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ABSTRACT

We have reevaluated analytically the vacuum polarization in Coulomb field using relativistic Coulomb wave functions by a new method. The result is finite giving the standard result in the lowest order of iteration. The formalism of selffield quantumelectrodynamics used here needs neither field quantization, nor renormalization. There are no infrared or ultraviolet divergences.

The attitude of some physicists towards renormalization and infinities in quantum field theory, once thought to be the most serious difficulty in all theoretical physics, seems to be gradually mellowed in recent years. One accepts and justifies the renormalization procedure by saying that we do not know the interactions at high energies (or short distances), hence at every order of perturbation theory we should express our ignorance by certain constants (renormalized parameters). (This is even sometimes presented as a virtue.) It is certainly true that we do not know all the physics at short distances, but the question is whether a given theory, say quantumelectrodynamics (QED), by itself is a mathematically consistent theory. We know that classical mechanics also does not describe the physics at high energies, but it does not lead by itself to infinities or renormalization. Therefore efforts must continue to understand the theory in other ways than the sole paradigm of field quantization, perturbation theory and renormalization.

One such approach that has been extensively studied in recent years is based on eliminating the electromagnetic field $A_\mu(x)$ between the coupled Maxwell and Dirac equations and to study the ensuing nonlinear action

$$
\begin{aligned}
W = & \int dx\, \bar{\Psi}(x) \left[\gamma^\mu(i\partial_\mu - eA_\mu^{\mathrm{ext}}) - m\right] \Psi(x) \\
& - \frac{e^2}{2} \int dx\, dy\, \bar{\Psi}(x)\gamma^\mu \Psi(x) \\
& \quad \times \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{k^2} \bar{\Psi}(y)\gamma_\mu \Psi(y)
\end{aligned}
\tag{1}
$$

This amounts to inserting the self-field $A_\mu^{\mathrm{self}}(x)$ of the electron's current into the Dirac equation together with a fixed external field $A_\mu^{\mathrm{ext}}(x)$, if present, where

$$
A_\mu^{\mathrm{self}}(x) = -e \int dy\, D(x-y)\bar{\Psi}(y)\gamma_\mu \Psi(y) \tag{2}
$$

and the covariant gauge $A^\mu_{,\mu} = 0$ is used. It is in fact the same procedure as the treatment of radiative processes in classical electrodynamics, the only difference being the use of the Dirac current $e\bar{\Psi}\gamma_\mu\Psi$, instead of the classical current $e\dot{x}_\mu$.

This approach has been applied to the nonrelativistic and relativistic Lamb shift[2], to spontaneous emission[3], to $(g-2)$ calculation[4], and other processes, reproducing all the results of QED[5].

Vacuum polarization is an important effect in quantumelectrodynamics both in practice and conceptually, because it represents the most divergent term in perturbation theory and enters significantly in the idea of a "running coupling constant" and renormalization group. The standard textbook treatment of vacuum polarization consists first in evaluating the photon propagator in perturbation theory, renormalizing the charge, then Fourier transforming it to get an effective potential and then taking the matrix elements of the potential between Coulomb wave functions. Field theoretic formulation of vacuum polarization was given by Schwinger[6]. Calculations with relativistic Coulomb wave functions were first performed in a classic paper by Wichmann and Kroll[7], and studied further, also numerically[8--16]. The work started by Wichmann and Kroll was never completed *analytically*, and to our knowledge the finiteness of vacuum polarization was not discussed. In a previous paper[17], we have applied the self-field-QED to this problem and obtained a closed expression for vacuum polarization. However, the choice of contours in some of the integrals was very complicated. Because the calculation involving the sum over infinitely many bound and continuum Coulomb functions is very complex and intricate a new method was thought to be necessary which we present now. The emphasize in this new method is a proof of the fundamental question of the finiteness of the theory and an unambiguous definition of all the integrals by their Mellin-Barnes transforms.[18]

In the present approach all QED effects follow organically from a single expression; we do not have to use results from separate Feynmann diagrams, such as a loop diagram. Thus the "virtual $e^+_-e^-$ pairs" are themselves in the Coulomb field, if one wishes to use this picture. To first order of iteration of the second self-energy term of eq. (1) one obtains a general expression for the energy shift of a level n, directly from the action without the intermediary of wave functions or amplitudes (in units $c = \hbar = 1$)

$$\begin{aligned}
\Delta E_n = {} & \frac{e^2}{2} \int dx\, \bar{\psi}_n(x)\gamma_\mu\psi_n(x) P \int \frac{dk}{(2\pi)^3} \int dy \frac{e^{ik\cdot(x-y)}}{k^2} \sum\!\!\!\!\!\!\int_s \bar{\psi}_s(y)\gamma^\mu\psi_s(y) \\
& - \frac{e^2}{2} \sum\!\!\!\!\!\!\int_s dx\, dy\, \bar{\psi}_n(x)\gamma_\mu\psi_s(x) \int \frac{dk}{(2\pi)^3} e^{ik\cdot(x-y)} \bar{\psi}_s(y)\gamma^\mu\psi_n(y) \left[\frac{1}{E_s - E - n - k} - \frac{1}{E_s - E_n + k}\right] \\
& - \frac{e^2}{2} \sum\!\!\!\!\!\!\int_{(s<\hbar)} \int dx\, dy\, \bar{\psi}_n(x)\gamma_\mu\psi S(x) \int \frac{dk}{(2\pi)^3} e^{ik\cdot(x-u)} \bar{\psi}_s(y)\gamma^\mu\psi_n(y) \frac{i\pi}{2k}\delta(E_s - E_n - k).
\end{aligned} \tag{3}$$

where ψ_n is a fixed level, and we sum over all levels ψ_s, discrete and continuous, of the external field (here Coulomb field). The first term is the contribution of vacuum polarization, the second that of self-energy (or Lamb-shift proper including the contribution of $(g-2)$), and the third term gives the spontaneous emission rate which has been evaluated exactly and analytically[3].

We now evaluate the first term in Eq. (3), vacuum polarization, which can also be interpreted as the interaction energy of two current distributions, in fact because of spherical symmetry, of two charge distributions. After using relativstic Coulomb wave functions and performing the spin algebra, we obtain[17]

$$\Delta E_n^{\mathrm{VP}} = 4\alpha(2J_n+1)\sum_{lm}\sum\!\!\!\!\!\!\!\int_s (2J_s+1)\int dr\, dr'\, V_l(r,r')\left(|f_n|^2+|g_n|^2\right)\left(|f_s|^2+|g_s|^2\right) \quad (4)$$

where we have introduced a potentail $V_l(r,r')$ by

$$V_;(r,r') \equiv \frac{2}{\pi}r^2r'^2\int_0^\infty j_\ell(kr)j_l(kr')dk = \frac{r^2r'^2}{2l+1}\frac{r_<^l}{r_>^{l+1}}. \quad (5)$$

and $f(r)$ and $g(r)$ radial Dirac wave functions, and the most difficult part of the calculation is the sum $\sum\!\!\!\!\!\!\int_s$over all discrete and continuous states. Here we use the method of Green's functions initiated by Whichmann and Kroll[7]. Because of completeness the Green's function involves both negative and positive energy solution and we have

$$e\sum_s\left(|f_s|^2+|g_s|^2\right) = \frac{e}{2}\sum_{E_s>0}\left(|f_s|^2+|g_s|^2\right) - \frac{e}{2}\sum_{E_s<0}\left(|f_s|^2+|g_s|^2\right) \quad (6)$$

since negative-energy solutions corresponds to positive-energy solutions with the sign of charged reversed. then the above sum can be represented as a well-known contour integral around the positive and negatie energy spectrum in the z (energy)-plane:

$$e\sum_s\left(|f_s|^2+|g_s|^2\right) = \frac{e}{2\pi i}\left[\int_{c+}+\int_{c-}\right]dz\, trK(r,r';z) \quad (7)$$

where $K(r,r';z)$ is the energy dependent Green's function of the radial Coulomb problem which is known.

The problem then reduces to the evaluation of the following expression

$$\begin{aligned}\Delta E_n^{\mathrm{VP}} &= \alpha\sum_{n_1n_2}A_{n_1n_2}\left[\int_{c_+}+\int_{c_-}\right]\frac{dz}{2\pi i}\frac{4i}{\sqrt{z^2-1}}\sum_{\kappa=1}^{\infty}2|\kappa|T_{\alpha\alpha'}\int_0^1 dt\, t^{\alpha-1}(1-t)^{2\gamma-\alpha} \\ &\times\int_0^\infty dt'\, t'^{\alpha'-1}(1+t')^{2\gamma-\alpha'}\int_0^\infty r^2dr\int_0^\infty r'^2dr'\frac{(2P_N)^3e^{-2P_nr}}{r'^2}\left(-2i\sqrt{z^2-1}r'\right)^{2\gamma} \quad (8)\\ &\times e^{2i\sqrt{z^2-1}(1-t+t')r'}\frac{r_<^\ell}{r_>^{\ell+1}}\cdot\frac{(2P_Nr)^{2\gamma_n+n_1+n_2-2}}{2\ell+1}\end{aligned}$$

where we have set

$$T_{\alpha\alpha'} = \left[\frac{iZ\alpha}{z^2-1)^{1/2}}\left[\frac{\delta_{\alpha,\gamma-iv}\delta_{\alpha',\gamma-iv}+\delta_{\alpha,\gamma+1-iv}\delta_{\alpha',\gamma+1-iv}}{\Gamma(\alpha)\Gamma(2\gamma+1-\alpha')}\right] - z\left[\frac{\delta_{\alpha,\gamma-iv}\gamma_{\alpha',\gamma+1-iv}}{\Gamma(\alpha)\Gamma(2\gamma+1-\alpha')} - \frac{\delta_{\alpha,\gamma+1-iv}\delta_{\alpha',\gamma-iv}}{\Gamma(\alpha')\Gamma(2\gamma+1-\alpha)}\right]\right]. \tag{9}$$

with $\gamma = (k^2-(Z\alpha)^2)^{1/2}, \nu = \frac{Z\alpha}{(z^2-1)^{1/2}}z$. The values of α and α' in (9) can be read off from the Kroenecker δ-functions. Here

$$A_{n_1 n_2} = \frac{\Gamma(2\gamma_n+n_r+1)\Gamma(n_1-n_r)\Gamma(n_2-n_r)}{2N_n(N_n-\kappa_n)\Gamma^2(-n_r)\Gamma(2\gamma_n+n_1+1)\Gamma(2\gamma_n+n_2+1)n_1!n_2!n_r!}$$

and γ_n, n_1, n_2, n_r are quantum numbers and

$$P_N = \frac{Z\alpha}{N_n}\ ,\ N_n = \left[n^2 - 2n_r(|\kappa| - \gamma_n)\right]^{1/2}$$
$$n_r = n - 1|\kappa_n|\ ,\ \kappa_n = \pm(J_n+1/2), \gamma_n = \left[\kappa_n^2 - (Z\alpha)^2\right]^{1/2}$$

The radial integrals

$$R = \int_0^\infty dr\, r^{-\ell+1} e^{-2P_N r}(2P_N r)^{2\gamma_n+n_1+n_2-2}\int_0^r dr' r'^{2\gamma+\ell} e^{2i(1-t+t')\sqrt{z^2-1}r'} + \int_0^\infty dr\, r^{\ell+2} e^{-2P_N r}(2P_N r)^{2\gamma_n+n_1-n_2-2}\int_r^\infty dr'\, r'^{2\gamma-\ell-1} e^{2i(1-t+t')\sqrt{z^2-1}r'} \tag{9}$$

can be exactly evaluated by converting them into the Mellin transforms of the hypergeometric functions and carefully analyzing the poles in the Mellin plane. From the asymptotic behavior we see that only the first term in (9) contributes and the result is, taking the contribution giving the lowest power in$(Z\alpha)$,

$$R = -\frac{\Gamma(2\gamma+3)}{2a^{2\gamma+3}}\ ,\ a = -2i(z^2-1)^{1/2}(1-t+t') \tag{9'}$$

Then we are left with

$$\Delta E_n^{\mathrm{VP}} = \alpha A_{0,0}\sum_{\kappa=1}^\infty |\kappa|\left[\int_{c_+} + \int_{c_-}\right]\frac{dz}{2\pi i}\frac{(2P_N)^3}{(z^2-1)^2}T_{\alpha\alpha'}\int_0^1 dt\, t^{\alpha-1}(1-t)^{2\kappa-\alpha}\int_0^\infty dt'\, t'^{\alpha-1} \times (1+t')^{2\kappa-\alpha'}\frac{\Gamma(2\kappa+3)}{(-2)(1-t+t')^{2\kappa+3}} \tag{10}$$

where we have approximated $\gamma \approx |\kappa|$ for $Z\alpha << 1$. Also for s-waves ($n_1 = n_2 = 0, n_r = n-1, \kappa_n = -1, \ell = 0$). We find $A_{n_1 n_2} = 1/2$ independent of n

In the previous paper[17], the t-integrals were done first and the summation $\sum_{\kappa=1}^{\infty}$ afterwards. The main point here is to do the reverse. The κ-summations can be reduced to hypergeometric functions.

Also dt-integration can be performed using hypergeometric functions. The remaining dt'-integration formally diverges at the lower limit $t = 0$. We change this limit to ε and convert the integrals again into a Mellin transform along the imaginary axis. A great many terms arising from (9) have been carefully analyzed term by term. The various integrands that occur are regular, or have on the left-hand plane one-double pole and one or two single poles. We can avoid by an appropriate choice of the contours the simple poles. The double poles give a finite and a logarithmic term in ε. These calculations are exteremly tedious and details will be given elsewhere.

The result for s-waves is

$$\Delta E_n^{\mathrm{VP}} = -2\alpha \left(\frac{Z\alpha}{N_n}\right)^3 \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \frac{z}{(z^2-1)^2} \left[A + i\frac{Z\alpha}{(z^2-1)^{1/2}} z \left(\frac{2}{5}\right) + B\ell n\varepsilon\right] \times \left(1 - \pi\frac{Z\alpha}{(z^2-1)^{1/2}} z\right)$$

where A and B are z-independent constants: $A = -417 + 15/56$ and $B = -762$. But A and B terms do not contribute after the z-integration, being odd in z.

Finally using

$$\frac{1}{2\pi i}\int dz \, \frac{z^2}{(z^2-1)^{5/2}} = -i/3\pi$$

we see that we have the finite result for the vacuum polarization energy shift

$$\Delta E_n^{\mathrm{VP}} = -\frac{4\alpha}{3\pi}\left(\frac{1}{5}\right) Z\alpha \left(\frac{Z\alpha}{N_n}\right)^3 + 0\left((Z\alpha)^5\right).$$

The terms higher order in $(Z\alpha)$ can in principle be evaluated if need be. There are of course other terms of that order in the Lamb-shift. Our method would also allow to take different contours to evaluate the shift in the limit of large $(Z\alpha)$, large $Z\alpha$-expansion.

We conclude that the divergences in quantumelectrodynamics are due to the use of plane waves in the intermediate states in the loop diagrams. The use of Coulomb wave functions alleviates this difficulty which act as a regularizing agent. In the limit when $(Z\alpha) \to 0$, the physically observable radiative effects disappear, as it should, for free particles.

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