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Bremsstrahlung in Self-Field QED

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We present a fully relativistic formulation of the theory of electron–nucleus Bremsstrahlung, within the context of self-field QED, as advanced recently by Barut and his co-workers. The Bremsstrahlung emission cross-section, reported here, is also shown to reduce to the standard correct nonrelativistic limit, in the dipole approximation.

1. INTRODUCTION

There has been a renewed interest lately[(1)] in the study of the elementary process of electron Bremsstrahlung from atoms. Most calculations reported to date, however, employ the standard second quantized version of perturbation theory with its Feyman graphs, plane waves and divergences, or are based upon the nonrelativistic dipole approximation. Within the S-matrix formulation, the electron–atom interaction has also been treated mainly in the Born approximation, good for high-energy electrons and, hence, low-energy emitted photons.

The aim of this paper is to present a finite, fully relativistic, reformulation of this important phenomenon. In this work, we formulate the theory of electron–nucleus Bremsstrahlung within the framework of the self-field approach, advanced recently by Barut *et al.* The expression we derive below for the Bremsstrahlung cross-section uses the fully relativistic, Dirac-Coulomb wave functions with their full Z dependence. It is also not limited to the dipole or Born approximations and, thus, contains all multipole contributions.

In the final analysis, our equations will be cast into a familiar form, bringing about formal resemblance to the conventional theory. Neverthe-

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less, it ought to be borne in mind that the approach here is different from the standard one in that it is semiclassical and essentially nonperturbative.

In the next section, the main result of the self-field approach to QED, pertaining to the emission and absorption of radiation, is briefly reviewed. The Bremsstrahlung cross section is next derived in Section 3. In Section 4, the dipole limit of conventional QED is recovered. Finally, some concluding remarks are given in the last section.

2. SELF-FIELD QED

The radiative processes of emission and absorption, among others, in the Coulomb problem (electron plus self-field in the Coulomb field of an atomic nucleus, the latter assumed pointlike and heavy) have been recently treated, semiclassically and nonperturbatively, by Barut and his coworkers.[2] In particular, the probability per unit time of emission and absorption of radiation by the electron emerges, in this theory, as (the negative of) twice the imaginary part of the following complex energy shift of level n of the quantum system:

$$\Delta\varepsilon_n = (\Delta\varepsilon_n^{\mathrm{VP}} + \Delta\varepsilon_n^{\mathrm{LS}}) + i\frac{\pi e^2}{4}\sum\!\!\!\!\!\!\int_s \int d^3r\, \bar{\psi}_n(\mathbf{r})\,\gamma^\mu\psi_s(\mathbf{r}) \int d^3r'\, \bar{\psi}_s(\mathbf{r}')\,\gamma_\mu\psi_n(\mathbf{r}')$$
$$\times \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k}\,[\delta(\varepsilon_s-\varepsilon_n+k)+\delta(\varepsilon_s-\varepsilon_n-k)] \tag{1}$$

where $\sum\!\!\!\!\!\!\int$ stands for a sum over the discrete and an integral over the continuous parts of the spectrum of the system. In Eq. (1), the γ^μ, where $\mu = 0, 1, 2, 3$, are the familiar Dirac gamma matrices, the Ψ's are the fully relativistic, Dirac–Coulomb wave functions, and $k \equiv |\mathbf{k}|$. Equation (1) is derived as a self-energy effect from the full QED action functional[2] of the quantum system in question, after the radiation field has been completely eliminated in favor of the matter field.

The real part of the energy shift [first term in (1)] gives the vacuum polarization and Bethe logarithmic parts of the Lamb shift and will not be considered any further here. The imaginary part, on the other hand, is related to the probabilities per unit time of emission (first delta function) and absorption (second delta function). The choice of the first dcelta function is useful for handling problems like spontaneous emission[2] from atoms and Bremsstrahlung, the subject of this paper, while the second delta function can be employed in treating the photoelectric effect[8] and such related phenomena as photo-ionization. Those delta functions evolve naturally from the analysis and fix conservation of energy, as in the standard theory, after the integration over k has been carried out.

Thus, for an emission process, the probability per unit time of a single transition $n \to s$ can be read off of (1) as the negative of twice its imaginary part. This probability is given by

$$w = -\frac{\pi}{2}\int \frac{d^3k}{k}\, T^{\mu}_{ns}(\mathbf{k})\, T_{sn}(-\mathbf{k})_{\mu}\, \delta(\varepsilon_s - \varepsilon_n + k) \tag{2}$$

where

$$T^{\mu}_{ns}(\mathbf{k}) = -e\int \frac{d^3r}{(2\pi)^{3/2}}\, \bar{\psi}_n(\mathbf{r})\, \gamma^{\mu}\psi_s(\mathbf{r})\, e^{i\mathbf{k}\cdot\mathbf{r}} \tag{3}$$

is a Fourier transform of the electron current $J^{\mu}_{ns} = -e\bar{\Psi}_n\gamma^n\Psi_s$, which can be regarded as a relativistic matrix element for the transition $n \to s$.

Upon integration over k, the differential probability per unit time for emission, from the electron, of radiation in the solid angle $d\Omega_k$ becomes

$$dw = -\frac{\pi}{2}\,\omega T^{\mu}_{ns}(\mathbf{k})\, T_{sn}(-\mathbf{k})_{\mu}\, d\Omega_k \tag{4}$$

where $\omega = k = \varepsilon_n - \varepsilon_s$, as dictated by the delta function.

Therefore, in this formulation of the elementary electron–atom Bremsstrahlung, the incident electron, of 4-momentum $p_n = (\varepsilon, \mathbf{p})$, is described by the wave function Ψ_n. It is subsequently scattered into the range d^3p' of continuum states Ψ_s with 4-momentum $p_s = (\varepsilon', \mathbf{p}')$. This process is accompanied by the emission of the energy difference in the form of radiation of 4-momentum $(\omega, \mathbf{k})$. Thus, Ψ_n and Ψ_s both belong to the continuum.

3. THE BREMSSTRAHLUNG CROSS-SECTION

Next, the differential probability per unit time is turned[4] into a cross-section for the emission of the Bremsstrahlung radiation into the solid angle $d\Omega_k$, accompanied by the scattering of the electron into the range of momentum states d^3p', by multiplying by $d^3p'/(2\pi)^3$ and dividing by the incident electron flux density Φ. Assuming, as usual, that the continuum wave functions are normalized to one particle per unit volume, the incident electron flux density $\Phi = v = p/\varepsilon$. Our final expression for the Bremsstrahlung cross-section then takes the form

$$d\sigma = -\frac{1}{16\pi^2}\frac{\omega\varepsilon'}{p}\,\omega T^{\mu}_{ns}(\mathbf{k})\, T_{sn}(-\mathbf{k})_{\mu}\, d\Omega_k\, d^3p' \tag{5}$$

where $p = |\mathbf{p}|$ and $p' = |\mathbf{p}'|$.

Another version of Eq. (5), whose nonrelativistic approximation is in use in practical calculations,[6] is the unpolarized cross-section, triply differentiable with respect to the photon energy and photon and scattered electron angles:

$$\frac{k}{Z^2}\frac{d^3\sigma}{dk\,d\Omega_k\,d\Omega'}=\left(\frac{k\varepsilon'}{4\pi Z}\right)^2\frac{p'}{p}\,T^{\mu}_{ns}(\mathbf{k})\,T_{sn}(\mathbf{k})_{\mu} \tag{6}$$

Equation (6) follows from (5) by noting that $\varepsilon'^2=p'^2+m^2$ implies $p'\,dp'=\varepsilon'\,d\varepsilon'$ and that $\omega=k=\varepsilon-\varepsilon'$ implies $dk=-d\varepsilon'$.

With the fully relativistic, Dirac–Coulomb continuum wave functions used in Eqs. (5) or (6), a general relativistic expression for the cross-section of any electron–atom Bremsstrahlung process can, in principle, be found.

4. THE DIPOLE LIMIT

For our expressions (5) and (6) to make any sense, however, they should reproduce the often quoted, nonrelativistic forms based upon the dipole approximation. This can be readily accomplished for (5) by first replacing the exponential term in Eq. (3) by unity. Next, the Dirac–coulomb wave functions are replaced by their nonrelativistic Schrödinger counterparts. The electron spin, lost as a result, has to be compensated[2,3] for by multiplying by a factor of 2. Then, the Dirac α-matrices are replaced by the components of the electron velocity operator, $\boldsymbol{\alpha}\rightarrow\mathbf{v}=\mathbf{p}/m$. Finally, taking the fine structure constant $\alpha=e^2/4\pi$, in the system of units we have been using, whereby, $\hbar=c=1$, we arrive at

$$T^{\mu}_{ns}(\mathbf{k})\,T_{ns}(-\mathbf{k})_{\mu}\rightarrow-|\mathbf{M}_{ns}|^2=-\frac{4\mu\alpha}{(2\pi)^3}\frac{|\mathbf{p}_{ns}|^2}{m^2} \tag{7}$$

where $\mathbf{p}_{ns}$ is the transition matrix element of the momentum operator. Thus, the nonrelativistic Bremsstrahlung cross-section, following from (5) in the dipole approximation, is[4]

$$d\sigma=\frac{\alpha}{(2\pi)^4}\frac{\omega}{mp}\,|\mathbf{e}\cdot\mathbf{p}_{ns}|^2\,d\Omega_k\,d^3p' \tag{8}$$

Here $\mathbf{e}$ is a polarization vector inserted by hand as usual. On the other hand, expression (6) should reduce, in the same approximation, to[6]

$$\frac{k}{Z^2}\left(\frac{d^3\sigma}{dk\,s\Omega_k\,d\Omega'}\right)_{\text{unpol}}=\frac{\alpha m^2}{(2\pi)^4}\frac{k^4}{Z^2}\frac{p'}{p}\left(|\mathbf{r}_{ns}|^2-\frac{|\mathbf{k}\cdot\mathbf{r}_{ns}|^2}{k^2}\right) \tag{9}$$

This can be easily arrived at from (6) by noting that $\omega = k$, $d^3p' = p'^2\, dp'\, d\Omega'$ and that, nonrelativistically, $\varepsilon' = p'^2/2m$. Hence $d\varepsilon' = (p'/m)\, dp'$. With the incident electron energy fixed, $k = \varepsilon - \varepsilon'$ implies $dk = -d\varepsilon'$. Furthermore, $d^3p' = p'^2\, dp'\, d\Omega'$. Finally, $\mathbf{p}_{ns} = m\mathbf{v}_{ns} = imk\mathbf{r}_{ns}$. Keeping all this in mind, we arrive at

$$\frac{k}{Z^2}\frac{d^3\sigma}{dk\, d\Omega_k\, d\Omega'} = \frac{\alpha m^2}{(2\pi)^4}\frac{k^4}{Z^2}\frac{p'}{p}\,|\mathbf{e}\cdot\mathbf{r}_{ns}|^2 \tag{10}$$

Equation (8) follows immediately from (9) upon averaging over the directions of $\mathbf{e}$, with the help of the identity

$$(\mathbf{r}_{ns}\cdot\mathbf{e})(\mathbf{r}_{ns}\cdot\mathbf{e}^*) = \tfrac{1}{2}(|\mathbf{r}_{ns}|^2 - |\mathbf{r}_{ns}\cdot\mathbf{n}|^2) \tag{11}$$

where $\mathbf{n} = \mathbf{k}/k$, and doubling the result[7] (two photon polarization states).

5. CONCLUDING REMARKS

In this paper, we have carefully tailored Barut's theory in order to elucidate the origin of an important phenomenon, Bremsstrahlung, and to derive it as a self-field effect, semiclassically and without having to second-quantize the electromagnetic field. We have derived a general Bremsstrahlung cross section in terms of the relativistic wave functions of the electron and with the full retardation effects included.

Work on the difficult integrals involved in the evaluation of the general Bremsstrahlung cross-section, (5) or (6), will be published elsewhere.[5] It is hoped that the final result will be applicable to all incident electron energies, and that it will be exact to all orders in $Z\alpha$, as it contains contributions from all multipoles, besides the dipole term.

In conclusion, we have demonstrated in this paper that, once more, Barut's novel approach to QED does agree with the conventional theory, in the limit where the latter is known quite well to be valid.

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