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ABSTRACT

From the coupled Maxwell-Dirac equations for two fermion fields ψ_1, ψ_2 we derive a covariant 2-body equation for the composite field $\Phi(x_1, x_2)$ in configuration space which includes radiative self energy effects. Both Coulomb gauge and covariant gauge have been used and their equivalence is proved. For the space S^1 we solve the two-body equation with mutual interactions exactly and obtain the mass spectrum in the case of massless fermions.

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1 introduction

Quantumelectrodynamics in one space and one time dimensions, known as the Schwinger model[1] in the massless case which is exactly soluble, has been the subject of a large number of investigations. A list of references is given elsewhere[2),3)]. QED in lower dimensions is useful as a simpler model, for discussion of topological effects, as well as in their own right when such lower dimensions may be physically realizable under certain conditions.

In these series of work we shall address ourselves to another aspect of QED in lower dimensions, in the present paper in 1+1 dimensions, namely the two (or many)-body problems. There is some discussion of many-body problems in 1+1-dimensions, however under instantaneous phenomenological (e.g. δ function) potentials [4]. We wish to use the exact QED-potentials in one-dimension and study the corresponding nonperturbative two-body equation.

The method we use consists in starting from two fermion fields ψ_1 and ψ_2 coupled by the usual electromagnetic minimal coupling. We eliminate the A_μ-field, introduce a composite field $\Psi(x_1, x_2)$ and derive a 2-body wave equation for this composite field including the radiative corrections. This procedure has been applied in 3+1-QED to positronium and H-atom, and the spectrum as well as radiative corrections have been obtained [5].

2 Two-body equation in the Coulomb gauge

1. We consider the system of two spinorial matter fields $\psi_1(x,t)$, $\psi_2(x,t)$ interacting via the electromagnetic field $A_\mu(x,t)$. The Lagrangian of the system is

$$\mathcal{L} = \sum_{\kappa=1}^{2} [\bar{\psi}_\kappa \gamma^\mu (i\partial_\mu - e_\kappa A_\mu)\psi_\kappa - m_\kappa \bar{\psi}_\kappa \psi_\kappa] - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} , \tag{2.1}$$

where $(\mu, \nu) = \overline{0,1}$, γ^μ are Dirac matrices which in 1+1-dimensional space-time are 2×2:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

We suppose that space is a circle of length L, $0 \le x < L$, so space-time manifold is a cylinder $S^1 \times R^1$. The fields ψ_κ are 2-component Dirac spinors, and $\bar{\psi}_\kappa = \psi_\kappa^* \gamma^0$. The coupling constants e_κ and the masses m_κ have the dimensions of a reciprocal length. ψ_κ the dimension of $1/\ell^{1/2}$ and A_μ is dimensionless.

On the circle boundary conditions for the field must be specified. We suppose the following ones

$$\begin{aligned} A_\mu(L,t) &= A_\mu(0,t) , \\ \psi_\kappa(L,t) &= e^{i2\pi\kappa_\kappa}\psi_\kappa(0,t), \ \kappa = 1,2 \end{aligned} \tag{2.2}$$

κ_1, κ_2 being arbitrary numbers. Moreover, we assume that $A_0(0,t) = 0$ (see later).
The Lagrangian (2.1) is invariant with respect to the local gauge transformations

$$\psi_\kappa \to \psi'_\kappa = \exp\{-e_\kappa \Lambda\}\psi_\kappa .$$

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda \ , \tag{2.3}$$

where $\Lambda(x, t)$ is a gauge function. It follows that the gauge transformations which respect the boundary conditions (2.2) must be of the form

$$\Lambda(L) = \Lambda(0) + \frac{2\pi}{e_2}\mathcal{N} . \quad \mathcal{N} \in \mathbf{Z} \ , \tag{2.4}$$

with

$$\frac{e_1}{e_2} = \mathcal{N}_0 \ , \tag{2.5}$$

$\mathcal{N}_0$ being a non-zero integer, positive or negative, so that the total charge is $e_2(1 + \mathcal{N}_0)$. Eq. (2.5) implies the charge quantization condition for our system.

We see that the gauge transformations under consideration are divided into topological classes characterized by the integer $\mathcal{N}$. If $\Lambda(L) = \Lambda(0)$ then the gauge transformation is topologically trivial and belongs to the $\mathcal{N} = 0$ class. If $\mathcal{N} \neq 0$ it is nontrivial and has winding number $\mathcal{N}$.

Given Eq. (2.4), the non-integrable phase

$$\Gamma(A) = \exp\{\frac{i}{2\pi}\int_0^L dx A_1(x, t)\}$$

is a unique gauge-invariant quantity that can be constructed from the gauge field[1),3)]. By a topologically trivial transformation we can make A_1 independent of x,

$$A_1(x, t) = b(t) \ .$$

i.e.obeying the Coulomb gauge $\partial_1 A_1 = 0$, then

$$\Gamma(A) = \exp\{\frac{iL}{2\pi}b(t)\} \ .$$

In contrast to $\Gamma(A)$, the line integral

$$b(t) = \frac{1}{L}\int_0^L dx A_1(x, t)$$

is invariant only under the topologically trivial gauge transformations. The gauge transformations from the $\mathcal{N}$th topological class shift b by $\frac{2\pi}{e_2 L}\mathcal{N}$. By a non-trivial gauge transformation of the form $g_{\mathcal{N}} = \exp\{i\frac{2\pi}{L}\mathcal{N}x\}$, we can then bring b into the interval $[0, \frac{2\pi}{e_e L}]$. The configurations $b = 0$ and $b = \frac{2\pi}{e_2 L}$ are gauge-equivalent, since they are connected by the gauge transformation g_1 from the first topological class. the gauge-field configuration space is therefore a circle with length $\frac{2\pi}{e_2 L}$ [2)].

2. The action of our system is

$$W = \int_{-\infty}^{\infty} dt \int_0^L dx \mathcal{L}(x, t) \ , \tag{2.6}$$

The electromagnetic field equations deduced from it are

$$\partial_\nu F^{\nu\mu} = J^\mu \ , \tag{2.7}$$

where we have introduced the total matter current

$$J^\mu = \sum_{\kappa=1}^{2} e_\kappa j^\mu_{(\kappa)} \ .$$

from Eq. (2.7) we have

$$\partial_\mu J^\mu = 0 \ ,$$

i.e. the total current is conserved.

The action of the electromagnetic field can be reexpressed by a partial integration, using (2.7) and the boundary conditions (2.2), as (see Appendix 2)

$$-\frac{1}{4}\int_{-\infty}^{\infty} dt \int_0^L dx F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}dt \int_0^L dx J^\mu A_\mu \ .$$

so the total action is

$$W[\psi, A] = \int_{-\infty}^{\infty} dt \int_0^L dx\{\sum_{\kappa=1}^{2} \bar\psi_\kappa(\gamma^\mu i\partial_\mu - m_\kappa)\psi_\kappa$$

$$- \frac{1}{2}J^\mu A_\mu\} \ . \tag{2.8}$$

If we solve Eq. (2.7), express A_μ in terms of J^μ and insert the expression obtained into the Eq. (2.8), then we get an action written only in terms of the matter fields. To perform actual calculations especially for radiative processes, it is much simpler and more direct to work with this action rather than with the equations of motion for matter.

The equations (2.7) take the form

$$E'(x, t) = J^0(x, t), \tag{2.9}$$

$$\dot E(x, t) = -J^1(x, t) \ .$$

where in Coulomb gauge

$$E \equiv F_{01} = \dot b(t) - A'_0(x, t)$$

is the electric field. The overdot indicates the time derivative, and the prime is the derivation with respect to x.

The first equation gives us the Gauss' law and the boundary conditions for $E(x, t)$:

$$Q \equiv \int_0^L dx J^0(x, t) = E(L, t) - E(0, t) \tag{2.10}$$

i.e. the electric field is single-valued only if the total electric charge is zero.

The equations (2.9) can be rewritten in the form

$$A''_0 = -J^0 \ , \qquad (a)$$

$$\ddot b = -\frac{1}{L}Q_5 \ , \qquad (b) \tag{2.11}$$

where $Q_5 \equiv \int_0^L dx J^1(x, t)$ is the axial charge.

Eq. (2.11a) with the boundary conditions $A_0(x+L,t) = A_0(x,t)$ and $A_0(0,t) = 0$ is solved by

$$A_0(x,t) = -\int_0^L dy \mathcal{D}(x,t|L) J^0(y,t) ,$$

where the Green's function is

$$\mathcal{D}(x,y|L) = \frac{1}{2}|x-y| + \frac{xy}{L} - \frac{1}{2}y , \tag{2.12}$$

i.e. $A_0(x,t)$ is completely determined by $J^0(x,t)$, while Eq. (2.11b) shows that the time evolution of the global gauge-field degree of freedom b is governed by the axial charge.

If we insert (2.12) into Eq. (2.8), we obtain the action in the Coulomb gauge as

$$\begin{aligned} W[\psi, A] = &\int_{-\infty}^{\infty} dt \int_0^L dx \sum_{\kappa=1}^{2} \bar\psi_\kappa(\gamma^\mu i\partial_\mu - m_\kappa)\psi_\kappa \\ &+\frac{1}{2}\int_{-\infty}^{\infty} dt \int_0^L dx \int_0^L dy J^0(x,t)\mathcal{D}(x,y|L)J^0(y,t) \\ &- \frac{1}{2}\int_{-\infty}^{\infty} dt \int_0^L dx J^1(x,t) b(t) . \end{aligned} \tag{2.13}$$

The last term represents the interaction of the matter currents with the global gauge-field degree of freedom b, while the middle term is a sum of current-current interactions containing both the mutual interaction terms, e.g.

$$\frac{e_1 e_2}{2}\int_{-\infty}^{\infty} dt \int_0^L dx \int_0^L dy \bar\psi_1(x,t)\gamma^0\psi_1(x,t)\mathcal{D}(x,y|L)\bar\psi_2(y,t)\gamma^0\psi_2(y,t) + (1 \leftrightarrow 2)$$

and the two self interaction terms

$$-\frac{e_i^2}{2}\int dx dy \bar\psi_i(x,t)\gamma^0\psi_i(x,t)\mathcal{D}(x,y|L)\bar\psi_i(y,t)\gamma^0\psi_i(y,t) ; \quad i = 1,2$$

3. We must now specify a variational principle for the matter fields. We could vary the action (2.13) with respect to individual fields ψ_1 and ψ_2 separately. This results in non-linear coupled Hartree-type equations for these fields. Instead, we use a relativistic configuration space formalism [5] to take into account the long-range quantum correlations. We define a composite field Φ by

$$\Phi(x_1,t|x_2,t) \equiv \psi_1(\kappa_1,t) \otimes \psi_2(x_2,t) . \tag{2.14}$$

This is a 4-component spinor field. The configuration space (x_1,x_2) is a square of side L $(0 \le x_1 < L,\ 0 \le x_2 < L)$, the opposite sides being identified in the way illustrated in Fig. 1, so the configuration space is a torus. In Fig. 2, we show moves which do not change, up to a phase, the field Φ defined on the torus. The corresponding boundary conditions are obtained from Eq. (2.2) as

$$\Phi(L,t|0,t) = \exp\{i2\pi\kappa_1\}\Phi(o,t|0,t) ,$$

$$\Phi(0,t|L,t) = \exp\{i2\pi\kappa_2\}\Phi(0,t|0,t) ,$$

$$\Phi(L,t|L,t) = \exp\{i2\pi(\kappa_1+\kappa_2)\}\Phi(o,t|0,t) .$$

We can rewrite our action (2.13) entirely in terms of the composite field Φ. In order to do this we multiply the kinetic energy terms with the normalization factors (which are constants of the motion)

$$\int_0^L dx \psi_\kappa^+(x,t)\psi_\kappa(x,t) = 1, \quad \kappa = 1 \text{ or } 2 .$$

We have to do this twice on the self-interaction terms. The resultant action in terms of the composite field is

$$\begin{aligned} W[\Phi, A] = &\int_{-\infty}^{\infty} dt \int_0^L dx_1 \int_0^L dx_2 \Phi(x_1,t|x_2,t)\{(\gamma\mu\pi_{(1)\mu} - m_1)\otimes\gamma^0 \\ &+ \gamma^0 \otimes (\gamma^\mu\pi_{(2)\mu} - m_2) + e_1 e_2(\gamma^0\otimes\gamma^0)\mathcal{D}(x_1,x_2|L)\}\Phi(x_1,t|x_2,t) . \end{aligned} \tag{2.15}$$

Here the spin matrices are written in the form of tensor products $\otimes$, the first factor always referring to the spin space of particle 1, the second to particle 2.

The generalized (kinetic) momenta $\pi_{(i)\mu}$ are given by

$$\pi_{(i)\mu} = p_{(i)\mu} + e_i A_{(i)\mu}^{self} , \tag{2.16}$$

with

$$p_{(i)\mu} \equiv i\frac{\partial}{\partial x_i^\mu} \text{ and}$$

$$A_{(1),0}^{self}(x,t) \equiv \varphi_{(1)}^{self}(x,t) = \frac{e_1}{2}\int_0^L dy \int_0^L dz \mathcal{D}(x,z|L)\Phi(z,t|y,t)(\gamma^0\otimes\gamma^0)\Phi(z,t|y,t) ,$$

$$A_{(2),0}^{self}(x,t) \equiv \varphi_{(2)}^{self}(x,t) = \frac{e_2}{2}\int_0^L dy \int_0^L dz \mathcal{D}(x,y|L)\Phi(z,t|y,t)(\gamma^0\otimes\gamma^0)\Phi(z,t|y,t)$$

$$A_{(1),1}^{self}(x,t) = A_{(2),1}^{self}(x,t) = -\frac{1}{2}b(t) ,$$

the self-potentials $\varphi_{(\kappa)}^{self}$ being non-linear integral expressions.

Let us note that the last term in (2.15) can also be put into the self-potentials $\varphi_{(\kappa)}^{self}$ one half for each particle; the total potentials then take the form:

$$\begin{aligned} \varphi_{(1)}^{self}(x,t) &\to \varphi^{self}(x,t) \\ \varphi_{(2)}^{self}(x,t) &\to \varphi^{self}(x,t) \end{aligned} \tag{2.17}$$

$$\begin{aligned} \varphi^{self}(x,t) \equiv \frac{1}{2}\int_0^L d \int_0^L dz\{e_1\mathcal{D}(x,z|L) + e_2\mathcal{D}(x,y|L)\} \\ \times\Phi(z,t|y,t)(\gamma^0\otimes\gamma^0)\Phi(z,t|y,t) . \end{aligned}$$

Now we require the action (2.15) to be stationary not with respect to the variation of the individual fields but with respect to the total composite field only. This is a weaker condition than before and leads to an equation for $\Phi(x_1t|x_2t)$ defined on the torus. This two-body equation is

$$\{(\gamma^\mu\pi_{(1),\mu} - m_1)\otimes\gamma^0 + \gamma^0\otimes(\gamma^\mu\pi_{(2),\mu} - m_2)$$

$$+e_1e_2(\gamma^0\otimes\gamma^0)\mathcal{D}(x_1,x_2|L)\}\Phi(x_1,t|x_2,t)=0\ . \tag{2.18}$$

Let us introduce center of mass and relative coordinates according to

$$\Pi=\pi_{(1)}+\pi_{(2)},\pi=\pi_{(1)}-\pi_{(2)}\ ,$$

$$x_+=\pi_1+x_2,x_-=x_1-x_2\ .$$

The configuration space (x_-,x_+) is again a torus, but with a circle length $2L(-L\le x_-<L,\ 0\le x_+<2L)$. The moves shown in fig. 2 induce the corresponding moves on this torus (see Fig. 3). The boundary conditions for the field $\Phi(x,t|x_+,t)$ are

$$\Phi(L,t|L,t)=\exp\{i2\pi\kappa_1\}\Phi(0,t|0,t)\ ,$$

$$\Phi(-L,t|L,t)=\exp\{i2\pi\kappa_2\}\Phi(o,t|0,t)\ ,$$

$$\Phi(0,t|2L,t)\lambda=\exp\{i2\pi(\kappa_1+\kappa_2)\}\Phi(0,t|0,t)\ .$$

The function $\mathcal{D}(x_1,x_2|L)$ can be rewritten as a sum of center of mass and relative parts:

$$\mathcal{D}(x_1,x_2|L)=\mathcal{D}_-(x_-|L)+\mathcal{D}_+(x_+|L)\ ,$$

$$\mathcal{D}_-(x_-|L)\equiv\frac{1}{2}|x_-|-\frac{1}{4L}x_-^2+\frac{1}{4}x_-\ ,$$

$$\mathcal{D}_+(x_+|L)\equiv\frac{1}{4L}x_+^2=\frac{1}{4}x_+\ ,$$

both parts being asymmetric. The functions $\mathcal{D}_-$ and $\mathcal{D}_+$ are illustrated in Fig. 4.

Eq. (2.187), without the self-field terms, for simplicity, becomes

$$\{\Gamma^\mu\Pi_\mu+\kappa^\mu\pi_\mu+(\gamma^0\otimes\gamma^0)\varphi(x_-,x_+|L)-$$

$$-m_1I\otimes\gamma^0-m_2\gamma^0\otimes I\}\Phi(x,t|x_+,t)=0\ ,$$

where we have introduced

$$\Gamma^\mu\equiv\frac{1}{2}(\gamma^\mu\otimes\gamma^0+\gamma^0\otimes\gamma^\mu) \tag{2.19a}$$

and

$$\kappa^\mu\equiv\frac{1}{2}(\gamma^\mu\otimes\gamma^0-\gamma^0\otimes\gamma^\mu)\ . \tag{2.19b}$$

We see now that κ^0 vanishes which means that the zero component of the relative momentum π_0 drops out of the equation automatically and we get

$$\{\Gamma^0\Pi^0-\Gamma^1\Pi^1-\kappa^1\pi^1+(\gamma^0\otimes\gamma^0)\varphi(x_-,x_+|L)-$$

$$-m_1I\otimes\gamma^0-m_2\gamma^0\otimes I\}\Phi(x_-,t|x_+,t))=0\ . \tag{2.20}$$

Thus we have only one time variable conjugate to the center of mass energy Π_0, one degree of freedom for the center of mass momentum Π^1 and one degree of freedom for the relative momentum Π^1. Since Π_0 is the "Hamiltonian" of the system, by multiplying (2.20) by Γ_0^{-1} we obtain the Hamiltonian form of the two-body equation

$$\Pi_0\Phi=\{\alpha\Pi^1+\frac{1}{2}(\alpha_1-\alpha_2)\pi^1-\varphi+m_1\beta_1\cdot I+m_2I\cdot\beta_2\}\Phi\ , \tag{2.21}$$

where we have defined

$$\alpha\equiv\frac{1}{2}(\alpha_1+\alpha_2)\ ,\alpha_1\equiv\gamma^5\otimes I,\quad\alpha_2\equiv I\otimes\gamma^5\ ,\beta_1=\beta_2\equiv\gamma^0\ .$$

Eq. (2.21) has the form of a generalized Dirac equation, now a 4-component wave equation.

An important property of the Hamiltonian obtained is that relative and center of mass terms are additive:

$$\Pi_0=H_{c.m.}+H_{rel}$$

$$H_{c.m.}\equiv\alpha\Pi^1-e_1e_2\mathcal{D}_+\ .$$

$$H_{rel}\equiv\frac{1}{2}(\alpha_1-\alpha_2)\pi^1-e_1e_2\mathcal{D}_-+m_1\beta_1\otimes I+I\otimes m_2\beta_2\ . \tag{2.22}$$

We shall continue our discussion of the properties of the two-body equation in the Coulomb gauge in Sec. 4.

3 Two-body equation in the Lorentz gauge

1. Let us construct the two-body equation in the Lorentz gauge $\partial_\mu\hat{A}^\mu=0$. This provides a covariant description for the two-body problem.

In the Lorentz gauge, the electromagnetic field equations (2.7) take the form

$$\Box\hat{A}^\mu=J^\mu \tag{3.1}$$

The solution of (3.1) is

$$\hat{A}^\mu(x,t)=\int_{-\infty}^{\infty}d\hat{t}\int_0^L dy\hat{\mathcal{D}}(x-y,t-\hat{t})J^\mu(y,\hat{t})+C^\mu(x,t)\ , \tag{3.2}$$

where $\hat{\mathcal{D}}(x,t)$ is a fundamental solution of the operator $\Box\equiv\partial_\mu\partial^\mu=\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}$:

$$\Box\hat{\mathcal{D}}(x,t)=\delta(x)\delta(t)\ , \tag{3.3}$$

$$\hat{\mathcal{D}}(x,t)=\frac{1}{2}\theta(t-|x|)\ ,$$

while $C^\mu(x,t)$ is a solution of the corresponding homogeneous equation:

$$\Box C^\mu(x,t)=0\ . \tag{3.4}$$

We denote the Lorentz-gauge electromagnetic field by $\hat{A}^\mu$ in order to distinguish it from the Coulomb gauge one A^μ.

A general solution to Eq. (3.4) can be represented as a superposition of functions depending only on $(x-t)$ or $(x+t)$. The boundary conditions $\hat{A}^\mu(0,t)=\hat{A}^\mu(L,t)$ and $\hat{A}_0(0,t)=0$ fix C^μ in Eq. (3.2) in the form

$$C^\mu=-\delta_0^\mu\int_{-\infty}^{\infty}d\hat{t}\int_0^L dy\hat{\mathcal{D}}(-y;t-\hat{t})J^0(y,\hat{t})\ ,$$

i.e. C^μ is a constant.

With Eq. (3.3), Eq. (3.2) then becomes

$$\dot{A}_\mu(x,t) = \frac{1}{2}\int_{-\infty}^{t} d\bar{t} \int_{x-(t-\bar{t})}^{x+(t-\bar{t})} dy J^\mu(y,\bar{t})$$

$$-\frac{1}{2}\delta_0^\mu \int_{-\infty}^{t} d\bar{t} \int_{-(t-\bar{t})}^{(t-\bar{t})} dy J^\mu(y,\bar{t}) \ . \tag{3.5}$$

The action is

$$W[\psi,\dot{A}] = \int_{-\infty}^{\infty} dt \int_0^L dx \{\sum_{\kappa=1}^{2} \bar{\psi}_\kappa(\gamma^\mu i\partial_\mu - m_\kappa)\psi_\kappa -$$

$$-\frac{1}{2}J^\mu \dot{A}_\mu\} \ . \tag{3.6}$$

If we insert (3.5) into the action and use the composite field Φ defined by Eq. (2.14), we obtain

$$W[\Phi,\dot{A}] = \int_{-\infty}^{\infty} dt \int_0^L dx_1 \int_0^L dx_2 \Phi(x_1,t|x_2,t)\{(\gamma^\mu \dot{\pi}_{(1),\mu} - m_1)\otimes\gamma^0$$

$$+\gamma^0 \otimes (\gamma^\mu \dot{\pi}_{(2),\mu} - m_2) + \frac{e_1 e_2}{2}((\gamma^\mu G_\mu \otimes \gamma^0) + (\gamma^0 \otimes \gamma^\mu \Gamma_\mu))\}$$

$$\times \Phi(x_1 t|x_2,t) \tag{3.7}$$

The generalized (kinetic) momenta are now

$$\dot{\pi}_{(i),\mu} = p_{(i),\mu} + e_i \dot{A}^{self}_{(i),\mu}$$

with self-potentials

$$\dot{A}^{self}_{(1),\mu} = \frac{e_1}{2}\Gamma_\mu \ ,$$

$$\dot{A}^{self}_{(2),\mu} = \frac{e_2}{2}G_\mu \ ,$$

where

$$\Gamma^\mu(x,t) \equiv \int_{-\infty}^{\infty} d\bar{t} \int_0^L dz \int_0^L d\bar{z} \Phi(z,\bar{t}|\bar{z},\bar{t})\{\delta_0^\mu(\gamma^0\otimes\gamma^0)\dot{\mathcal{D}}(-z,t-\bar{t})-$$

$$-(\gamma^\mu\otimes\gamma^0)\dot{\mathcal{D}}(x-z,t-\bar{t})\}\Phi(z,\bar{t}|\bar{z},\bar{t}) \ ,$$

$$G^\mu(x,t) \equiv \int_{-\infty}^{\infty} d\bar{t} \int_0^L dz \int_0^L d\bar{z} \Phi(z,\bar{t}|\bar{z},\bar{t})\{\delta_0^\mu(\gamma^0\otimes\gamma^0)\dot{\mathcal{D}}(-\bar{z},t-\bar{t})-$$

$$-(\gamma^0\otimes\gamma^\mu)\dot{\mathcal{D}}(x-\bar{z},t-\bar{t})\}\Phi(z,\bar{t}|\bar{z},\bar{t}) \ .$$

Note that the mutual interaction term in (3.7) can be rewritten as

$$\frac{e_1 e_2}{2}\int_{-\infty}^{\infty} dt \int_0^L dx_1 \int_0^L dx_2 \Phi(x_1 t|x_2,t)((\gamma^\mu G_\mu \otimes\gamma^0) + (\gamma^0\otimes\gamma^\mu\Gamma_\mu))$$

$$\times\Phi(x_1,t|x_2,t) =$$

$$= -\frac{e_1 e_2}{2}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \int_0^L dx_1 \int_0^L dx_2 \bar{\Phi}(x_1 t_1|x_2,t_2)(\gamma^0\otimes\gamma^0)$$

$$\times(\dot{\mathcal{D}}(x_1 - x_2; t_1 - t_2) + \dot{\mathcal{D}}(x_1 - x_2; t_2 - t_1) - \dot{\mathcal{D}}(-x_2, t_1 - t_2)$$

$$-\dot{\mathcal{D}}(x_1,,t_2 - t_1))\Phi(x_1 t_1|x_2,t_2)$$

where the field $\Phi(x_1,t_1|x_2,t_2)$ is composed of the matter fields taken at different times:

$$\Phi(x_1,t_1|x_2,t_2) = \psi_1(x_1,t_1)\psi_2(x_2,t_2) \ .$$

Again, as in the Coulomb gauge, this term can be put into the self-potentials $\dot{A}^{self}_{(i),\mu}$. The self-potentials then take the form

$$\dot{A}^{self}_{(1),\mu} \to \dot{A}^{self}_{(1),\mu} + \frac{e_2}{2}G_\mu = \frac{1}{2}(e_1\Gamma_\mu + e_2 G_\mu) \ ,$$

$$\dot{A}^{self}_{(2),\mu} \to \dot{A}^{self}_{(2),\mu} + \frac{e_1}{2}\Gamma_\mu = \frac{1}{2}(e_1\Gamma_\mu + e_2 G_\mu) \ . \tag{3.8}$$

The two-body equation is

$$\{(\gamma^\mu\dot{\pi}_{(1),\mu} - m_1)\otimes\gamma^0 + \gamma^0\otimes(\gamma^\mu\dot{\pi}_{(2),\mu} - m_2) \cdot$$

$$+\frac{e_1 e_2}{2}(\gamma^\mu G_\mu\otimes\gamma^0 + \gamma^0\otimes\gamma^\mu\Gamma_\mu)\}\Phi(x_1,t|x_2,t) = 0 \ , \tag{3.9}$$

which looks like that in the Coulomb gauge picture, the difference being in the form of self-potentials and mutual interaction.

2. The physical results are, of course, independent of the choice of gauge. Let us show now that the Lorentz gauge and Coulomb gauge pictures are gauge-equivalent, i.e. related by a gauge transformation from the local gauge symmetry group in the problem.

We transform the matter fields as follows

$$\psi_\kappa(x,t) \xrightarrow{\lambda} \exp\{-i\frac{e_\kappa}{2}\lambda(x,t)\}\psi_\kappa(x,t) \ .$$

The equation (3.6) which is invariant under this transformation takes the form

$$W[\psi,\dot{A}] \xrightarrow{\lambda} W(\psi,\dot{A} + \partial\lambda] \ .$$

If we choose λ in such a way that

$$\dot{A}_\mu + \partial_\mu\lambda = A_\mu \tag{3.10}$$

where A_μ satisfies the Coulomb gauge condition $\partial_1 A_1 = 0$, $A_1(x,t) = b(t)$, then we get the action in the Coulomb gauge:

$$W[\psi,\dot{A} + \partial\lambda] = W[\psi,A] \ .$$

From Eq. (3.10) we have

$$\lambda(x,t\ ) = -\int_+^x dz \dot{A}_1(z,t) + b(t)\cdot x$$

and

$$\lambda(L,t) = \lambda(0,t) = 0$$

that describes a topologically trivial gauge transformation.

Let us perform the transformation (3.10) in the action (3.7) written in terms of the composite field Φ. For simplicity, we consider the case when the mutual interaction term is put into the self-potentials, so that the latters are given by Eq. (3.87). The composite field is transformed as

$$\Phi(x_1, t|x_1, t) \xrightarrow{\Lambda} \exp\{-\frac{i}{2}(e_1\lambda(x_1, t) + e_2\lambda(x_2, t)\}\Phi(x_1 t|x_2, t) ,$$

while the self-potentials change as follows

$$\hat{A}^{self}_{(i),1} \xrightarrow{\Lambda} \hat{A}^{self}_{(i),1} - \frac{1}{2}\partial_\mu\lambda . \tag{3.11}$$

Substituting λ from Eq. (3.10) into Eq. (3.11), and using Eqs. (2.12) and (3.5), we get that

$$\hat{A}^{self}_{(i),\mu} \xrightarrow{\Lambda} \hat{A}^{self}_{(i),1} - \frac{1}{2}(A_1 - \hat{A}_1) = -\frac{1}{2}b(t) \tag{3.12a}$$

and

$$\hat{A}^{P}self_{(i),0} \xrightarrow{\Lambda} \hat{A}^{self}_{(i),0} - \frac{1}{2}(A_0 - \hat{A}_0) = \varphi^{self} \tag{3.12b}$$

Thus, under the gauge transformation (3.10) the self-potentials in the Lorentz gauge are transformed to the one in the Coulomb gauge. The Lorentz gauge action (3.7) is therefore transformed to the Coulomb gauge one given by Eq. (2.15).

To study the role of the global gauge-field degree of freedom b it is more convenient to work is the Coulomb gauge. Through the rest of the paper we shall use just the Coulomb gauge description.

4 Analysis of two-body equation

4.1 Massless case

There are three types of interactions which distinguish the action (2.15) from the free one: i) interaction described by the self-potentials $\varphi^{self}_{(\kappa)}$, ii) interaction between the matter fields and global gauge-field degree of freedom $b(t)$ and iii) the Coulomb interaction with potential φ. All these interactions influence the spectrum. The first and second ones are responsible for radiative processes. We start first with the action and corresponding two-body equation without the self-field potentials $\varphi^{self}_{(\kappa)}$. These potentials will be considered in the next section.

Let us find the eigenfunctions and the spectrum of the first-quantized Hamiltonian (2.22). The equation for the eigenfunctions is

$$(\alpha\Pi^1 + \frac{1}{2}(\alpha_1 - \alpha_2)\pi^1 + m_1\beta_1 \cdot I + I \cdot m_2\beta_2)\Phi = (E + \varphi)\Phi , \tag{4.1}$$

where

$$\Pi^1 = 2i\frac{\partial}{\partial x_+} - \frac{1}{2}(e_1 + e_2)b ,$$

$$\pi^1 = 2i\frac{\partial}{\partial x_-} - \frac{1}{2}(e_1 - e_2)b .$$

Here we treat $b(t)$ as an external time-dependent field and postpone its consideration as a self-potential also to the next section.

If we denote the components of the composite field Φ as

$$\Phi^{11} \equiv \chi_1 , \quad \Phi^{12} \equiv \chi_2 .$$

$$\Phi^{21} \equiv \chi_3 , \quad \Phi^{22} \equiv \chi_4 .$$

(see Fig. 5), then for the zero masses $m_1 = m_2 = 0$ Eq. (4.1) reduces to the system of four equations:

$$2i\frac{\partial}{\partial x_+}\chi_1 - ((\varphi + E) + \frac{1}{2}(e_1 + e_2)b)\chi_1 = 0 .$$

$$2i\frac{\partial}{\partial x_+}\chi_4 + ((\varphi + E) - \frac{1}{2}(e_1 + e_2)b)\chi_4 = 0 . \tag{4.2a}$$

$$2i\frac{\partial}{\partial x_-}\chi_2 - (\varphi + E) + \frac{1}{2}(e_1 - e_2)(e_1 - e_2)\chi_2 = 0 .$$

$$2i\frac{\partial}{\partial x_-}\chi_3 + ((\varphi + E) - \frac{1}{2}(e_1 - e_2)b)\chi_3 = 0 . \tag{4.2b}$$

The presence of the global gauge-field degree of freedom shows itself in all four equations. For $e_1 = -e_2$, b drops out of the first pair of the equations, and for $e_1 = e_2$ of the second one.

In the absence of the Coulomb interaction ($\varphi = 0$), we see from these equations that

$$\chi_1(-E, -e_1, -e_2) = \chi_4(E, e_1, e_2) .$$

$$\chi_2(-E, -e_1, -e_2) = \chi_3(E, e_1, e_2) .$$

i.e. the negative energy solutions of χ_1 and χ_2 coincide correspondingly with the positive energy solutions of χ_4 and χ_3 of opposite charges. Therefore we may consider either positive and negative energy solutions of χ_1 and χ_2 or only positive energy solutions of all four equations as physical particles.

This interpretation of the negative energy solutions has to be slightly modified in the presence of the Coulomb interaction. In this case we have

$$\chi_1^*(E, -e_1, -e_2) = \chi_4(e, e_1, e_2) . \tag{4.3}$$

$$\chi_2^*(E, -e_1, -e_2) = \chi_3(E, e_1, e_2) .$$

Again only half of all solutions correspond to physical particles.

With the boundary and normalization conditions, namely ($i = \overline{1, 4}$)

$$\chi_i(L|L) = \exp\{i2\pi\kappa_1^{(i)}\}\chi_i(0, 0) ,$$

$$\chi_i(-L|L) = \exp\{i2\pi\kappa_2^{(i)}\}\chi_i(0, 0) ,$$

$$\chi_i(0, 2L) = \exp\{i2\pi(\kappa_1^{(i)} + \kappa_2^{(i)})\}\chi_i(0, 0$$

and

$$\int_{-L}^{L} dx_- \int_0^{2L} dx_+ \chi_i^*(x_-|x_+)\chi_i(x_-|x_+) = 1$$

(no summation over i), Eqs. (4.2) are easily solved by eigenfunctions

$$\chi^c_{1,n} = \frac{1}{2L}\exp\{-\frac{i}{2}e_1e_2I_1(x_-,x_+) - \frac{i}{2}(E^c_{1,n} + \frac{1}{2}(e_1+e_2)b)(x_+ - \frac{L}{2})\} ,$$

$$\chi^c_{2,n} = \frac{1}{2L}\exp\{-\frac{i}{2}e_1e_2I_2(x_-,x_+) - \frac{i}{2}(E^c_{2,n} + \frac{1}{2}(e_1-e_2)b)(x_- + \frac{L}{2})\} \tag{4.4}$$

and the eigenvalues

$$E^c_{1,n} = -\frac{1}{12}e_1e_2L + \frac{2\pi}{L}n - \frac{1}{2}(e_1+e_2)b ,$$

$$E^c_{2,n} = -\frac{1}{6}e_1e_2L + \frac{2\pi}{L}n - \frac{1}{2}(e_1-e_2)b\ , n \in \mathbf{Z} \tag{4.5}$$

where

$$I_1(x_-,x_+) \equiv \frac{1}{2}x_+\mathcal{D}_+(x_+|L) + (x_+ - \frac{L}{2})\mathcal{D}_-(x_-|L) - \frac{1}{24L}(x_+^3 - \frac{1}{2}L^3) ,$$

$$I_2(x_-,x_+) \equiv \frac{1}{2}x_-\mathcal{D}_-(x_-|L) + (x_- + \frac{L}{2})\mathcal{D}_+(x_+|L) + \frac{1}{24L}(x_-^3 + \frac{1}{2}L^3) .$$

The eigenfunctions $\chi^c_{3,n}$ and $\chi^c_{4,n}$ are obtained from Eq. (4.4) by making use of the relation (4.3), the corresponding spectrums being

$$E^c_{3,n} = -\frac{1}{6}e_1e_2L + \frac{2\pi}{L}n + \frac{1}{2}(e_1-e_2)b ,$$

$$E^c_{4,n} = -\frac{1}{12}e_1e_2L + \frac{2\pi}{L}n + \frac{1}{2}(e_1+e_2)b .$$

The superscript "c" indicates that the eigenfunctions $\chi^c_{i,n}$ and the eigenvalues $E^c_{i,n}$ represent the solution of our two-body problem in the presence of Coulomb interaction, but without the self-field potentials.

Eqs. (4.2) fix also the phases $\kappa_1^{(i)}$, $\kappa_2^{(i)}$ as follows

$$\kappa^{(1)}_{1,n} = -\kappa^{(1)}_{2,n} = \frac{n}{2} - \frac{e_1e_2}{32\pi}L^2 ,$$

$$\kappa^{(2)}_{1,n} = \kappa^{(2)}_{2,n} = \frac{n}{2} - \frac{e_1e_2}{32\pi}L^2 ,$$

$$\kappa^{(3)}_{1,n} = \kappa^{(3)2,n} = \frac{n}{2} + \frac{e_1e_2}{32\pi}l^2 ,$$

$$\kappa^{(4)}_{1,n} = -\kappa^{(4)}_{2,n} = \frac{n}{2} + \frac{e_1e_2}{32\pi}L^2 ,$$

so that the boundary conditions for $\chi^c_{1,n}$ and $\chi^c_{2,n}$, for instance, are

$$\chi^c_{1,n}(L,L) = (-1)^n e^{-i2\pi\kappa_0}\chi^c_{1,n}(0,0) ,$$

$$\chi^c_{1,n}(-L,L) = (-1)^n e^{i2\pi\kappa_0}\chi^c_{1,n}(0,0) ,$$

$$\chi^c_{1,n}(0,2L) = \chi^c_{1,n}(0,0) ,$$

and

$$\chi^c_{2,n}(L,L) = (-1)^n e^{-2\pi\kappa_0}\chi^c_{2,n}(0,0) ,$$

$$\chi^c_{2,n}(-L,L) = (-1)^n e^{-i2\pi}\kappa_0\chi^c_{2,n}(0,0) ,$$

$$\chi^c_{2,n}(0,2L) = e^{-i4\pi\kappa_0}\chi^c_{2,n}(0,0) ,$$

where $\kappa_0 \equiv \frac{e_1e_2}{32\pi}L^2$.

We see that the spectrums $E^c_{i,n}$ depend on the global gauge field degree of freedom b. As b increases from 0 to $\frac{2\pi}{e_2L}$, then for $\mathcal{N}_0 > 1$ say, the energies of $E^c_{1,n}$ and $E^c_{2,n}$ decrease by $\frac{\pi}{L}(\mathcal{N}_0+1)$ and $\frac{\pi}{K}(\mathcal{N}_0-1)$ and the energies of $E^c_{3,n}$ and $E^c_{4,n}$ increase by $\frac{\pi}{L}(\mathcal{N}_0-1)$ and $\frac{\pi}{L}(\mathcal{N}_0+1)$ correspondingly. Some of energy levels change sign. however, all the spectrums at the configurations $b=0$ and $b=\frac{2\pi}{e_2L}$ must be the same, namely , the integers, since thee gauge-field configurations are gauge equivalent. this means that $\mathcal{N}_0$ must be an odd integer. The precise form of the charge quantization condition is then

$$\frac{e_1}{e_2} = \text{ odd integer } . \tag{4.6}$$

We see also that the Coulomb interaction shifts the discrete energy spectrums by a constant value. this value is different for $E^c_{1,n}$ and $E^c_{2,n}$, the difference being equal to $\frac{1}{12}e_1e_2L$. on b. In terms of $C_{i,n}$, the normalization conditions for $\chi^c_i(x_-,t|x_+,t)$ become

$$\sum_n C^*_+ i,n)(t)C_{i,n}(t) = 1 .$$

4.2 Discussion

The main results in this first part of a study of the two-body problems in spinor quantum electrodynamics on the circle are a quantization of the ratio of the charges, e_2/e_1 = odd integer, the derivation of a two-body configuration space wave equation including the non-linear and nonlocal radiative self-energy effects, the solutions of the equations for mutual interactions. The spectrum is discrete and exhibits shifts due to spin-orbit and spin-spin interactions.

Appendix 1

Maxwell's equations in 1+1-dimensions

The only nonvanishing field component is $F^{01} = \frac{\partial A^0}{\partial x} - \frac{\partial A'}{\partial E}$, have $B = 0$. From $\nabla \cdot E = \rho = J^0$ we have

$$\frac{\partial E}{\partial x} = J^0, \tag{1}$$

hence the Gauss's law takes the form

$$\int_0^L \frac{\partial E}{\partial x} = \int_0^L J^0 dx = Q = E(L) - E(0) \tag{2}$$

In particular the electric field on S^1 must have a jump if $Q \neq 0$.

Faraday's law $\nabla_\wedge E = -\frac{1}{c}\frac{\partial E}{\partial t}$ yields

$$J(L) - J(0) = -i\frac{\partial}{\partial t}(E(L) - E(0)) = -\frac{\partial Q}{\partial t} \tag{3}$$

so that for $Q = \text{const.}$, $J(x)$ is continuous.

The current conservation law $\frac{\partial J^0}{\partial t} + \frac{\partial J}{\partial x} = 0$, also yields (3).

The Lorentz gauge condition $\frac{\partial A^0}{\partial t} + \frac{\partial A'}{\partial x} = 0$ together with (1) implies the wave equations

$$\frac{\partial^2 A^0}{\partial x^2} - \frac{\partial A^1}{\partial t \partial x} = \frac{\partial^2 A^0}{\partial x^2} - \frac{\partial^2 A^0}{\partial t^2} = J^0$$

and

$$\frac{\partial E}{\partial t} = \frac{\partial^2 A^0}{\partial x \partial t} - \frac{\partial^2 A^1}{\partial t^2} = -\frac{\partial^2 A^1}{\partial x^2} - \frac{\partial^2 A^1}{\partial t^2} = -J$$

The Lienard-Wiechot potential for a point charge is

$$A_i(x,t) = e\int ds\,\dot{x}_i(s) D(x - x(s)) = \frac{e}{2}\int ds\,\dot{x}_i(s)\theta(x^0 - x^0(s) - |x - x(s)|)\ , \quad i = 0, 1\ .$$

Appendix 2

Surface Terms

We have by partial integration

$$-\frac{1}{4}\int dx F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}\int dx A_{\nu,\mu}F^{\mu\nu}$$

$$= -\frac{1}{2}\int dx[\frac{\partial}{\partial x^\mu}(A_\nu F^{\mu\nu}) - A_\nu F^{\mu\nu}_{,\mu}] + \frac{1}{2}\int dx A_\nu J^\nu + \text{ surface term}$$

Thus the surface term in the action is

$$-\frac{1}{2}\int_V dx(A_\nu F^{\mu\nu})_{,\mu} = -\frac{1}{2}\int_S d\sigma_\mu(A_\nu F^{\mu\nu})$$

There are 4 boundary surfaces bounded by $t = t_1$ and $t = t_2$, and $x = x_1$ and $x = x_2$; $y = y_1$ and $y = y_2$ and $z = z_1$ and z_2.

In 1+1 dimensions we have

$$-\frac{1}{2}\int_S d\sigma_\mu(A_\nu F^{\mu\nu}) = -\frac{1}{2}\int_{t_1}^{t_2} dx(A^1 E) + \frac{1}{2}\int_{x_1}^{x_2} dt(A^0 E)\ .$$

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Figure Captions

Fig.1 The coordinate space $(0 \leq x_1 < L, 0 \leq x_2 < L)$ is a square of side L (Fig. 1(a)). The points $(x_1, x_2 = 0)$ and $(x_1, x_2 = L)$ for all x_1 as well as the points $(x_1 = 0, x_2)$ and $(x_1 = L, x_2)$ for all x_2 are identified. These points are connected by the dashed and waved lines correspondingly. So the square is topologically equivalent to a torus $S^1 \times S^1$, both circles being of the same length L (Fig. 1(b)).

Fig.2 The moves which do not change a physical state of our system. Move (1): move particle 1 anticlockwise (say) around the first circle from the point $A(x_1 = 0, x_2 = 0)$ to the point $(x_1 = L, x_2 = 0)$. Move (2): move particle 2 anticlockwise around another circle from A to the point $(x_1 = 0, x_2 = L)$. Move (3): Move (1) + Move (2).

Fig. 3 The moves which are induced on the torus $(-L \leq x_- < L, 0 \leq x_+ < 2L)$ by the moves shown in Fig. 2. The move from $A(x_- = 0, x_+ = 0)$ to the point $(x_- = L, x_+ = L)$ (Fig. 3(a)) corresponds to the move (1), in Fig. 2, the move from the same point A to $(x_- = -L, x_+ = L)$ (Fig. 3(b)) corresponds to the move (2) and the move to the point $(x_- = 0, x_+ = 2L)$ (Fig. 3(c)) to the move (3).

Fig.4 The relative $\mathcal{D}_-(x_-|L)$ and center of mass $\mathcal{D}_+(x_+|L)$ parts of the function $\mathcal{D}(x_1, x_2|L)$ (Figs. 4(a) and 4(b) correspondingly). $\mathcal{D}_-(-\frac{L}{2}|L) = \frac{1}{16}L$, $\mathcal{D}_+(\frac{L}{2}|L) = -\frac{1}{16}L$.

Fig.5 A schematic representation of the two-body system states described by $\chi_i (i = \overline{1,4})$. The first and second components of the two-components fields ϕ_κ are represented by $\uparrow$ and $\downarrow$ correspondingly.

(a): $\chi_1 = \Phi^{11} = \psi_1^{(1)}\psi_2^{(1)}$,

(b) $\chi_2 = \Phi^{12} = \psi_1^{(1)}\psi_2^{(2)}$,

(c) $\chi_3 = \Phi^{21} = \psi_1^{(2)}\psi_2^{(1)}$,

(d): $\chi_4 = \Phi^{22} = \psi_1^{(2)}\psi_2^{(2)}$

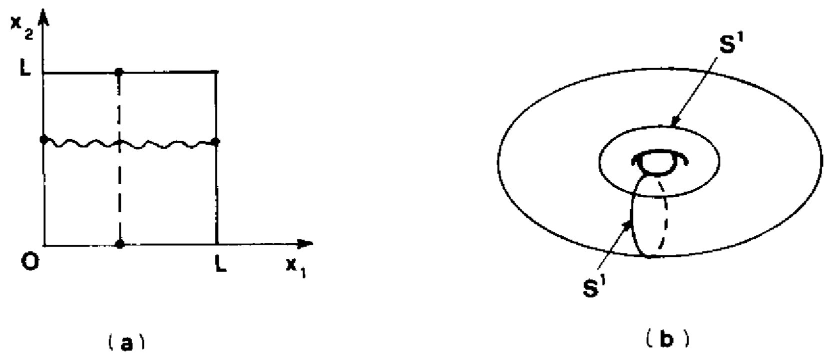

Fig.1

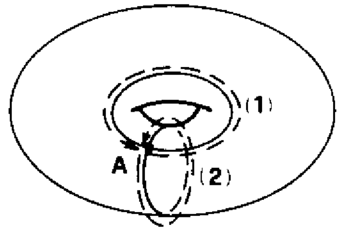

Fig.2

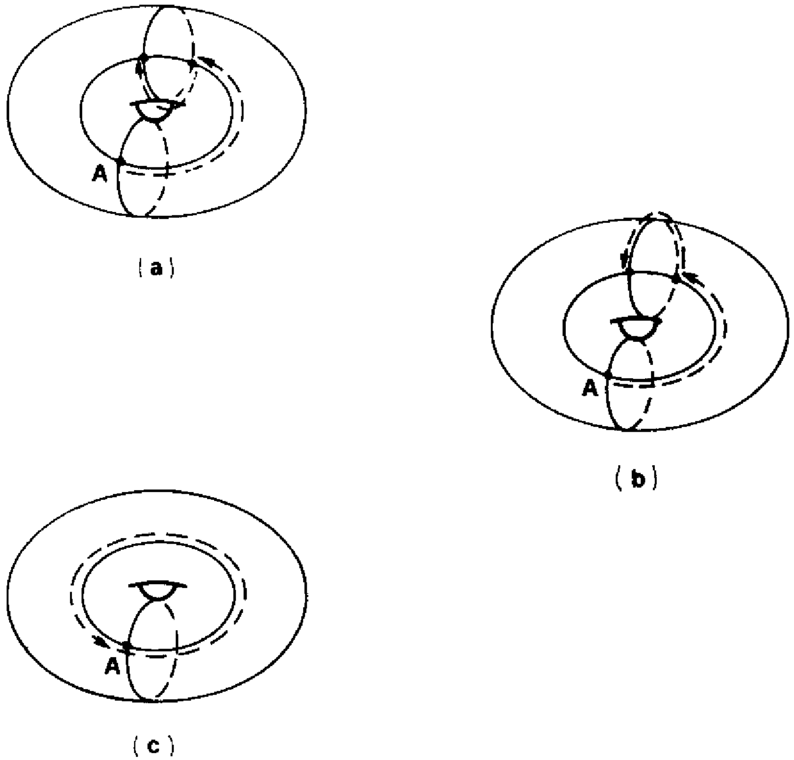

Fig.3

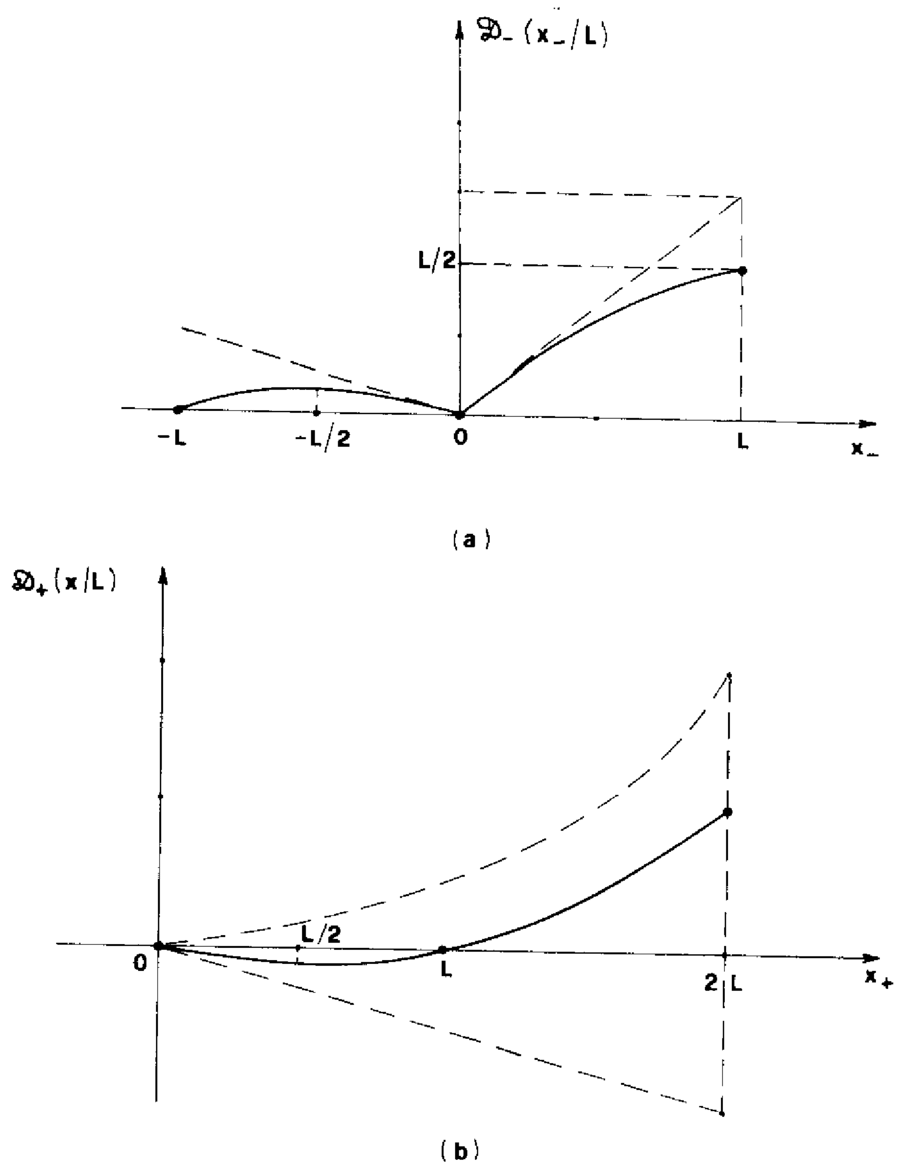

Fig.4

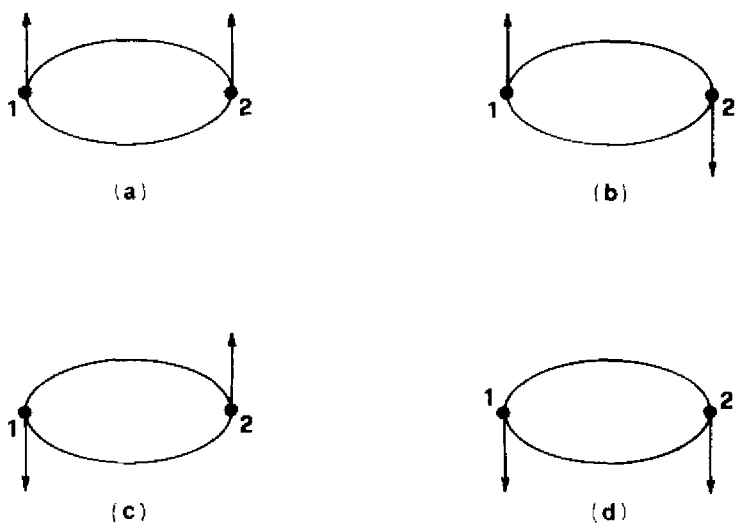

Fig.5