Relativistic theory of the Lamb shift in self-field quantum electrodynamics

A. O. Barut, J. Kraus,* Y. Salamin,† and N. Ünal‡
Department of Physics, University of Colorado, Boulder, Colorado 80309
(Received 26 December 1991)

The self-field QED is based on a nonlinear action in the ψ field alone obtained by eliminating the electromagnetic field A_μ. This approach, previously applied to spontaneous emission, vacuum polarization, and $g-2$ and to the low-energy part of the Lamb shift, is here extended to the fully relativistic calculation of the Lamb shift. It is shown that, to first order of iteration, it gives the same result as the quantized theory of Mohr [Ann. Phys. (N.Y.) **88**, 26 (1974); **88**, 52 (1974)]. The renormalization procedure is different and simpler.

PACS number(s): 12.20.Ds, 11.10.Gh

I. INTRODUCTION

The general formulation of "self-field quantum electrodynamics" in which the electromagnetic field A_μ has been eliminated between the coupled Maxwell-Dirac equations, and where one studies a nonlinear integro-differential equation for the electron's ψ field alone [1–3], has been applied to spontaneous emission [4], $(g-2)$ calculation [5], and to nonrelativistic Lamb shift [2]. The purpose of this work is to continue the analysis to include the relativistic or high-energy part of the Lamb shift. The general formula for energy shifts looks at first quite different than the corresponding QED result. But we show here, to first order of iteration of the nonlinear integro-differential equation, that the self-field QED gives exactly the same result as the QED calculation of Mohr [6]. The renormalization procedure is, however, simpler.

There is a natural way, in first-quantized Dirac theory, to discuss the negative-energy solutions [3]: They are equivalent to the positive-energy solutions of the mass conjugate equation. In the case of the Coulomb problem, they are equivalent to the positive-energy solutions with the sign of charge reversed. Consequently we can extend the sum into the negative-energy states in order to make use of the Coulomb Green's function.

II. REVIEW OF THE SELF-FIELD ELECTRODYNAMICS

It has recently been shown [3] that some of the most important effects in quantum electrodynamics can be attributed to the electron's self-energy. In this section we present a brief review of this theory in which the electromagnetic field is related to its source, the electron, and which is thus closer in spirit than conventional QED to the intuitively clear classical theory of electrodynamics. The idea is to set up the QED action for an electron in an arbitrary external electromagnetic field $A_\mu^{\rm ext}$, together with its own self-field $A_\mu^{\rm self}$. Employing Heaviside units, where $\hbar=c=1$, and taking $dx\equiv d^4x$ we write the action in the following form:

$$W=\int dx\,[\overline{\Psi}(\gamma^\mu i\partial_\mu-m)\Psi+J^\mu A_\mu-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}]\ . \quad (1)$$

In this expression, the first term is the kinetic action of the Dirac electron, where $\Psi(x)$ is the first-quantized electron matter field at the space-time point $x\equiv(x_0,\mathbf{x})$ and γ_μ stands for the familiar Dirac γ matrices. The second term describes the interaction of the electron with the electromagnetic field. In this term, the electron current is given by

$$J^\mu=-e\overline{\Psi}\gamma^\mu\Psi \quad (2)$$

and the total electromagnetic field is given by

$$A_\mu=A_\mu^e+A_\mu^s \quad (3)$$

with the superscripts e and s standing for *external* and *self*, respectively. Here A_μ^e is treated as a given nondynamical function. In other words, the external field is due to sources that are far away from the region of the dynamics under consideration. The last term in Eq. (1) is what we call the self-energy, where the electromagnetic field tensor $F_{\mu\nu}=A_{\nu,\mu}^s-A_{\mu,\nu}^s$ satisfies the Maxwell equations:

$$F^{\mu\nu}_{\ \ ,\nu}=-J^\mu\ . \quad (4)$$

We use Eq. (4) in order to put Eq. (1), after a single integration by parts has been performed on the last term, into the following form:

$$W=\int dx\,\{\overline{\Psi}[(i\partial_\mu-eA_\mu^e)-m]\Psi+\tfrac{1}{2}J^\mu A_\mu^s\}\ . \quad (5)$$

In the next step, we complete the elimination of A_μ^s from the action by inserting into (5) the solution of the wave equation in the covariant gauge $A^\mu_{\ ,\mu}=0$:

$$\Box A_\mu^s=J_\mu=-e\overline{\Psi}\gamma_\mu\Psi\ , \quad (6)$$

namely,

$$A_\mu^s(x)=-e\int dy\,D_{\mu\nu}(x-y)\overline{\Psi}(y)\gamma^\nu\Psi(y)\ . \quad (7)$$

In (7), $D_{\mu\nu}(x-y)$ is the *causal* Green's function in the covariant gauge $A^\mu_{\ ,\mu}=0$, which we take as

$$D_{\mu\nu}(x-y)=-g_{\mu\nu}\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2}\ . \quad (8)$$

Thus Eq. (5) now becomes

$$W=\int dx\,\overline{\Psi}(x)[\gamma^{\mu}(i\partial_{\mu}-eA^{e}_{\mu})-m]\Psi(x)$$
$$-\frac{e^2}{2}\int dx\,dy\,\overline{\Psi}(x)\gamma^{\mu}\Psi(x)$$
$$\times\frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2}\overline{\Psi}(y)\gamma_{\mu}\Psi(y)$$
$$=W_0+W_1\,. \tag{9}$$

Next, we introduce the following Fourier expansion for the electron matter field Ψ in the time variable:

$$\Psi(x)=\sum\!\!\!\!\!\!\int_{n}\psi_n(x)e^{-iE_nx_0}\,. \tag{10}$$

Hence the current has the Fourier expansion

$$j_{\mu}(x)=-e\sum_{n,m}j^{nm}_{\mu}(\mathbf{x})e^{i\omega_{nm}t}\,,\quad \omega_{nm}=E_n-E_m\,.$$

The Fourier coefficients in this expansion will be given an interpretation at a later stage. We now substitute for the Ψ's in Eq. (9) the expression (10) and carry out the time and timelike integrations over k_0, y_0, and x_0, in this order for convenience. After all this has been done the linear kinetic energy piece of the total action becomes ($\not{A}=\gamma^{\mu}A_{\mu}$)

$$W_0=2\pi\sum\!\!\!\!\!\!\int_{n}\int d^3x\,\overline{\psi}_n(\mathbf{x})(\gamma^0E_n-\gamma\cdot\mathbf{p}-e\not{A}^{e}-m)\psi_n(\mathbf{x})\,, \tag{11a}$$

while the nonlinear self-energy piece will read

$$W_1=-2\pi\frac{e^2}{2}\sum\!\!\!\!\!\!\int_{n,m,r,s}\delta(E_n-E_m+E_r-E_s)$$
$$\times\int d^3x\,\overline{\psi}_n(\mathbf{x})\gamma^{\mu}\psi_m(\mathbf{x})\int d^3y\,\overline{\psi}_r(\mathbf{y})\gamma_{\mu}\psi_s(\mathbf{y})$$
$$\times\int\frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{2}\left[\frac{i\pi}{k}[\delta(E_r-E_s+k)+\delta(E_r-E_s-k)]\right.$$
$$\left.+\mathrm{P}\frac{1}{2k}\left[\frac{1}{E_r-E_s-k}-\frac{1}{E_r-E_s+k}\right]\right]\,. \tag{11b}$$

Here P stands for the principal value integral and $\sum\!\!\!\!\!\!\int$ implies a sum over the discrete part and an integration over the continuum part of the system's spectrum. In carrying out the k_0 integration, the contour is closed in the upper half plane for $y_0>x_0$ where it encloses the simple pole at $k_0=-k\,(k\equiv|k|)$, and in the lower half plane for the case $y_0<x_0$ where it encloses the pole at $k_0=+k$; Θ functions are used in order to distinguish between the two cases. The y_0 integrations turn out to be simply Fourier transforms of the Θ functions which give rise to the principal value integrals and the δ functions in (11b).

Now, the δ function, $\delta(\omega_{nm}+\omega_{rs})$, can be satisfied by the two choices (1) $\omega_{nm}=0$, hence $\omega_{rs}=0$; and (2) $\omega_{nm}=-\omega_{rs}$. With this, W_1 becomes

$$W_1=-2\pi\frac{e^2}{2}\sum\!\!\!\!\!\!\int_{n,s}\int d^3x\,\overline{\psi}_n(\mathbf{x})\gamma^{\mu}\psi_n(\mathbf{x})\int d^3y\,\overline{\psi}_s(\mathbf{y})\gamma_{\mu}\psi_s(\mathbf{y})$$
$$\times\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left[\frac{i\pi}{2k}[\delta(k)+\delta(-k)]+\mathrm{P}\frac{1}{2k}\left[-\frac{1}{k}-\frac{1}{k}\right]\right]$$
$$-2\pi\frac{e^2}{2}\sum\!\!\!\!\!\!\int_{n,s}\int d^3x\,\overline{\psi}_n(\mathbf{x})\gamma^{\mu}\psi_s(\mathbf{x})\int d^3y\,\overline{\psi}_s(\mathbf{y})\gamma_{\mu}\psi_n(\mathbf{y})$$
$$\times\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left[\frac{i\pi}{2k}[\delta(E_s-E_n+k)+\delta(E_s-E_n-k)]\right.$$
$$\left.+\mathrm{P}\frac{1}{2k}\left[\frac{1}{E_s-E_n-k}-\frac{1}{E_s-E_n+k}\right]\right]\,. \tag{12}$$

In Eq. (12) the term proportional to $\delta(k)+\delta(-k)=2\delta(k)$ will not contribute as a result of the integration over k.

The common procedure of minimizing the total action, consisting of the two pieces given by Eqs. (11), with respect to ψ_n will result in a Dirac equation for our electron that is nonlinear and in which $\psi_n(x)$ plays the role of the electrons' wave function. This can be seen without even having to go through the actual derivation. In this approach we do not deal with this nonlinear equation directly, instead we shall study the piece of the total action giving to the nonlinearity and which, it turns out, contains the radiative corrections in the Coulomb problem. To begin with, the exact solutions to the nonlinear equations when put back into the total action will make it assume its minimum value; namely, zero. Therefore if we write the solution of such an equation as $\psi_n=\psi^c_n+\delta\psi_n$ corresponding to the energy eigenvalue $E_n=E^c_n+\delta E_n$, where ψ^c_n belongs to the complete set of solutions to the

Dirac equation in the appropriate external field A_μ^e, and E_n^c in its eigenenergy, then in a first iteration of the action we can drop the correction $\delta\psi_n$. When this is done, along with the demand that the total action vanishes in this first iteration, the linear piece, W_0, will contribute a factor $2\pi\sum_n \delta E_n$ and W_1 will be evaluated in terms of the set of functions $\{\psi_n^c(\mathbf{x})\}$. Finally we get

$$W_1^{(1)} = -2\pi \sum_n \delta E_n \ , \tag{13}$$

where the superscript on $W_1^{(1)}$ is added to indicate that we are considering a first iteration of the action. From (12) and (13) we immediately identify the shift in the nth energy level as a sum of three terms having the following physical interpretations. (From here on we shall drop the superscript c on ψ_n.)

(1) *Vacuum polarization:*

$$\Delta E_n^{\mathrm{VP}} = -\frac{e^2}{2} \sum\!\!\!\!\!\!\int_s \int d^3x\, \bar{\psi}_n(\mathbf{x})\gamma^\mu\psi_n(\mathbf{x}) \times \mathrm{P}\left[\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^2}\right] \times \int d^3y\, \bar{\psi}_s(\mathbf{y})\gamma_\mu\psi_s(\mathbf{y}) \ . \tag{14}$$

(2) *Spontaneous Emission and Absorption:*

$$\Delta E_n^{\mathrm{SE}} = -\frac{e^2}{2} \sum\!\!\!\!\!\!\int_s \int d^3x\, \bar{\psi}_n(\mathbf{x})\gamma^\mu\psi_s(\mathbf{x}) \int d^3y\, \bar{\psi}_s(\mathbf{y})\gamma_\mu\psi_n(\mathbf{y}) \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \left[\frac{i\pi}{2k}[\delta(E_s-E_n+k)+\delta(E_s-E_n-k)]\right] . \tag{15}$$

(3) *The Lamb shift:*

$$\Delta E_n^{\mathrm{LS}} = \frac{e^2}{2} \sum\!\!\!\!\!\!\int_s \int \bar{\psi}_n(\mathbf{x})\gamma^\mu\psi_s(\mathbf{x}) \int d^3y\, \bar{\psi}_s(\mathbf{y})\gamma_\mu\psi_n(\mathbf{y}) \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{2k} \mathrm{P}\left[\frac{1}{E_s-E_n-k} - \frac{1}{E_s-E_n+k}\right] . \tag{16}$$

The vacuum polarization term has been treated elsewhere [7] and the spontaneous emission term has also been exactly evaluated [4]. In the next section we study in detail the last item in the list above.

III. THE LAMB SHIFT

A. The dipole limit

Our starting point in the relativistic calculation of the Lamb shift based on self-energy is going to be Eq. (16) of the preceding section. Here we make a little digression in order to show that Bethe and Salpeter's formula [8] can in fact be obtained from (16) in the dipole approximation (DA). Before we do that though, one should remember that in (16) full relativistic Dirac-Coulomb wave functions are to be used. In any nonrelativistic limit that involves the use of nonrelativistic wave functions, an overall factor of 2 must be included [4] in order to compensate for the electron's spin which is relativistic in origin. Moreover, if one is willing to accept the interpretation of the first term as resulting from virtual transitions to the negative-energy levels, then one should drop it in any nonrelativistic approximation. In the DA, the exponential factor in (16) is approximated by unity. This translates physically into taking the radiation wavelength to be much greater than the atomic dimensions, or $k|\mathbf{x}-\mathbf{y}| \ll 1$. In the next step, we make the expansion:

$$\int d^3x\, \bar{\psi}_n\gamma^\mu\psi_s \int d^3x'\bar{\psi}_s\gamma_\mu\psi_n = \left[\int \psi_n^\dagger\psi_s d^3x\right]\left[\int \psi_s^\dagger\psi_n d^3x'\right] - \left[\int \psi_n^\dagger\boldsymbol{\alpha}\psi_s d^3x\right]\cdot\left[\int \psi_s^\dagger\boldsymbol{\alpha}\psi_n d^3x'\right] .$$

The first term on the right vanishes as a result of the orthogonality of the wave functions, while the second term is immediately recognized as the scalar product of the expectation values of the velocity operator ($\mathbf{v}=c\boldsymbol{\alpha}$, $c=1$) between the states n and s. Therefore

$$\int \bar{\psi}_n\gamma^\mu\psi_s d^3x \int \bar{\psi}_s\gamma_\mu\psi_n d^3x' = -\mathbf{v}_{ns}\cdot\mathbf{v}_{sn} \ .$$

On the other hand, the Heisenberg equations of motion give $\mathbf{v}_{ns}=i\mathbf{p}_{ns}/m$ and $\mathbf{v}_{sn}=i\mathbf{p}_{sn}/m$. Putting all of this back into Eq. (16), we get

$$\Delta E_n^{\mathrm{LS}}(\mathrm{DA}) \approx -\frac{\alpha}{8\pi^2 m^2} \sum\!\!\!\!\!\!\int_s \int_0^\infty k\,dk\, d\Omega_k \frac{\mathbf{p}_{ns}\cdot\mathbf{p}_{sn}}{\omega+k} \ , \tag{17}$$

where $\omega \equiv E_s - E_n$. With the polarization taken care of as usual, the angular integration in (17) gives $8\pi/3$. If we also multiply by the factor of 2 mentioned above, which comes out automatically in the relativistic calculation, we arrive at

$$\Delta E_n^{\mathrm{LS}}(\mathrm{DA}) \approx -\frac{2}{3\pi}\alpha\frac{1}{m^2}\int_0^\infty k\,dk \sum\!\!\!\!\!\!\int_s \frac{\mathbf{p}_{ns}\cdot\mathbf{p}_{sn}}{E_s-E_n+k} \ . \tag{18}$$

This is precisely Bethe's result, provided that some finite cutoff λ is introduced as usual to replace the infinite upper limit on the integration over k. We now move on to present our derivation for the relativistic Lamb-shift formula.

B. The relativistic calculation

Thus we start with our general expression for the Lamb shift of the nth level (with $e^2/4\pi=\alpha$):

$$\Delta E_n = 2\pi\alpha \int d\mathbf{x}\, \bar{\psi}_n(\mathbf{x})\gamma^\mu \sum\!\!\!\!\!\!\int_s d\mathbf{y}\, \psi_s(\mathbf{x})\bar{\psi}_s(\mathbf{y})\gamma^\mu\psi_n(\mathbf{y}) \int \frac{d\mathbf{k}}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{2k} \mathrm{P}\left[\frac{2k}{(E_s-E_n)^2-\mathbf{k}^2+i\varepsilon}\right] . \tag{19}$$

We introduce in this expression a contour integral using

$$\sum\!\!\!\!\!\!\int_s \psi_s(\mathbf{x})\bar{\psi}_s(\mathbf{y}) = -\int_{C_F} \frac{dz}{2\pi i} \sum\!\!\!\!\!\!\int_s \frac{\psi_s(\mathbf{x})\bar{\psi}_s(\mathbf{y})}{z-E_s} = -\int_{C_F}\frac{dz}{2\pi i} G(\mathbf{x},\mathbf{y};z) . \tag{20}$$

The contour integration replaces the z dependence of the wave function $\psi_z(\mathbf{x})$ by E_s.

We can extend the sum $\sum\!\!\!\!\!\int_s$ over intermediate states in our Fourier expansion (10) also to the negative-energy solutions in order to be able to introduce the energy-dependent Green's function $G(\mathbf{x},\mathbf{y};z)$ of the relativistic Coulomb problem, because the negative-energy solutions are equivalent to positive-energy solutions with the sign of e changed.

It is instructive to note at this point that (19) with (20) can be written in a four-dimensional form

$$\Delta E_n = i\alpha 4\pi \int d(t_2-t_1) \int \frac{dz}{2\pi i} \int d\mathbf{x}\, d\mathbf{y}\, \bar{\psi}_n(\mathbf{x})\gamma_\mu i G(\mathbf{x},\mathbf{y};z)\gamma^0\gamma^\mu\psi_n(\mathbf{x}) e^{+iz(t_2-t_1)} e^{-iE_n(t_2-t_1)}$$
$$\times \int_{C_F} \frac{dk^0}{2\pi}\frac{d\mathbf{k}}{(2\pi)3} \frac{e^{-ik_0(t_2-t_1)+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k_0^2-\mathbf{k}^2+i\varepsilon}$$

which in turn can be written in terms of the causal propagators S_F and D_F as

$$\Delta E_n = -i\alpha\pi \int d(t_2-t_1) \int d\mathbf{x}_2 d\mathbf{x}_1 \bar{\psi}_n(x_2)\gamma_\mu S_F(x_2,x_1) \times \gamma^\mu \psi_n(x_1) D_F(x_2-x_1) . \tag{21}$$

This expression has the standard form of the Lamb shift.

We now return to Eq. (19) with (20), and perform first the angular integrations in k space,

$$\Delta E_n = -4\alpha \int \frac{dz}{2\pi i} d\mathbf{x}\, d\mathbf{y}\, \bar{\psi}_n(\mathbf{x})\gamma^\mu G(\mathbf{x},\mathbf{y};z)\gamma^0\gamma_\mu\psi_n(\mathbf{y}) \times \sum_{l,m} Y_{lm}(\hat{\mathbf{x}})Y^*_{lm}(\hat{\mathbf{y}}) \times \int \frac{k dk}{2} \mathrm{P}\left[\frac{2k}{(z-E_n)^2-k^2+i\varepsilon}\right] \times j_l(kx) j_l(ky) ,$$

then the k integration

$$\Delta E_n = -4\alpha \int_{C_Z} \frac{dz}{2\pi i} d\mathbf{x}\, d\mathbf{y}\, \bar{\psi}_n(\mathbf{x})\gamma^\mu G(\mathbf{x},\mathbf{y};z)\gamma^0\gamma_\mu\psi_n(\mathbf{y}) \times \sum_{l,m} Y_{lm}(\hat{\mathbf{x}})Y^*_{lm}(\hat{\mathbf{y}}) \times \left[-\frac{i\pi}{2}\omega j_l(\omega r_<) h_l^{(2)}(\omega r_>)\right] , \tag{22}$$

where $r_<$ is the smaller one of $(|\mathbf{x}|,|\mathbf{y}|)$ and $r_>$ is the larger one of $(|\mathbf{x}|,|\mathbf{y}|)$, and

$$\omega^2 = (z-E_n)^2 + i\varepsilon \text{ or } \omega = [(z-E_n)^2]^{1/2} + i\varepsilon' .$$

The contour of integration of (3.8) in the z plane is shown in Fig. 1. The Green's function $G(\mathbf{x},\mathbf{y};z)$ has the poles corresponding to the bound states, plus the cuts beginning at $\pm m$ corresponding to positive and negative continuous spectra.

In addition the factor $f(\omega) = h_l^{(2)}(\omega r_>) j_l(\omega r_<)$ has cuts coming from ω at $z = E_n + i\varepsilon'$ and $z = E_n - i\varepsilon'$, as shown.

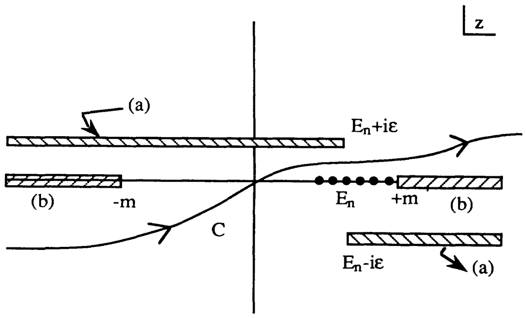

FIG. 1. Contour C in the z plane. (a) Cuts coming from $f(\omega)=h_l^{(2)}(\omega r_>) j_l(\omega r_<)$, Eq. (22). (b) Cuts from $G(\mathbf{x},\mathbf{y};z)$. Dots are poles of $G(\mathbf{x},\mathbf{y};z)$.

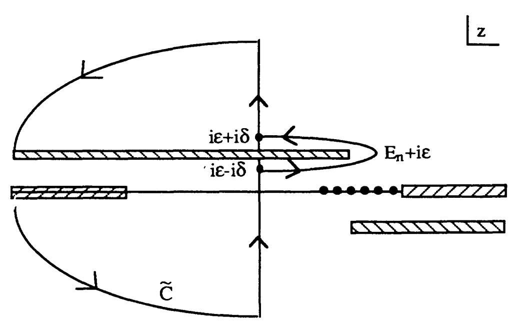

FIG. 2. Deformed contour $\tilde{C}$.

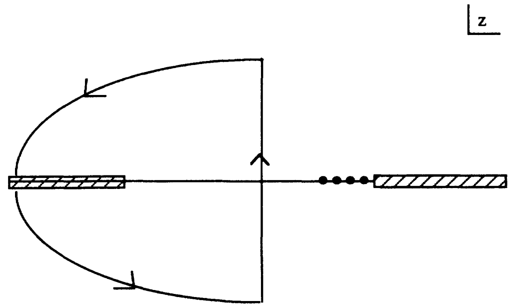

FIG. 3. High-energy part of the deformed contour $\tilde{C}$ in the limit $\delta \rightarrow 0$.

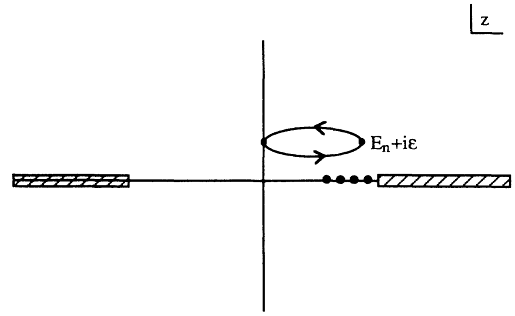

FIG. 4. Low-energy part of the deformed contour $\tilde{C}$ in the limit $\delta \rightarrow 0$.

Next we deform the contour C_F into $\tilde{C}_F$ as shown in Fig. 2. There are two small parameters ε and δ and two types of integrals:

$$\text{(a)} \quad \lim_{\delta \to 0} \left[\int_{-i\infty}^{i(\varepsilon - i\delta)} + \int_{i(\varepsilon+\delta)}^{i\infty} \right] = \int_{-i\infty}^{+i\infty} . \tag{23}$$

This is the integral along the imaginary axis (Fig. 3).

$$\text{(b)} \quad \lim_{\varepsilon \to 0} \left[\lim_{\delta \to 0} \left[\int_{i\varepsilon - i\delta}^{E_n + i\varepsilon - i\delta} A(z) dz + \int_{E_n + i\varepsilon + i\delta}^{i\varepsilon + i\delta} A(z) dz \right] \right] = \int_0^{E_n} [A(z_-) - A(z_+)] dz$$
$$= \int_0^{\infty} \text{discontinuity } A(z) dz . \tag{24}$$

This is the integral and around the cut shown in Fig. 4 from 0 to the particular energy level E_n. These two contributions correspond to the so-called-high-energy and low-energy parts of the Lamb shift, respectively.

(a) The first integral (23) is

$$\Delta E_n(\text{high energy}) = 4\alpha \int_{-i\infty}^{+\infty} \frac{da}{2\pi i} \int d\mathbf{x}\, d\mathbf{y}\, \bar{\psi}_n(\mathbf{x}) \gamma_\mu G(\mathbf{x},\mathbf{y};z) \gamma^0 \gamma^\mu \psi_n(\mathbf{y}) \sum_{l,m} \left[\frac{i\pi\omega}{2} \right] j_l(\omega r_<) h_l^{(2)}(\omega r_>) \Bigg|_{\varepsilon=0} Y_{lm}(\hat{\mathbf{x}}) Y_{lm}(\hat{\mathbf{y}})$$
$$= \alpha \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \int d\mathbf{x}\, d\mathbf{y}\, \gamma_\mu G(\mathbf{x},\mathbf{y};z) \gamma^0 \gamma^\mu \psi_n(\mathbf{y}) \frac{e^{-i\omega|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} d\Omega_K , \tag{25}$$

where we have written the sum over l and m as an angular integral over $\hat{\mathbf{k}}$.

(b) The second integral (24) gives

$$\Delta E_n(\text{low-energy part}) = -4\alpha \int_0^{E_n} \frac{dz}{2\pi i} d\mathbf{x}\, d\mathbf{y}\, \bar{\psi}_n(\mathbf{x}) \gamma^\mu G(\mathbf{x},\mathbf{y};z) \gamma_0 \gamma_\mu \psi_n(\mathbf{y})$$
$$\times \sum_{l,m} \psi_{lm}(\hat{\mathbf{x}}) \psi_{lm}^*(\hat{\mathbf{y}}) \left[-\frac{i\pi}{2} \omega_- j_l(\omega_- r_<) h_l^{(2)}(\omega_- r_>) + \frac{i\pi}{2} \omega_+ j_l(\omega_+ r_<) h_l^{(2)}(\omega_+ r_>) \right] ,$$

where

$$\omega_- \cong -(z - E_n) = -\omega ,$$
$$\omega_+ \cong -(z - E_n) = \omega .$$

Therefore

$$\Delta E_n(\text{low-energy part}) = -4\alpha \int_0^E \frac{dz}{2\pi i} i\pi \int d\mathbf{x}\, d\mathbf{y}\, \bar{\psi}_n(\mathbf{x}) \gamma_\mu G(\mathbf{x},\mathbf{y};z) \gamma_0 \gamma^\mu \psi_n(\mathbf{y}) \sum_{l,m} Y_{lm}(\hat{\mathbf{r}}) j_l(\omega r_<) j_l(\omega r_>) Y_{lm}^*(\hat{\mathbf{y}})$$

or, again writing $\sum_{lm}$ as an angular integration

$$\Delta E_n(\text{low } E) = \frac{\alpha}{\pi} \int_0^{E_n} dz \int d\mathbf{x}\, d\mathbf{y}\, \bar{\psi}_n(\mathbf{x}) \gamma_\mu G(\mathbf{x},\mathbf{y};z) \times \gamma_0 \gamma^\mu \psi_n(\mathbf{y}) \times \frac{\sin[(E_n - z)|\mathbf{x}-\mathbf{y}|]}{|\mathbf{x}-\mathbf{y}|} . \tag{26}$$

Both contributions (25) and (26) agree exactly with the expansions of Mohr [6].

We perform, as in classical electrodynamics, the renormalization as follows. The theory contains two parameters α and $(Z\alpha)$, and there are two limits which can be taken independently: (a) $\alpha \rightarrow 0$, $(Z\alpha)$ fixed, (b) $(Z\alpha) \rightarrow 0$, α fixed. For the first limit, we obtain the Dirac-Coulomb problem without self-energy directly from action. In the second case, we obtain a free particle with self-energy. But since the free particle's self-energy is already in the mass, any remainder ΔE, as $(Z\alpha)\to 0$, α fixed, must be subtracted.

ACKNOWLEDGMENTS

One of us (N.U.) was partially supported by a NATO collaborative Research grant and by TUBITAK.

*Present address: Hochgernstrasse 24, D-8221 Bergen, Germany.

†Present address: Department of Physics, Birzeit University, Birzeit, Israel.

‡Present address: Department of Physics, Dicle University, Diyarbaker, Turkey.

[1] A. O. Barut and J. Kraus, Phys. Rev. D **16**, 161 (1988); Found. Phys. **13**, 189 (1983).

[2] A. O. Barut and J. F. van Huele, Phys. Rev. A **32**, 3187 (1985).

[3] See the review in *The Electron. New Theory and Experiment,* edited by D. Hestenes and A. Weingartshofer (Kluwer, Dordrecht, 1991), pp. 105–148.

[4] A. O. Barut and Y. Salamin, Phys. Rev. A **37**, 2284 (1988).

[5] A. O. Barut and J. Dowling, Z. Naturforsch. Teil A **44**, 1051 (1989).

[6] P. Mohr, Ann. Phys. (N.Y.) **88**, 26 (1974); **88**, 52 (1974).

[7] A. O. Barut and N. Ünal, Phys. Rev. D **41**, 3822 (1990).

[8] H. A. Bethe and E. E. Salpeter, in *Atoms I,* edited by S. Flügge, Encyclopedia of Physics Vol. 35 (Springer, Berlin, 1957), p. 88.