslant R < 2L)$. The function $D^{\rm sym}(x_1^-,x_2^-|L)$ can be rewritten as a sum of center of mass and relative parts:

$$D^{\rm sym}(x_1^-,x_2^-|L) = D_-(r|L) + D_+(R|L),$$

$$D_-(r|L) \equiv \frac{1}{2}|r| - \frac{1}{4L}r^2,$$

$$D_+(R|L) \equiv \frac{1}{4L}R^2.$$

Equation (3.7) becomes

$$P^-\Phi^{++}(r,x^+|R,x^+) = \{-e^2(D_-(r) + D_+(R)) - m\phi^{\rm self}\}\Phi^{++}(r,x^+|R,x^+). \tag{3.8}$$

Equation (3.8) is a Hamiltonian form of the bound state equation. We have in this equation only one time variable conjugate to the center of mass energy $P^-$; the relative energy $p^-$ does not enter.

## IV. ANALYSIS OF BOUND STATE EQUATION

Let us define the operator $M^2 = P^+P^-$, so that its eigenvalues correspond to the invariant mass spectrum of the theory. By acting on Eq. (3.8) by $P^+$ we get the Hamiltonian form of the bound state equation in terms of $M^2$:

$$M^2\Phi^{++} = \left\{ -4ie^2\frac{\partial D_+(R)}{\partial R} + 4i[e^2 D_-(r) + e^2 D_+(R) - m\phi^{\rm self}]\frac{\partial}{\partial R}\right\}\Phi^{++}. \tag{4.1}$$

To find eigenfunctions and eigenvalues of $M^2$ we must solve the equation

$$M^2\Phi^{++} = \sigma\Phi^{++}, \tag{4.2}$$

where $\sigma$ has the dimension $\langle \text{energy}\rangle^2$.

With the normalization condition

$$\int_0^{2L} dR \int_{-L}^{L} dr \Phi^{\star,++}(r,R)\Phi^{++}(r,R) = 1,$$

Equation (4.2) is solved by the eigenfunctions

$$\Phi_n^{++}(r,R) = C(m,\phi)\frac{f_m(r)}{R^2 + d_m(r)}\exp\left\{ i\frac{L}{e^2}\sigma_n \mathcal{F}(d_m(r),R)\right\}, \tag{4.3}$$

where

$$d_m(r) \equiv -\frac{4L}{e^2}m\phi + 4LD_-(r),$$

$\phi$ is a fixed value of the self-potential $\phi^{\rm self}$, $f_m(r)$ is an arbitrary function

$$\mathcal{F}[d_m(r),R] = \begin{cases} \dfrac{1}{\sqrt{d_m(r)}}\arctan\left(\dfrac{R}{\sqrt{d_m(r)}}\right) & \text{for } d_m(r) > 0, \\ \dfrac{1}{2\sqrt{|d_m(r)|}}\ln\left|\dfrac{\sqrt{|d_m(r)|} - R}{\sqrt{|d_m(r)|} + R}\right| & \text{for } d_m(r) < 0, \end{cases}$$

and

$$\mathcal{F}(d_m(r),R) = -\frac{1}{R} \quad \text{for } d_m(r) = 0.$$

The normalization constant $C(m,\phi)$ is

$$C(m,\phi) = \left( \int_0^{2L} dR \int_{-L}^{L} dr \frac{f_m^2(r)}{[R^2 + d_m(r)]^2}\right)^{-1/2}.$$

The eigenfunctions $\Phi_n^{++}$ become singular at those points of the configuration space $(r,R)$ where

$$R^2 + d_m(r) = 0.$$

However, if

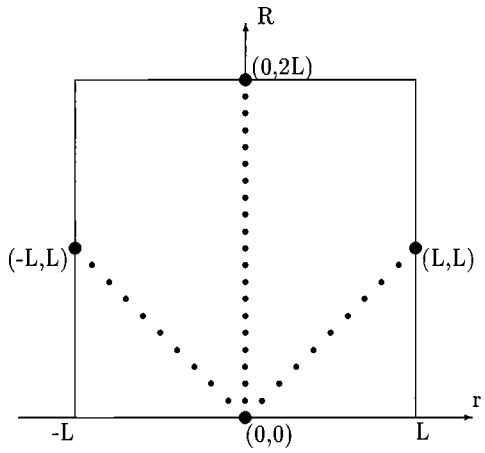



FIG. 1. A schematic representation of the boundary conditions for the composite matter field on the configuration space ($-L \leqslant r < L$, $0 \leqslant R < 2L$). Each two boundary points connected by a dotted line are related also by the corresponding boundary condition.

$$\phi > \frac{5}{4}\frac{e^2 L}{m}, \tag{4.4}$$

then all singularity points are outside of the torus ($-L \leqslant r < L$, $0 \leqslant R < 2L$). There are no singularity points also for negative values of $\phi^{\mathrm{self}}$. For any nonzero $m$ we can therefore choose $\phi$ in such way that the condition (4.4) is fulfilled and the eigenfunctions $\Phi_n^{++}$ are not singular. For $m=0$, the condition (4.4) is not already valid, and the eigenfunctions are singular at one point of the configuration space, namely, ($r=0$, $R=0$).

### A. Spectrum

The spectrum of eigenvalues $\sigma_n$ is fixed by the boundary conditions. From the boundary conditions for the individual fields $\psi_+$ we can deduce in general three boundary conditions for the composite field $\Phi^{++}$ (see Fig. 1):

$$\Phi^{++}(L|L) = \exp\{i2\pi\kappa_+\}\Phi^{++}(0|0),$$

$$\Phi^{++}(-L|L) = \exp\{i2\pi\kappa_+\}\Phi^{++}(0|0),$$

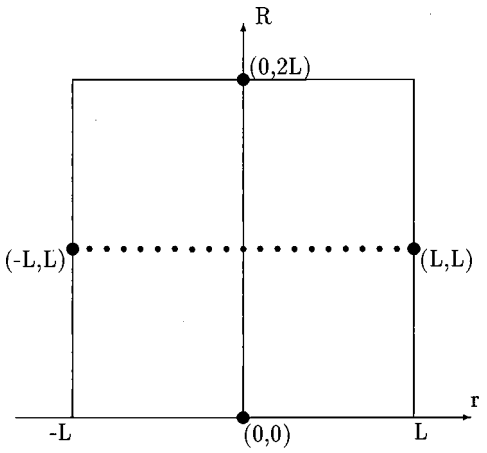
$$\Phi^{++}(0|2L) = \exp\{i4\pi\kappa_+\}\Phi^{++}(0|0).$$

Nevertheless, only part of them are really valid. Indeed, the preexponential in the solution (4.3) is not a constant and depends on the relative coordinate $r$. Only those boundary points at which the values of the preexponential coincide can be used in the boundary conditions. Let us first put $f_m(r)=1$. Then the preexponential $1/[R^2+d_m(r)]$ takes the same value at two boundary points $(-L,L)$ and $(L,L)$. So we have one boundary condition [see Fig. 2(a)]

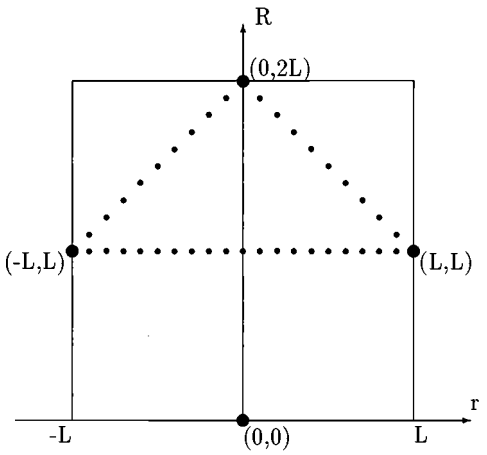
$$\Phi^{++}(-L,L) = \Phi^{++}(L,L) \tag{4.5}$$

which can be considered as an equation for $\sigma$, while $\kappa_+$ remains arbitrary.

For the solution (4.3) with $f_m(r)=1$, from Eq. (4.5) we get




(a)




(b)

FIG. 2. A schematic representation of the boundary conditions for the solution (4.3). (a): $f_m(r)=1$, (b): $f_m(r)\neq 1$.

$$\frac{L}{e^2}\sigma_n\{\mathcal{F}[d_m(-L),L]-\mathcal{F}[d_m(L),L]\} = 2\pi n, \quad n \in \mathcal{Z}.$$

Since $d_m(-r)=d_m(r)$, the expression in the squared brackets vanishes and $\sigma_n$ drops out of the boundary conditions. In the case $f_m(r)=1$ we therefore cannot derive any information concerning the spectrum.

For the general case $f_m(r)\neq 1$, we can use one boundary point more in the boundary conditions. If we take $f_m(r)$ with the boundary values connected as

$$f_m(-L) = f_m(L),$$

$$\frac{f_m(L)}{f_m(0)} = \frac{L^2+d_m(L)}{4L^2+d_m(0)},$$

then the preexponential $f_m(r)/[R^2+d_m(r)]$ takes the same value at three boundary points $(-L,L)$, $(L,L)$, and $(0,2L)$. The corresponding boundary conditions are [see Fig. 2(b)]

$$\Phi^{++}(0|2L)=\exp\{i2\pi\kappa_+\}\Phi^{++}(L|L),$$

$$\Phi^{++}(0|2L)=\exp\{i2\pi\kappa_+\}\Phi^{++}(-L|L),$$

$$\Phi^{++}(L|L)=\Phi^{++}(-L|L).$$

Again $\sigma_n$ drops out of the third boundary condition, while the first two ones coincide and give us for $\kappa_+=0$ the spectrum

$$\sigma_n=\frac{2\pi e^2}{\alpha_m}n, \quad n\in\mathcal{Z}, \tag{4.6}$$

which is linear in $n$, where

$$\alpha_m\equiv L\{\mathcal{F}[d_m(0),2L]-\mathcal{F}[d_m(L),L]\},$$

and $n$ must be taken positive for $\alpha_m>0$ and negative for $\alpha_m<0$. Equations (4.3) and (4.6) represent a solution of the invariant mass squared operator eigenvalue problem for a fixed value of $\phi^{\rm self}$.

### B. Massless case

For the massless case, there are no self-energy terms in the bound state equation. The eigenfunctions of $M^2$ become

$$\Phi_n^{(0),++}=C(0,0)\frac{f_0(r)}{R^2+d_0^{(0)}(r)} \times\exp\left\{i\frac{L}{e^2}\sigma_n^{(0)}\mathcal{F}^{(0)}[d_0^{(0)}(r),R]\right\}, \tag{4.7}$$

where $d_0^{(0)}(r)=4LD_-(r)$ is positive for all nonzero $r$ and vanishes at $r=0$. The superscript (0) means that Eq. (4.7) is the solution of the invariant mass squared operator eigenvalue problem without the self-energy terms.

We easily evaluate $\alpha_0^{(0)}$ as

$$\alpha_0^{(0)}=-\frac{1}{2}\left(1+\frac{\pi}{2}\right)<0,$$

so that the spectrum takes the form

$$\sigma_n^{(0)}=\frac{8\pi e^2}{\pi+2}n, \quad n=0,1,2,\ldots. \tag{4.8}$$

For the boundary values of $f_0(r)$ we have

$$f_0(-L)=f_0(L),$$

$$f_0(L)=\frac{1}{2}f_0(0).$$

One possible choice of $f_0(r)$ is

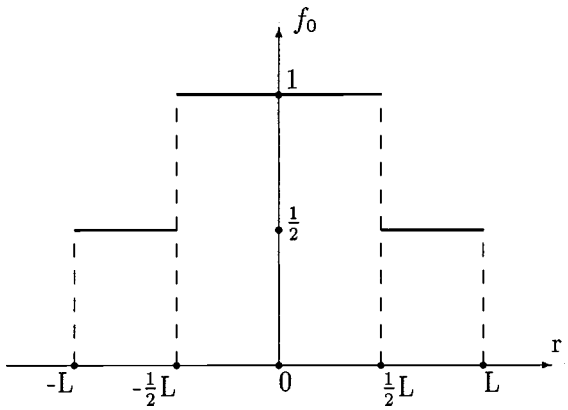



FIG. 3. A special choice of the function $f_0(r)$: $f_0(r)=\theta[(L^2/4)-r^2]+\frac{1}{2}\theta[r^2-(L^2/4)]$.

$$f_0(r)=\theta\left(\frac{L^2}{4}-r^2\right)+\frac{1}{2}\theta\left(r^2-\frac{L^2}{4}\right)$$

(see Fig. 3). With this choice the normalization constant takes the value

$$C(0,0)=L\left(\frac{1}{4}\ln\frac{3}{2}+\ln\frac{11}{7}\right)^{-1/2}\approx 1{,}35L.$$

## V. DISCUSSION

(1) In our study of the bound state problem in two-dimensional QED we have tried to combine the advantages of both the self-field approach and the light-front formulation. The self-field approach allows us to construct a relativistic bound state equation. Bound states are described by a composite matter field which is a bilinear combination of the original matter field and therefore four-component. On the light front only one of these components is nonzero, so the bound state equation is simply onecomponent.

We have derived the Hamiltonian form of the bound state equation for the invariant mass squared operator. The equation includes a self-potential which enters the mass term. For a fixed value of the self-potential, we have solved the eigenvalue problem of the invariant mass squared operator and found its spectrum as well as the corresponding eigenfunctions. The invariant mass spectrum turns out to be discrete and linear. For the massless matter fields, when the self-potential contribution vanishes, we have evaluated the invariant mass spectrum explicitly.

(2) Our solution of the invariant mass squared operator eigenvalue problem is not complete. In particular, in the massless case the eigenfunctions are singular at one of the boundary points, so only a part of the boundary conditions can be employed. Since just the boundary conditions determine the spectrum, some information about the spectrum is lost. As a result, the invariant mass spectrum obtained in our work differs from the well-known one of the second-quantized massless $QED_2$ on light front (the light-front Schwinger model) [5].

There are several puzzles or problems regarding the light-front formulation, e.g., the null plane and missing degrees of

freedom, causality and boundary conditions, the zero mass limit, and so on (see, for example, Ref. [6]). Many aspects of the Schwinger model on light front such as the anomaly and the $\theta$ vacuum cannot be understood without worrying about these problems [7]. Ignoring the problems works only in one case when we evaluate the spectrum. The spectrum of physical bosons of the light-front Schwinger model is reproduced exactly without a complete formulation in which all these problems are solved.

In the self-field approach we first find the eigenfunctions of the two-body Hamiltonian or the invariant mass squared operator, and then impose the boundary conditions and determine the spectrum. An exact expression for the spectrum cannot be therefore reproduced unless a complete construction of the two-body Hamiltonian on the light front is given.

In the usual equal-time formulation of the self-field QED we have only one time variable in the relativistic bound state equation [2,3]. We could start in principle with the field

$$\Phi(x^1_{(1)},x^0_{(1)}|x^1_{(2)},x^0_{(2)})=\psi_1(x^1_{(1)},x^0_{(1)})\otimes\psi_2(x^1_{(2)},x^0_{(2)})$$

composed of the matter fields $\psi_1$ and $\psi_2$ taken at different times. However, only the center of mass energy conjugate to the time $t\equiv x^0_{(1)}+x^0_{(2)}$ enters the bound state equation and contributes to the two-body Hamiltonian. The relative energy conjugate to the relative time $\tau\equiv x^0_{(1)}-x^0_{(2)}$ drops out of this equation automatically. That is why, without loss of generality, we put $x^0_{(1)}=x^0_{(2)}$ from the beginning.

In the light-front formulation the situation seems to be quite different. The time and spatial variables are mixed, so both the light-front center of mass and relative energies can contribute to the bound state equation. In the present work, following the equal-time formulation prescription we have taken the matter fields in the ansatz (3.1) at the same light-front time, and the relative energy contribution was lost. A generalization of the ansatz (3.1) is then obvious: we should take the matter fields at different light-front times.

As shown in Ref. [6], infrared regularization using a finite volume and a careful treatment of the boundary surfaces are required to construct a light-front theory that is equivalent to the equal-time theory. The Hamiltonian and other conserved charges obtained in this way are guaranteed to be identical to the ones we would construct in the equal-time formulation. We believe that with the finite box regularization and the generalized ansatz for the composite matter field it will be possible to perform a complete construction of the light-front two-body Hamiltonian in our model. If so, this will allow us to derive exact and complete expressions for the invariant mass spectrum and the eigenfunctions.

(3) For nonzero $m$, the solution (4.3) is a formal one because it contains an undetermined value of the self-field potential.

In the self-field quantum electrodynamics the self-field potentials are calculated by iteration procedure. To lowest order of iteration we solve the Hamiltonian eigenvalue problem without the self-energy terms ($\phi^{\rm self}_{(0)}\equiv\phi_0=0$). Next we substitute the eigenfunctions obtained into the expressions for the self-field potentials and calculate these potentials explicitly. For our model with $\phi^{\rm self}$ independent of spatial coordinate, we would get that $\phi^{\rm self}_{(1)}\equiv\phi_1$ is simply a number.

To the next order of iteration we find the solution of the eigenvalue problem already with the potential $\phi_1$. Using the new eigenfunctions, we calculate $\phi^{\rm self}_{(2)}\equiv\phi_2$, then find the eigenfunctions and eigenvalues corresponding to $\phi_2$ and so on, the potentials $\phi_2$, $\phi_3$, . . . , depending on $m$.

If there is a small parameter in the theory, then we can often stop the iteration procedure already after the first order. In our model we could take the mass $m$ as such a parameter and consider the mass contribution to the bound state eigenfunctions and eigenvalues as small corrections to the corresponding eigenfunctions and eigenvalues for the vanishing mass. To do actual calculations in the massive case we therefore need again the complete solution of the eigenvalue problem for the massless case. The mass corrections in the second-quantized light-front $QED_2$ were calculated in Ref. [8].

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