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The Classical Lamb Shift: Why Jackson is Wrong![1]

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I provide here a classical calculation of the Lamb shift that is of the same order of magnitude as the quantum Bethe result. This contradicts Jackson's claim that a classical calculation can not get the Lamb shift right—even to within an order of magnitude.

1. INTRODUCTION

At a recent NATO ASI summer school held in Edirne, Turkey,[(1)] a discussion developed on how it seemed to be possible to use classical methods to calculate atomic spontaneous emission rates, but not the Lamb shift. This point of view has particularly been emphasized in Jackson's book, *Classical Electrodynamics*, in which he flat-out states that a classical argument based on radiation-reaction theory and the Abraham–Lorentz equation will give the correct Einstein A coefficient for spontaneous decay, but gives the wrong value for the Lamb shift.[(1)] This bit of folklore is often touted as some sort of proof that classical methods cannot reproduce elements of quantum electrodynamics, even in a limiting sense. Usually, as Jackson does, the blame is placed on electromagnetic vacuum fluctuations—a manifestly nonclassical phenomenon—that is claimed to be the "true" origin of the Lamb shift. But people claim that vacuum fluctuations are also the cause of atomic spontaneous emission—how come classical radiation-reaction theory is right in that case? "Just good luck," is the usual

[1] I would like to dedicate this paper to the memory of Asim O. Barut. Asim was a great physicist, as well as my advisor, colleague, and friend. Things just won't be the same without him.

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reply. This seems an unsatisfactory state of affairs, for several reasons. To me, *the correspondence principle* presents the biggest objection to the notion that there is no classical analog to the Lamb shift. In the limit of large quantum numbers, one would expect the quantum electrodynamical Lamb shift to go over into a classical frequency pulling that an oscillating macroscopic charge experiences by interacting with its self-field. An additional disturbing fact is that classical theory is quite often used to compute cavity corrections to the Lamb shift,[2] and at least for excited atoms the results agree exactly with full QED calculations.[3] Why then can the classical approach get the cavity-induced Lamb shift right, but not the free-space shift? As we shall see, there is nothing wrong with the classical approach to the free-space Lamb shift; it is *Jackson* who is wrong.

2. JACKSON'S ARGUMENT

This argument is given in Jackson,[1] and can even be found in the book by Barut.[4] The derivation is based on the belief that the most general nonrelativistic equation of motion for a classically-charged harmonic oscillator of frequency ω_0 is given by[1-4]

$$\tau\dddot{x} - \ddot{x} - \omega_0^2 x = 0 \tag{1}$$

the Abraham–Lorentz–Dirac (ALD) equation, where

$$\tau = \frac{2}{3}\frac{e^2}{mc^3} \tag{2}$$

is the usual radiation-reaction time constant, and x is the position coordinate. Assuming a solution of the form $x = x_0 e^{-\alpha t}$ yields a characteristic equation

$$\tau\alpha^3 + \alpha^2 + \omega_0^2 = 0 \tag{3}$$

This has one real root, $-\alpha_0$, and two complex conjugate roots, $\alpha_\pm$. The real root leads to the unphysical "runaway" solution $x = x_0 e^{\alpha_0 t}$, which is discarded. The two remaining roots correspond to damped harmonic oscillations. In the limit of slow oscillations, $\omega_0 \ll 1/\tau \cong 1.67\times 10^{23}$ Hz, the complex roots may be approximated as[1]

$$\alpha_\pm \cong \frac{\Gamma}{2} \pm i(\omega_0 + \Delta\omega_c) \tag{4}$$

where

$$\Gamma = \omega_0^2 \tau \tag{5a}$$

$$\Delta\omega_c = -\tfrac{5}{8}\omega_0^3 \tau^2 \tag{5b}$$

The most general solution in this approximation can be written

$$x(t) = x_0 e^{-(\Gamma/2)t} \cos(\omega_0 + \Delta\omega_c) \tag{6}$$

if x_0 is the oscillator position at $t = 0$. Clearly, this indicates a damped and frequency-shifted oscillation. The classical decay constant Γ differs from the correct QED decay rate Γ_{ij} for the $|i\rangle \to |j\rangle$ transition only by the quantum mechanical oscillator strength f_{ij} that is on the order of unity or less. Hence

$$\Gamma_{ij} = f_{ij}\Gamma \tag{7}$$

and we see that the classical calculation is correct, to within at least an order of magnitude. Clearly, in addition to the classical decay constant Γ, there is apparently a classical frequency pulling $\Delta\omega_c$, Eq. (5b). However, Jackson argues that this shift is many orders of magnitude too small to be considered the classical analog of the Lamb shift $\Delta\omega_q$. The Bethe calculation of the quantum Lamb shift gives

$$\frac{\Delta\omega_q}{\omega_0} \sim \omega_0 \tau \log\left(\frac{mc^2}{\hbar\omega_0}\right) \sim \omega_0 \tau \tag{8}$$

if we take $\omega_0 \cong |\omega_i - \omega_j|$ in a correspondence-principle limit. However, the classical quantity from Eq. (5b) becomes

$$\frac{|\Delta\omega_c|}{\omega_0} \sim (\omega_0 \tau)^2 \tag{9}$$

where, remember, we have assumed $\omega_0 \tau \ll 1$. Hence, it appears that the so-called classical Lamb shift is one order of $\omega_0 \tau$ too small to be the analog of the Bethe shift. For this reason, Jackson claims that the "real" quantum shift $\Delta\omega_q$, Eq. (8), arises from a completely different mechanism. Somehow, radiation reaction and the self-field can be held responsible for the classical shift, but "vacuum fluctuations" are responsible for the much larger quantum Lamb shift.

3. THE CORRECT CLASSICAL SHIFT

Several years ago I mentioned this puzzle to John Sipe; that apparently classical calculations can give the correct cavity QED Lamb shift, but not the free-space one. Sipe told me that, in fact, Jackson was wrong and that he (Sipe) had shown that a classical Lamb shift could be derived in free space that was of the same order of magnitude as the Bethe result. Unfortunately, Sipe never sent me this reference, so in Edirne, when Barut asked me how Sipe's calculation might go, I produced the following original and covariant derivation and presented it on the board.

Let us work with the classical limit of Barut's self-field approach to electrodynamics. In covariant form, the self-field action density has the form

$$W = \frac{1}{2c^3} \iint dx^4\, dy^4\, j^{\mu}(x)\, D_{\mu\nu}(x - y)\, j^{\nu}(y) \tag{10}$$

where $j^{\mu} = [\rho, \mathbf{j}]$ is the electric source current four-vector, $x = x^{\mu} = [ct, \mathbf{x}]$ and $y = y^{\mu} = [cu, \mathbf{y}]$ are space-time four vectors, and $D_{\mu\nu}$ is the Green's function in the radiation gauge. In the full-quantum Barut theory, $j^{\mu}(x)$ is the appropriate Dirac current. However, here, we take it to be the classical current associated with a classical point dipole oscillating at frequency ω_0. In particular,

$$\rho(\mathbf{x}, t) = 0 \tag{11a}$$

$$\mathbf{j}(\mathbf{x}, t) = \omega_0 \mathbf{p} e^{i\omega_0 t} \delta(\mathbf{x}) \tag{11b}$$

where $\mathbf{p}$ is the point-dipole moment. Since $j_0 = c\rho = 0$, Eq. (10) for the action becomes

$$W = \frac{1}{2c} \iint dt\, du \iint d^3x\, d^3y\, \mathbf{j}(\mathbf{x}, t) \cdot \overleftrightarrow{\mathbf{D}}[\mathbf{x} - \mathbf{y}; t - u] \cdot \mathbf{j}(\mathbf{y}, t) \tag{12}$$

where $\mathbf{j}$ is given above in Eq. (11b), and the dyadic Green's function in the radiation gauge has the form[6]

$$\overleftrightarrow{\mathbf{D}}[\mathbf{x} - \mathbf{y}; t - u] = -\frac{4\pi}{(2\pi)^4} \int \frac{d\omega}{c} \int d^3k\, \frac{e^{-i\omega(t-u)} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{|\mathbf{k}|^2 + i\varepsilon} [\overleftrightarrow{\delta} - \hat{\mathbf{k}}\hat{\mathbf{k}}] \tag{13}$$

Hence, the action W can be written as

$$W = -\frac{1}{2} \frac{p^2 \omega_0^2}{c^2} \iint dt\, du \int \frac{d\omega}{(2\pi)} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i(\omega_0 - \omega)t} e^{i(\omega + \omega_0)u}}{\omega^2/c^2 - |\mathbf{k}|^2 + i\varepsilon} [1 - (\hat{\mathbf{p}} \cdot \hat{\mathbf{k}})^2] \tag{14}$$

where we have carried out the spatial integration, and $\hat{\mathbf{k}}$ and $\hat{\mathbf{p}}$ are unit vectors. Carrying out the angular $\hat{\mathbf{k}}$ integration, the ω integration, and the two temporal integrals yields

$$W = -\frac{p^2\omega_0^2}{(2\pi)^2}\frac{8\pi^2}{3c^2}\int dk\,\frac{k^2\delta(\omega_0)}{\omega_0^2/c^2 - k^2 + i\varepsilon} \tag{15}$$

where here $k = |\mathbf{k}|$, and I have used $\delta(2\omega_0) = \delta(\omega_0)/2$. Now, according to the Barut prescription, I can extract a frequency shift $\Delta\omega$ from the action W via[5]

$$W = 2\pi\delta(\omega_0)\frac{\Delta\omega}{\omega_0}\mathsf{E} \tag{16}$$

where $\mathsf{E} = mx_0^2\omega_0^2/2$ is the total invariant energy of the dipole, treated as a harmonic oscillation of mass m and maximum extension x_0, and $\Delta\omega$ is the shift of frequency of oscillation. Hence, from Eq. (15), we have

$$\mathsf{E}\frac{\Delta\omega}{\omega_0} = -\frac{4\pi}{3}\frac{p^2\omega_0^2}{(2\pi)^2 c^2}\int dk\left\{\frac{1}{2}\left[\frac{\omega_0/c}{\omega_0/c - k + i\varepsilon} - \frac{\omega_0/c}{\omega_0/c + k + i\varepsilon}\right] - 1\right\} \tag{17}$$

The contour integral selects out only the positive frequency solution. The infinite factor $\int dk$ contributes to mass renormalization, required even classically.[10] The renormalized shift $\Delta\tilde{\omega}$ then is, implicitly,

$$\mathsf{E}\frac{\Delta\tilde{\omega}}{\omega_0} = -\frac{1}{3}\frac{p^2\omega_0^3}{c^3\pi}\int dk\left\{\mathsf{P}\left[\frac{1}{\omega_0/c - k}\right] - i\pi\delta(\omega_0/c - k)\right\} \tag{18}$$

where P stands for principal part, and an extra factor of two has been added to account for the two degrees of polarization. Writing $\Delta\tilde{\omega} = \Delta\Omega + i\Gamma/2$, we have $\Gamma = p^2\omega_0^4/(3c^3\mathsf{E})$. If we take $p = ex_0/\sqrt{2}$, and recall that $\mathsf{E} = \frac{1}{2}m\omega_0^2 x_0^2$, then we have $\Gamma = \omega_0^2\tau$, in agreement with the Jackson result, Eq. (5a), and hence with the quantum Einstein A coefficient.

However, the frequency shift $\Delta\Omega$ is *not* the same as Jackson's "classical" Lamb shift. Already, from Eq. (18), we can see that $\Delta\Omega \equiv \mathrm{Re}\{\Delta\tilde{\omega}\}$ has the form of the Bethe logarithm for the Lamb shift.[6] A cut-off K for the integral can be arrived at classically by noting that wavelengths $\lambda < r_0$, the classical electron radius, should not contribute much to the shift. Hence $k < 1/r_0 = mc^2/e^2$ is a reasonable cut-off, and we get

$$\frac{\Delta\Omega}{\omega_0} = \frac{1}{2\pi}\omega_0\tau\ln\left[\frac{2}{3\omega_0\tau}\right] \tag{19}$$

where the argument of the logarithm is very large, since $\omega_0 \tau \ll 1$. This expression, I would claim, is then the classical Lamb shift. It certainly does not agree with Jackson's result in Eq. (5b). We can see that the classical Lamb shift of Eq. (19) is comparable to or greater than the classical linewidth of Eq. (5a), just as is the quantum Lamb shift. Hence, Jackson's classical shift, $\Delta\omega$ of Eq. (5b), is an order of $\omega_0 \tau$ too small. Clearly, the shift of Eq. (19) is the correspondence-principle limit of the quantum Bethe logarithm calculation of the shift, Eq. (8) (See Ref. 7.)

4. SUMMARY AND CONCLUSIONS

The question then becomes, what has Jackson done wrong? An expression for a classical Lamb shift similar to Eq. (19) was derived independently by Sipe and co-workers in a noncovariant formalism.[9] These authors point out that, classically, the decay of a classical dipole arises from the interaction of the dipole with the out-of-phase portion of its own self field, and the frequency shift from the interaction with the in-phase part. However, the assumption that the ALD formula used in Jackson's calculation, Eq. (1) is the correct equation of motion must then be in fact *incorrect*. Somehow the first-order, in-phase contribution to the classical shift, Eq. (19), is missing in the ALD, and only the much smaller higher-order shift that Jackson gets, Eq. (5b), remains. It is not clear to me at this time why this term is missing from standard derivations of the ALD equation. However, these results would seem to indicate that the ALD equation, as used in Jackson and Barut and other standard texts, is not a complete classical equation of motion for a charged particle interacting with its own field.

So in conclusion, I have provided a covariant calculation of the classical Lamb shift and confirm that it agrees with the nonrelativistic calculation of Sipe and co-worker[9] and not the calculation of Jackson.[2] Since the Sipe calculation was published as an aside in a larger work on cavity-induced Lamb shifts, I felt it was important to give a covariant account of the calculation standing on its own to double-check this important foundational result. It is important in that it explicitly refutes the conventional wisdom, supported by Jackson, that vacuum fluctuations are required to get anything close to the physical Lamb shift. It is also important in that it indicates that the covariant ALD equation of motion is not a complete description of the physics of point classical charge—even within ordinary, relativistic, classical electrodynamics. I should also point out that, while there is some debate on whether the quantum version of the Barut self-field approach to QED is equivalent to the traditional second-quantized formalism, it can be shown that the covariant *classical* version of Barut's self-field formalism is fully equivalent to covariant, relativistic *classical* electrodynamics.[6] As a final point, one might ask why this relativistic but classical treatment does not give a result that is better than the nonrelativistic but quantum treatment of Bethe. In other words, why does my covariant classical account not give something closer to the finite result of fully second-quantized and covariant relativistic QED? Here I think the answer lies in the nature of the electron's self-field. The self-field is present in much the same form in nonrelativistic classical electrodynamics, relativistic classical electrodynamics, nonrelativistic quantum electrodynamics, and fully relativistic and covariant QED. This is why the free-space spontaneous emission rate is the same in all these theories, as this arises the same way as the residue of a simple pole for a contour integral. In addition, the Bethe logarithm shows up in all these calculations of the Lamb shift in much the same way as the principle part of these integrals, but only in the fully QED calculation does the cut-off arise as a natural part of the theory and is not added in *ad hoc*. It might be possible to make the introduction of a cut-off in my classical calculation less *ad hoc* by splitting virtual photon energies up into a relativistic and nonrelativistic regimes, as is done in the full QED calculation, but it is not clear if it is worth the trouble to do this. In the fully QED theory, logarithmic singularities—which give infinite shifts to individual energy levels––are rendered finite only when energy-level differences are considered, as must be done in any quantum theory. However, in a classical theory, there are no energy levels and hence no level differences to compute. (Possibly, using *the correspondence principle*, something could be learned by taking energy-level differences in the fully QED calculation and then comparing them to the classical calculation in the limit of large quantum numbers.) The total energy of oscillation is fixed in a classical theory and assumed to be measurable—at least in principle. In addition, some of these finite, cut-off independent contributions to the fully relativistic QED Lamb shift arise from things such as vacuum-polarization Feynman diagrams that have no obvious analog in a classical theory. I am afraid that an entirely classical theory can do much better than what Jackson claims, but I do not believe it can never compete with a fully relativistic theory of QED.

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