

# BRIEF HISTORY AND RECENT DEVELOPMENTS IN ELECTRON THEORY AND QUANTUMELECTRODYNAMICS

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**ABSTRACT.** Major steps in the hundred years history of the electron concerning its selfenergy due to its own electromagnetic field are outlined and the present status and revival of the selfenergy and radiative problems of the electron are discussed.

## 1. HISTORY OF THE ELECTRON

### 1.1. Concept and Discovery

The first and the foremost of all elementary particles, the electron, has not been discovered suddenly, as the muon, for example. It was “in the air” for a long time. 1990 is a good date to remember the hundredth anniversary of its conception. The evidence came from at least three widely different phenomena.

Its study in the cathode rays can be said to begin with the work of the mathematician J.Plücker in 1858. Plücker is also a pioneer in differential geometry in introducing the rays as coordinates rather than the points (Plücker coordinates) which anticipates the space of all light rays or light cones. Wilhelm Hittorf (1869) talks about the straightline trajectories of the negatively charged “glowrays”. Other important names in the early studies of cathode rays are W. Crookes (1879), H.Hertz (1881), P.Lenard, J.Perrin, W.Wien and J.J.Thomson.

Secondly, from Faraday’s equivalence law for electrolysis, Helmholtz in 1881 concluded that each ion must carry a multiple of an elementary charge. Independently, the Irish physicist G.Johnstone Stoney in 1881 also talks of fundamental units of both positive and negative electric charges. Already in 1874 Stoney had the idea of an “atom” of electricity. And it was Stoney who introduced the name “electron” a bit later in 1894.

The third phenomenon in which the electron appears independently was its identification in radioactivity by Jean Becquerel (1 March 1896).

The first precise determination of the ratio  $e/m$  were made by Peter Zeeman (31 October 1896), E.Wiechert (7 Jan. 1897) and J.J.Thomson (30 April 1897). Zeemann also gave an explanation of what is now called the Zeeman effect on the basis of

the electron-hypothesis. These early experimental developments we may call the first period of electron's history.

The second phase, up to the discovery of the wave properties of the electron, is the history of the electron as a relativistic particle, the electron according to Lorentz, and the models by Abraham and Lorentz. The relativistic equations of motion derived by Poincaré and Einstein in 1904 and 1905 have been verified e.g the increase of the effective mass with velocity.

The third period of electron's history opens with the unexpected wave and spin properties of the electron and leads to the picture according to Schrödinger and Dirac, the electron described by a wave equation, and ends with the QED picture of the electron. It is remarkable that such a seemingly simple object as an electron has produced so many surprises. In view of the appearance of heavy leptons, like muon and tau which are very much like the electron, I am tempted to conjecture a fourth period in which we may be again surprised by a nonperturbative internal structure of the electron. The perturbative treatment of electron interactions by Feynman graphs has somewhat diminished the preeminence of the electron; it is just one of the many "elementary" particles. But these results have not solved the structure problem. In the last sentence of his famous review article on Quantum Theory of Radiation<sup>1</sup> Fermi writes: "In conclusion, we may therefore say that practically all the problems in radiation theory which do not involve the structure of the electron have their satisfactory explanation; while the problems connected with the internal properties of the electron are still very far from their solution".

Because the structure of the electron is the main topic of these lectures I would like also to review briefly the history of the selfenergy of the electron.

## 1.2. History of Selfenergy

It is remarkable that the force law between two charged particles in motion including the magnetic force was written down as early as 1875 by R. Clausius, and this in "relativistic" form namely, what would correspond to an interaction Lagrangian of the form

$$L = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \frac{e_1e_2}{4\pi\epsilon_0} \left( 1 - \frac{1}{c^2} \mathbf{v}_1 \cdot \mathbf{v}_2 \right) \frac{1}{|r_1 - r_2|}$$

This is not surprising because the laws of relativity were essentially deduced later from electrodynamics. J.J.Thomson in 1881 confirmed this law of force (up to a factor) from Maxwell equations. In 1892 H.A.Lorentz began to combine the action of the particles with that of the electromagnetic field culminating in his *Theory of the Electrons* as described in his famous 1904 Encyclopedie article and in his book. This is essentially the present "classical" electron theory, the first comprehensive selfconsistent treatment of charged particles and their electromagnetic field in mutual interaction. We shall see that the basic tenets of this theory remains unchanged today, only the way we describe matter has undergone several changes over the years. Because the selfenergy of the electron for a point particle is infinite at the position of the particles, Lorentz and Abraham tried to model the electron as an extended charge distribution in order to understand whether the mass of the electron is wholly of electromagnetic origin, that is an electromagnetic origin of the concept of mass and an electromagnetic view of all

matter. The advent of the relativity principle, independently formulated by Poincaré and Einstein, put a temporary end to these endeavours by elevating the concept of the rest mass to the level of an invariant, like the velocity of light. Also the covariant laws of mechanics were promoted to the same footing as the Maxwell's equations; in fact every classical theory could be relativized. Even if we accept an unexplained invariant rest mass the problems of the infinite selfenergy and the structure of the electron remained. Dirac in 1938 returned to the classical electron theory and extracted the covariant form of the selfenergy contribution to the motion of the electron. This is the radiation reaction force in addition to the external force and the resultant equation is now known as the Lorentz-Dirac equation. But an infinite renormalization was necessary by putting part of the selfenergy into the rest mass. Dirac returned to the classical electron theory in 1962 by modelling it as an extensible charged shell held together by a surface tension. The parameter of the surface tension can be eliminated in terms of the mass and charge so that this model has also only two parameters, rest mass and charge, just like a point particle. Unfortunately, to my knowledge, the relativistic motion of such a shell in space-time, its radiation reaction and renormalization have not been studied at all.

The reason might be that meanwhile we have developed first a wave mechanics for the electron, then a relativistic Dirac wave equation incorporating the new spin property of the electron and finally a quantumelectrodynamics so that classical models seemed to be obsolete. However, as exemplified by the above quote from Fermi the selfenergy problem remained. At first it might be thought that the description of the electron by an extended wave function instead of a point charge might alleviate the selfenergy infinity. Schrödinger himself in 1926 tried to include additional radiation reaction or selfenergy terms to his famous equation, but obtained, due to an incomplete treatment, wrong results. In the meantime the statistical interpretation of the wave function became the dominant paradigm; the self energy terms were dropped, an independent quantized radiation field was introduced and the self energy infinities were absorbed by renormalization into mass and charge. So the expected finite, closed and selfconsistent electrodynamics did not materialize.

### 1.3. The Importance of the Structure and Selfenergy of the Electron

It is with this background that I wish to reexamine the problem of the selfenergy of the electron. As Lorentz had already prophesied, "in speculating on the structure of these minute particles we must not forget that there may be many possibilities not dreamt of at present". It is not only a problem of having a mathematically sound finite theory of the electron, but a question of curiosity which may have farreaching consequences : what is really an electron? What is the structure which gives its spin? Why must there be a positron? What is mass? Why and how does the electron manifests wave properties? And what is the interaction between two electrons or between an electron and a positron at short distances? The Dirac equation gives a mathematical description of these questions, except the last one, but not a clear intuitive picture.

These questions, in fact the history of the electron, show that although there has been tremendous progress in details and applications of the electron theory, the progress in the foundations of physics is very slow. We are normally busy with our calculations

and with building models. But at occasions like this, which I welcome, we may reflect on these larger issues, fundamental ideas and unifying concepts. I think some of the most fundamental and soluble problems of basic physics at the present time are: (1) a clear understanding of the wave and particle properties of electron, photon and other particles within simple logic, without paradoxes, as objective material properties; (2) the completion of quantum electrodynamics; a self-consistent intuitive rendering of the interactions of matter and electromagnetic field without infinities; (3) different forms of matter: how many really fundamental particles do we have, the electron-muon puzzle, for example; and (4) how many distinct fundamental interactions do we have to introduce, and how do we unify the four seemingly different forces? Into a large mathematical framework containing all these, or showing that they are just different manifestations of a single interaction? For example, the chemical force and alpha decay, although extremely weak, have been shown to be different manifestations of the electromagnetic force. It may seem incredulous, but I think we may have a much better understanding of these four basic problems on the basis of electron's structure and specially on the basis of the strong selfenergy effects between electron and positron at short distances. This would be a fulfillment of Einstein's statement "You know, it would be sufficient to really understand the electron". The framework for this program, I think, exists, but we must justify it with precise mathematical calculations.

## 2. SOME RECENT RESULTS FROM ELECTRON THEORY

### 2.1 Selfenergy of the Point Electron: The Lorentz-Dirac Equation

Because the basic postulate of both the classical electrodynamics and the selffield quantum electrodynamics, the action principle, is essentially the same, we begin with the longstanding apparent problem of causality violation and runaway solutions of the Lorentz-Dirac equation. The classical action for spinless charged point particles in terms of invariant time parameters is given by

$$A = \int d\tau p^\mu \dot{x}_\mu - \int \left[ e A_\mu j^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] dx$$

where the current is

$$j_\mu(x) = \int d\tau \delta(x - x(t)) \dot{x}(\tau)$$

The action leads to Maxwell's equations and to the particle equations

$$\begin{aligned} F'^{\nu}_{\mu\nu} &= -j_\mu \\ m \ddot{x}_\mu &= e F_{\mu\nu} x^\nu \end{aligned}$$

Each particle sees the external field of all other particles plus its own selffield which is a necessary consequence of the self-consistent action principle. The selffield is obtained from the Lienard-Wiechert potential

$$A_\mu(x) = e \int d\tau \dot{x}_\mu(t) \delta(x - x(t)) D(x - x(t))$$

A regularization or renormalization is necessary, because the selffield at the position of the point particle is not defined, which can be done in different ways. The result is the covariant equation

$$m\ddot{x}_\mu = eF_{\mu\nu}\dot{x}^\nu + \frac{2}{3}e^2 (\ddot{x}_\mu + (\ddot{x})^2\dot{x}_\mu)$$

which contains all the radiative processes for point particles in a closed nonperturbative manner which, I think, is one of the most important feature of this equation. However, the complete consistency of this equation has been questioned. This comes about if one tries to solve this equation in regions where the external field vanishes. The nonlinear term by itself then leads to preacceleration and to runaway solutions, the exponential increase of  $x$  with time. I have recently pointed out that renormalization of the theory has two aspects : one is putting an infinite inertial term  $\lim (1/u)\dot{x}^\mu$ , into the mass term, the other to make sure that when the external field vanishes the charge must move like a free particle with an experimental mass. In renormalizing a theory we must know beforehand to what we are renormalizing. This means that in regions where the external field is zero the correct solution must coincide with that for a free particle. The so-called causality violation and runaway solutions all have been shown in the examples where the external field is switched on and off at finite times. On the basis of explicit solutions it has been shown that with the above physical requirement no preacceleration or runaway solutions arise<sup>2</sup>. Furthermore, with the radiated power and the finite change of mass correctly included the Lorentz Dirac equation conserves energy<sup>3</sup> removing some other doubts expressed in the literature. This does not mean that we should be completely happy with the Lorentz Dirac equation. First of all it does not contain spin which however will be added in the next Section. But the infinity in the mass renormalization is still with us. But then the electron has wave and spin properties which are not yet in the point particle model.

## 2.2. Classical Relativistic Spinning Electron

The spin properties of the electron are very well described by the quantum Dirac equation. For many practical applications of the Dirac equation there is no need to make a model of spin. But eventually it becomes important to understand the physical mechanism underlying the spin or magnetic moment degree of freedom of the electron.

Dirac<sup>4</sup> has found (by chance as he says) his equation without quantizing of an existing classical model. Ever since there was no lack of effort to find an intuitive model of this remarkable relativistic spinning particle. A model of spin may help to discuss possible excited states of the electron, the existence of antiparticles, heavy leptons, and perhaps shed some light on Pauli exclusion principle, and short distance extrapolation of electrodynamics.

If a spinning particle is not quite a point particle, nor a solid three dimensional top, what can it be? What is the structure which can appear under probing with electromagnetic fields as a point charge, yet as far as spin and wave properties are concerned exhibits a size of the order of the Compton wave length? I want to describe a model in which a point charge performs as its natural motion a helix which gives an effective structure and size scale to the particle and accounts for the spin as the

intrinsic angular momentum of the helix, and the frequency of the helical motion determines an internal clock and attributes a mass to the particle. Furthermore the sense of the orientation of the helix is related to the particle-antiparticle duality. And all this already in a purely classical framework.

### The Action

The phase space consists of the usual conjugate pair of variables  $(x_\mu(\tau), p_\mu(\tau))$  plus another conjugate pair of internal variables  $(\bar{z}(\tau), z(\tau))$ . Here  $\tau$  is an invariant time parameter,  $\bar{z}$  and  $z$  are 4-component classical c-number spinors, thus in  $C_4$ . We could have used the real and imaginary parts of  $z$ , but for a symplectic formulation the spinor form is much more economical and elegant. The notation is such that

$$\bar{z} = z^+ \gamma \cdot n \quad , \quad \gamma \cdot n \equiv \gamma^\mu n_\mu$$

where  $n^\mu$  is the normal to a space-like surface  $\Sigma$ . The action is the integral of Cartan's symplectic 1-form

$$\omega = pdq - \mathcal{H}d\tau$$

where  $\mathcal{H}$  is the "Hamiltonian" with respect to  $\tau$  or the mass operator. In our case we have explicitly

$$\omega = pdx + i\lambda\bar{z}\gamma \cdot nz - \mathcal{H}d\tau$$

with

$$\mathcal{H} = \bar{z}\gamma^\mu z (p_\mu - eA_\mu)$$

so that the action including that of the coupled electromagnetic field  $A$  is<sup>5</sup>

$$\int w = \int (i\lambda\bar{z}\gamma \cdot nz + p\dot{x} - \pi_\mu \bar{z}\gamma^\mu z) d\tau - \frac{1}{4} \int dx F_{\mu\nu} F^{\mu\nu}$$

where we have introduced the kinetic momenta

$$\pi_\mu = p_\mu - eA_\mu$$

### Properties of the Particle and Solutions

1. There are only two fundamental constants in the theory: the coupling constant  $e$ , the charge, and the constant  $\lambda$  of dimension of action ( $\hbar$ ) multiplying  $z$  for dimensional reasons. The mass  $m$  will enter as the value of an integral of motion in the solutions.
2. The system is integrable. Two integrals of motion are

$$\begin{aligned} \mathcal{H} &= \pi_\mu \bar{z}\gamma^\mu z \quad , \quad \dot{\mathcal{H}} = 0 \\ \mathcal{N} &= \bar{z}\gamma \cdot nz \quad , \quad \dot{\mathcal{N}} = 0 \end{aligned}$$

We can choose  $\mathcal{N} = 1$  (normalization), and  $\mathcal{H} = m$ .

3. The equations of motion are

$$\begin{aligned} \dot{\bar{z}}\gamma \cdot n &= \frac{i}{\lambda} \bar{z}\pi \quad , \quad \pi = \gamma^\mu (p_\mu - eA_\mu) = \gamma^\mu \pi_\mu \\ \gamma \cdot n \dot{z} &= -\frac{i}{\lambda} \pi z \\ \dot{\pi}_\mu &= eF_{\mu\nu} \dot{x}^\nu \\ \dot{x}_\mu &= \bar{z}\gamma_\mu z \end{aligned}$$

4. The velocities  $\dot{x}_\mu$  are thus constrained by the internal variables (similar to the rolling condition of a rigid body on a surface) by  $\dot{x}_\mu = \bar{z}\gamma_\mu z$ . In particular

$$\dot{x}_0 = \bar{z}\gamma_0 z = \frac{dx_0}{d\tau}$$

which for  $n = (1000)$  is equal to  $\mathcal{N} = 1$ . Hence  $\tau$  has the meaning of proper time in the frame determined by  $n$ . It is also possible to rewrite the action so that  $\tau$  has the meaning of the proper time of the center of mass.

5. Solutions for a free particle. We give only the solution for  $\dot{x}_\mu$ .<sup>6</sup>

$$\dot{x}_\mu(\tau) = \dot{x}_\mu(0) + \ddot{x}_\mu(0) \frac{\sin 2p\tau}{2p} + 2 \frac{\sin^2 p\tau}{p^2} V_\mu$$

where

$$V_\mu = \delta_{\mu 0} (mp_0 - m^2) - p_\mu(m - p_0)$$

Thus

$$\begin{aligned} \dot{x}_0(\tau) &= \dot{x}_0(0) + \ddot{x}_0(0) \frac{\sin^2 p\tau}{2p} + 2 \sin^2 p\tau \\ \vec{\dot{x}}(\tau) &= \vec{\dot{x}}(0) + \vec{\ddot{x}}(0) \frac{\sin^2 p\tau}{2p} + \frac{\vec{p}}{p_0 + m} \sin^2 p\tau \end{aligned}$$

Compare this with the solution when the invariant parameter is taken to be the proper time of the center of mass:

$$\begin{aligned} \dot{x}_0 &= \frac{p_0}{m} + \left( \dot{x}_0(0) - \frac{p_0}{m} \right) \cos 2m\tau + \frac{\ddot{x}_0(0)}{2m} \sin 2m\tau \\ \vec{\dot{x}} &= \frac{\vec{p}}{m} + \left( \vec{\dot{x}}(0) - \frac{\vec{p}}{m} \right) \cos 2m\tau + \frac{\vec{\ddot{x}}(0)}{2m} \sin 2m\tau. \end{aligned}$$

6. We see in either of the above forms the helical motion of the velocities, hence after integration, of the coordinates  $x_\mu$ . This is the natural motion of the particle and the helix does not radiate. The frequency of the helical; motion is  $2m$  in the proper time of the center of mass.

7. Instead of  $\bar{z}$  and  $z$  we can introduce the more physical velocity and spin variables. Defining  $S_{\mu\nu} = \frac{1}{4}\bar{z}[\gamma_\mu\gamma_\nu]z$  we have the dynamical system

$$\begin{aligned}\dot{x}_\mu &= v_\mu \\ \dot{v}_\mu &= 4S_{\mu\varrho}\pi^\varrho \\ \dot{\pi}_\mu &= eF_{\mu\varrho}v^\varrho \\ \dot{S}_{\mu\nu} &= \pi_\mu u_\nu - \pi_\nu u_\mu\end{aligned}$$

These equations are identical in form to the Heisenberg equations of the Dirac equation.

8. The system is symplectic given by the Poisson brackets

$$\begin{aligned}\{f, g\} &= \left( \frac{\partial f}{\partial x^\alpha} \frac{\partial g}{\partial p_\alpha} - \frac{\partial g}{\partial x^\alpha} \frac{\partial f}{\partial p_\alpha} \right) - i \left( \frac{\partial f}{\partial z} \frac{\partial g}{\partial \bar{z}} - \frac{\partial g}{\partial z} \frac{\partial f}{\partial \bar{z}} \right) \\ \{z, i\bar{z}\} &= 1 \quad , \quad \{x^\mu, p_\nu\} = \delta^\mu_\nu\end{aligned}$$

Consequently the equations of motion have the Poisson bracket as well as the Hamiltonian forms

$$\begin{aligned}\dot{x}_\mu &= \{x_\mu, \mathcal{H}\} = \partial\mathcal{H}/\partial p^\mu \\ i\dot{\bar{z}} &= \{i\bar{z}, \mathcal{H}\} = -i \partial\mathcal{H}/\partial z\end{aligned}$$

similarly for all the other dynamical variables.

9. Quantization can be performed in three different forms.

- (i) Canonical quantization: This is the replacement of Poisson brackets by the commutators and of the dynamical variables by the Heisenberg operators. And as we mentioned above the resultant equations of motion coincide with those of the Dirac theory. Thus we have a correct classical model of the relativistic spinning electron.
- (ii) Path Integral quantization. There is a longstanding problem of how to obtain discrete quantum spin values from continuous classical spin variables by path integration. This problem has been solved with our classical spin variables  $\bar{z}$  and  $z$ . In fact it is possible to formulate precisely the whole of QED perturbation theory directly from classical particle trajectories by path integration<sup>7</sup>. The Dirac propagator, by the way, has the physical interpretation that its matrix element corresponds to a path integral not only with the endpoints fixed, but also with the initial and final spin components fixed at  $\alpha$  and  $\beta$ :

$$K_{\alpha\beta} = \left( \frac{1}{\gamma \cdot p - m} \right)_{\alpha\beta} = \int_{x_{a,\alpha}}^{x_{b,\beta}} \mathcal{D}(x)\mathcal{D}(p)\mathcal{D}(z)\mathcal{D}(\bar{z})e^{1/\hbar \int d\tau\omega}$$

- (iii) Schrödinger quantization. Here we start from a function  $\phi(\bar{z}, x; \tau)$  in the configuration space and represent the conjugate momenta by first order differential operators. The function  $\phi$  satisfies

$$i \frac{\partial}{\partial \tau} \phi(\bar{z}, x; \tau) = \mathcal{H}\phi(\bar{z}, x; \tau)$$

where  $\mathcal{H}$  is now the differential operator

$$\mathcal{H} = \bar{z}\gamma^\mu \frac{\partial}{\partial z} \left( i \frac{\partial}{\partial x^\mu} - eA_\mu \right)$$

We expand  $\phi$  on both sides of this equation in powers of  $\bar{z}_\alpha$  as

$$\phi(\bar{z}, x, \tau) = \phi(x, \tau) + \bar{z}^\alpha \psi_\alpha(x, \tau) + \bar{z}^\alpha \bar{z}^\beta \psi_{\alpha\beta}(x, \tau) + \dots$$

Acting with the operator  $\mathcal{H}$  and comparing the coefficients of  $\bar{z}_\alpha$  the first equation is the Dirac equation

$$i \frac{\partial}{\partial \tau} \psi_\alpha = \gamma^\mu (i\partial_\mu - eA_\mu) \psi_\alpha$$

10. Excited States of Zitterbewegung. The Schrödinger quantization gives in addition to Dirac equation a set of higher spin equations<sup>8</sup>. The next one is

$$i \frac{\partial}{\partial \tau} \psi_{\alpha\beta} = (\gamma^\mu \otimes I + I \otimes \gamma^\mu) (i\partial_\mu - eA_\mu) \psi_{\alpha\beta} \equiv \beta^\mu D_\mu \psi$$

where the matrices  $\beta$  satisfy the Kemmer algebra

$$\beta^\mu \beta^\lambda \beta^\nu + \beta^\nu \beta^\lambda \beta^\mu = \beta^\mu \delta^{\lambda\nu} + \beta^\nu \delta^{\lambda\mu}$$

the irreducible parts of which are, as is well known, the 5 and 10—dimensional matrices representing spin 0 and spin 1 particles. The general  $\beta$ -matrices are of the form

$$\beta^\mu(\gamma) = \gamma^\mu \otimes I \otimes I \dots + I \otimes \gamma^\mu \otimes I \dots + \dots + I \otimes I \otimes \dots \otimes \gamma^\mu$$

and satisfy the commutation relations

$$[(\beta(\gamma), \beta(\gamma'))] = \beta([\gamma, \gamma'])$$

where  $\gamma, \gamma'$  are the elements of a Lie algebra.

There are also supersymmetric structure given by the maps, for example,

$$\begin{aligned} \psi_\alpha &\rightarrow \psi_{\alpha\beta} \\ \gamma^\mu &\rightarrow \gamma^\mu \otimes I + I \otimes \gamma^\mu \\ (A &\rightarrow A \otimes A) \end{aligned}$$

11. Lorentz-Dirac equation with radiation reaction and spin. Now that we have included the correct spin terms into our classical equation we can evaluate self-energies and radiation reaction. The generalized Lorentz-Dirac equation so derived reads<sup>9</sup>

$$\dot{\pi}_\mu = eF_{\mu\nu}^{\text{ext}} u^\nu + e^2 \tilde{g}_{\mu\nu} \left( \frac{2}{3} \frac{\ddot{v}^\nu}{v^2} - \frac{9}{4} \frac{(v \cdot \dot{v}) \dot{v}^\nu}{v^4} \right)$$

where

$$\tilde{g}_{\mu\nu} = \tilde{g}'_{\mu\nu} - \frac{1}{v^2} v_\mu v_\nu$$

We have a new term coming from spin with coefficient  $-\frac{9}{4}$  in addition to the standard Larmore term. In the spinless limit  $v^2 \rightarrow 1, v \cdot \dot{v} \rightarrow 0, \dot{\pi}_\mu \rightarrow m\dot{x}_\mu$ , we obtain back the Lorentz Dirac equation.

12. Generalization to curved spaces is obtained by adding to the momenta a spin connection  $\Gamma_\mu : \pi_\mu = p_\mu - eA_\mu + i\bar{z}\Gamma_\mu z$ . The action in terms of  $\pi_\mu$  is the same as before. The equations of motion are  $(\lambda = 1)^{10}$ .

$$\begin{aligned} \dot{\bar{z}}\gamma \cdot n &= i(\bar{z}\dot{\pi} + i\bar{z}\Gamma_\mu v^\mu) \\ \gamma \cdot n\dot{z} &= -i(\pi z + iv^\nu\Gamma_\nu z) \\ \dot{x}_\mu &= \bar{z}\gamma_\mu z = v_\mu \\ \dot{\pi}_\mu - \Gamma_{\mu\nu}^\alpha \pi_\alpha v^\nu &= eF_{\mu\nu}v^\nu - \frac{1}{2}R_{\alpha\beta\mu\nu}S^{\alpha\beta}v^\nu \end{aligned}$$

Papapetrou equations for spin follow.

13. Generalizations to spinning strings and membranes can be obtained by introducing world sheet variables and functions  $x^\nu(\sigma^\alpha, \tau)$  and  $\bar{z}(\sigma^\alpha, \tau)$ .<sup>11</sup>
14. Internal Algebras SO(5) and SO(6). If one separates center of mass and relative coordinates and momenta one can exhibit a remarkable algebraic structure of the zitterbewegung which shows also the interesting geometry of the internal phase space. Again this structure is the same for the classical and quantum case. The brackets of the dynamical variables close to a SO(6) algebra. We only indicate here how the brackets of the relative conjugate coordinates is related to spin<sup>12</sup>:

$$\{\mathcal{Q}^\mu, \mathcal{Q}^\nu\} = \frac{1}{m^2}S^{\mu\nu}, \quad \{\mathcal{Q}^\mu, \mathcal{P}^\nu\} = -\tilde{g}^{\mu\nu}\frac{\mathcal{H}}{m}, \quad \{\mathcal{P}^\mu, \mathcal{P}^\nu\} = 4m^2S^{\mu\nu}$$

15. Two and many-body equations with spin and radiation reaction. In analogy to recently established covariant many-body equations in quantum electrodynamics we can also derive many -body equations for classical spinning particles. The method consists of defining composite spinors from the tensor product of the spinors for each particles,  $Z = z_1 \otimes z_2$  rewrite the action in terms of these composite spinors and derive their equations. For the two body problem one obtains in particular the following covariant Hamiltonian<sup>13</sup>:

$$H = Z\vec{\beta}Z \cdot \vec{P} + \bar{Z}(\vec{\alpha}_1 - \vec{\alpha}_2)Z \cdot \vec{p} - e_1e_2 \frac{\bar{Z}(1 - \alpha_1 \cdot \alpha_2)Z}{r} + M$$

where  $\vec{\beta} = \vec{\gamma} \cdot 1 + 1 \cdot \vec{\gamma}$  and  $\vec{\alpha}$ 's are the Dirac matrices.

### 2.3 Modelling Mass and Wave-Particle Duality

In the previous Section we have presented an intuitive model for spin, but the mass entered the theory as a constant of integration for which we do not have a more physical representation. Moreover after quantization we have the Heisenberg equations of motion for the dynamical variables, but how do we picture the operator-valued Heisenberg equations? We picture the classical equations of motion by trajectories, but not so for the Heisenberg equations. In this Section we shall present a complementary classical wave approach to the internal structure of the particle which gives us more insight into the mass and wave properties of the quantum particle. At the end then we can attempt a synthesis between the wave and particle approaches to pinpoint what a "quantum" particle is.

The wave approach is based on the explicit construction of localized oscillating nonspreading wavelets which move like relativistic particles. This is best explained in the simplest case of ordinary scalar linear wave equation

$$\square\phi = 0$$

We look for a localized solution initially at a point  $x_0$  of the form

$$\phi(\vec{x}, t) = F(\vec{x} - \vec{x}_0)e^{-i\Omega t}$$

The wave form oscillates with an internal frequency  $\Omega$ ; it is not static and this is a crucial point. The function  $F$  then satisfies the Helmholtz equation

$$\Delta F + (\Omega/c)^2 F = 0$$

whose solutions in spherical coordinates are

$$F(\vec{x} - \vec{x}_0) = \sum_{\ell m} C_{\ell m} \frac{1}{\sqrt{r}} J_{\ell+1/2} \left( \frac{\Omega}{c} r \right) P_{\ell}^m(\cos \theta) e^{im\varphi}, \quad r = \sqrt{(\vec{x} - \vec{x}_0)^2}$$

As an example, the simplest spherically symmetric solution is

$$F = C_{00} \frac{1}{\sqrt{r}} J_{1/2} \left( \frac{\Omega}{c} r \right)$$

We can associate a size  $c/\Omega$  to the large central region of the wavelet, although it has a small but infinitely long oscillating tail (Fig.1). The moving solution is obtained by a Lorentz transformation with parameters

$$\vec{\beta} = \frac{v}{c}, \quad \gamma^2 = (1 - \beta^2)^{-1}$$

and has the form<sup>13</sup>

$$\phi(\vec{x}, t) = F(r_{\perp}(\vec{x} - \vec{x}_0, )t) e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

where

$$r_{\perp}^2 = (r_{\mu}n^{\mu}) - (r_{\nu}n^{\nu}) \quad , \quad n^{\mu} = \gamma(1, \vec{\beta}) \quad , \quad r^{\mu} = (t, \vec{x} - \vec{x}_0)$$

or

$$r_{\perp} = \left[ (\vec{x} - \vec{x}_0)^2 c^2 \beta^2 \gamma^2 t^2 - 2\gamma^2 ct (\vec{\beta} \cdot (\vec{x} - \vec{x}_0)) + \gamma^2 (\vec{\beta} \cdot (\vec{x} - \vec{x}_0))^2 \right]^{1/2}$$

is the space-like distance on the surface  $\sum$  perpendicular to  $n^{\nu}$ . We see now the de Broglie phase with  $\vec{k} = \frac{\Omega}{c}\gamma\vec{\beta}$  and  $\omega = \gamma\Omega$  and the dispersion relation

$$(\omega/c)^2 - \vec{k}^2 = (\Omega/c)^2$$

The group velocity of the wave lump is  $v$ , whereas the phase velocity is  $u = \frac{\omega}{k}$  satisfying  $uv = c^2$ .

We have here the remarkable result that the dispersion relation for the localized solution of the massless wave equation  $\square\phi = 0$  exactly coincides with that of the plane wave solution of the massive Klein-Gordon equation  $(\square + \frac{m^2c^2}{\hbar^2})\phi = 0$ . Historically, when wave properties of the electron were discovered one has added by hand a mass term to the wave equation; it would have been equally possible to consider the localized solutions of the massless equation of the above type. We thus obtain an identification between the internal frequency  $\Omega$  and the mass  $m$ :

$$\frac{mc}{\hbar} = \frac{\Omega}{c} \quad \text{or} \quad \Omega = \frac{mc^2}{\hbar}$$

and it is here that, for dimensional reasons, the Planck's constant enters. Thus the concept of mass is now related to the internal oscillations of a wave lump, the frequency of an internal clock,  $\Omega$  and without such internal oscillations we could neither construct localized solutions, nor make a Lorentz transformation. We shall see again this connection between mass and frequency when we calculate the total field energy contained in the lump. It is given, in the rest frame of the lump, by

$$\mathcal{E} = N_E \int \left\{ \left| \frac{1}{c^2} \frac{\partial\phi}{\partial t} \right|^2 + (\nabla\phi)^2 \right\} d^3x$$

where  $N_E$  is a normalization constant, for dimensional reasons since a free field has no scale. In order to obtain a finite energy we take a superposition of different frequencies  $\Omega$  in the neighborhood of some basic frequency  $\Omega$  with a distribution function  $f(\Omega)$ . Using the orthogonality properties of the Bessel functions we then obtain a finite energy which, normalizing the charge of the complex scalar field, becomes proportional to the frequency  $\Omega$  in the rest frame. In the moving frame we obtain

$$\mathcal{E} \sim \omega \quad , \quad \vec{P} \sim \vec{k}$$

and the relativistic relation

$$\mathcal{E}^2 = c^2 \vec{P}^2 + m^2 c^4$$

In the nonrelativistic limit the wave equation goes over into the Schrödinger equation and our solution into

$$\psi = F(\vec{x} - \vec{x}_0 - \vec{v}t) e^{i(m\vec{v}\cdot\vec{x} - \frac{\hbar^2 k^2}{2m}t)} e^{-i\Omega t}$$

One can verify directly that this localized solution satisfies the Schrödinger equation with  $F$  given as before and with the dispersion relation

$$\omega - \frac{\hbar k^2}{2m} = \Omega$$

It is the Galilei-boosted rest frame solution  $\psi = F(x - x_0) e^{-i\Omega t}$ . The phase differs from the usual Schrödinger phase by the constant rapid mass oscillations  $e^{-i\Omega t}$  which drops out in the calculation of phase differences in interference experiments. The energy and momentum calculated in the same manner as above are given by

$$\mathcal{E} = \hbar\omega \quad , \quad \vec{\mathcal{P}} = \hbar\vec{k}$$

In the presence of a potential  $V(x)$  in the Schrödinger equation one can generalize the above localized solutions into the form<sup>14</sup>

$$\psi = F(\vec{x} - \vec{x}_0 - g(t)) \Psi(\vec{x}, t) e^{i\hbar(x,t)}$$

where  $F$  again is the localization function and  $\Psi$  is the usual solution of the Schrodinger equation in the given potential. In this form we obtain a new deterministic interpretation of a single quantum particle, as well as the statistical interpretation of in repeated experiments as follow. In the limit  $\Omega \rightarrow \infty$  the localization function  $F$  approaches  $\delta(x - x_0 - g(t))$  hence we get a classical trajectory of a point particle. On the other hand if we average over the parameters of the solution, for example over  $x_0$  and  $v_0$ ,  $F$  approaches unity and we are left with  $\Psi$ . Thus  $\Psi$  represents the typical or averaged behaviour of the particle in repeated experiments, and this is the standard Born statistical interpretation of quantum theory applicable only to repeated experiments. But now inbetween these two limits we get something new, namely a description of a single particle which has both particle-like and wave-like behaviour. Further details about these interpretation questions are given elsewhere<sup>15</sup>.

Similar localized solutions have been obtained for the electromagnetic field<sup>16</sup>, the spinor Dirac field<sup>17</sup> and for the linearized gravity<sup>18</sup>. With these highly localized solutions of wave equations we have modelled the mass in terms of the internal oscillations. We also modelled the wave-particle duality because the wave lumps move like relativistic particles and have the correct wave properties in phase and in dispersion relations. The frequency of the internal oscillations is the same as that of the helical motion of the spinning particle model of Section II. In fact if we imagine a point charge at a distance  $\hbar/mc$  from the center of our wavelet solution, it would perform a helical motion during the time evolution of the solution. We may thus view the localized solution as the “quantum” picture of the helical solution, or as the picture of the Heisenberg equations of motion.

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### 3. SELF-FIELD QUANTUMELECTRODYNAMICS

#### 3.1 Introduction

There are many approaches to radiative processes, or more generally, to electromagnetic interactions of charged particles. We should welcome this multitude because different ways of looking at the same physical phenomena can only bring clarity and hopefully enlightenment. I list those different formulations which are definite and more or less complete:

- (i) Second quantized quantum field theory, or the perturbative QED<sup>1</sup>.
- (ii) The  $S$  matrix theory of electromagnetic interactions, either from unitarity, analyticity and successive pole approximation<sup>2</sup>, or from regularization of the product of distributions<sup>3</sup>. Both of these lead to the renormalized perturbation theory with particles on the mass shell.
- (iii) Path integral method. Either path integrals of Maxwell–Dirac fields<sup>4</sup>, or path integrals directly from the classical particle trajectories<sup>5</sup>.
- (iv) Source theory<sup>6</sup>.
- (v) Selffield quantumelectrodynamics.

Of these only the selffield approach is in the long tradition of classical radiation theory and classical electrodynamics and is the subject of these lectures.

It is often stated that a large number of radiative phenomena conclusively show that the electromagnetic field, and further the electron's field, is quantized as a system of infinitely many oscillators with their zero point energies. The radiative phenomena are listed in Table I. We shall show that all these processes can also be understood and calculated in the selffield approach which does not quantize the fields. The quantum properties of the electromagnetic field are reduced here to the quantum properties of the source. One avoids thereby some of the difficulties of the quantized fields, such as the infinite zero point energy and other infinities of the perturbative QED.

TABLE I: RADIATIVE PROCESSES

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Spontaneous emission
Lamb shift
Anomalous magnetic moment
Vacuum polarization
Casimir effect between parallel plates
Casimir Polder potentials
Planck-distribution law for blackbody radiation
Unruh effect
QED in cavities
$e^+ - e^-$ system:
positronium spectrum
positronium annihilation
pair production and annihilation
$e^+ - e^-$ scattering
Relativistic many body problem with retardation
Electron - photon system:
photoelectric effect
Compton effect
Bremsstrahlung

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This lecture tells the story of the developments of selffield QED and it is good to begin from the beginning, namely the classical electrodynamics.

### 3.2 Classical Electrodynamics

The selfconsistent treatment of coupled matter and electromagnetic field goes back to H.A. Lorentz<sup>7</sup>. The electromagnetic field has as its source all the charged particles which in turn move in this total electromagnetic field. We have thus the Maxwell's equations coupled to the equations for matter:

$$\left. \begin{array}{l}
 1) \quad F_{\mu\nu}{}^{\nu} = -j_{\mu} \\
 2) \text{ Equation of motion of matter in the field } F
 \end{array} \right\} \quad (1)$$

These equations, both, can be derived from a single action principle. It has the general form

$$W = \int [\text{Kinetic energy of matter} - j_\mu A^\mu - 1/4 F_{\mu\nu} F^{\mu\nu}] \quad (2)$$

The last term is the action density of the field and the middle term represents the interaction of the matter current with the field.

We shall keep this general framework throughout also for quantum electrodynamics. The only change will be in the specific form of the current or how we describe the matter, the electron.

Classical electrodynamics *per se* is usually associated with the current of point particles moving along worldlines. But we can have more general extended sources of currents, as we shall see. For a number of point particles the current is given by

$$j_\mu(x) = \sum_i e_i \int ds_i \dot{x}_{i\mu}(s_i) \delta(x - x_i(s_i)) \quad (3)$$

Hence the fundamental equations are

$$F_{\mu\nu}'{}^\nu = -j_\mu = - \sum_i e_i \int ds_i \dot{x}_i(s_i) \delta(x - x_i(s_i)) \quad (4)$$

Here  $s_i$  are invariant time-parameters on the worldlines of the particles, and dots represent differentiation with respect to these times.

The equations of motions of the worldlines are

$$m_i \ddot{x}_{i\mu} = e_i F_{\mu\nu} \dot{x}_i^\nu, \quad i = 1, 2, 3, \dots \quad (5)$$

It is essential for the selfconsistency of our system that the field  $F$  entering the last equation is the field produced by all the particles including the particle  $i$ , namely the selffield. Hence we divide  $F$  into two parts

$$m_i \ddot{x}_{i\mu} = e_i F_{\mu\nu}^{(\text{other particles})} \dot{x}_i^\nu + e_i F_{\mu\nu}^{\text{self}} \dot{x}_i^\nu \quad (6)$$

The selffield can be obtained from the Lienard-Wiechert potential

$$A_\mu(x) = \int dx_\mu(s) D(x - x(s)) = e \int ds \dot{x}_\mu(s) D(x - x(s)) \quad (7)$$

but is formally infinite at the position of the particle. It must be treated properly, for example, by analytic continuation onto the world line<sup>8</sup>. This leads to the final Lorentz-Dirac equation for each particle (in natural units  $c = \hbar = 1$ )

$$m \ddot{x}_\mu = e F_{\mu\nu}^{\text{ext}} \dot{x}^\nu + \frac{2}{3} e^2 (\ddot{x}_\mu + (\ddot{x})^2 \dot{x}_\mu) \quad (8)$$

This is the basic nonperturbative equation of classical electrodynamics. Here  $m$  is now the renormalized mass. Furthermore we must find solutions of this equation which have the property that whenever the external force is zero the electron moves like a free particle,  $m\ddot{x}_\mu = 0$ , that is the second term must vanish together with the external field. This is part of the renormalization program. The important feature of this equation is that all radiative effects are now expressed in a closed, we repeat, nonperturbative way. The price we pay for this is that the equation is not only nonlinear but also contains the third derivatives. The selffield approach to quantumelectrodynamics has the goal of finding the analogous nonlinear, nonperturbative equation in the case of quantum currents. It is clear that radiative effects like the Lamb shift, anomalous magnetic moment, spontaneous emission, etc. have their counterparts also in classical electrodynamics.

As a second example of a classical current we consider the classical model of the Dirac electron which describes a spinning and charged relativistic point particle. In this model the worldline of the point particle is a helix, called *zitterbewegung*, and the orbital angular momentum of the helix in the rest frame of the center of mass accounts for the spin and the magnetic moment of the particle. The generalization of the Lorentz-Dirac equation for this case has recently been given<sup>9</sup>:

$$\dot{\pi}_\mu = eF_{\mu\nu}^{\text{ext}}v^\nu + e^2 \left( g_{\mu\nu} - \frac{v_\mu v_\nu}{v^2} \right) \left[ \frac{2}{3} \frac{\ddot{v}^\nu}{v^2} - \frac{9}{4} \frac{(v \cdot \dot{v})\dot{v}^\nu}{v^4} \right] \quad (9)$$

where

$$\pi_\mu = p_\mu - eA_\mu, \quad v = \dot{x} \quad \text{and} \quad v^2 \neq 1 \quad \text{due to spin.}$$

There are other classical models of the electron. A remarkable one is due to Lees<sup>10</sup> and Dirac<sup>11</sup> in which a charged shell is held stable with a surface tension. In the equilibrium position the surface tension can be expressed in terms of the mass of the electron so that this model has exactly again two parameters, mass and charge, like the point worldline. The Lorentz-Dirac equation for this model to my knowledge has not been worked out yet.

### 3.3 Schrödinger and Dirac Currents Quantumelectrodynamics

Quantumelectrodynamics has the same two basic equations (1). Only the form of the current  $j$  is different. According to Schrödinger and Dirac the electron is described not by a worldline but by a field  $\psi(x, t)$  and the basic coupled equations (1) become

$$F_{\mu\nu}^\nu = -j_\mu \quad , \quad F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$$

and

$$(\gamma^\mu i\partial_\mu - m)\psi(x) = e\gamma^\mu\psi(x)A_\mu(x) \quad (10)$$

for the relativistic Dirac case, and

$$i\frac{\partial\psi}{\partial t} = \left( -\frac{1}{2m} \left[ (\vec{p} - e\vec{A})^2 \right] + eA_0 \right) \psi \quad (11)$$

for the nonrelativistic Schrödinger case. The currents for a number of electrons is

$$j_\mu(x) = \sum_i e_i \bar{\psi}_i(x) \gamma_\mu^{(i)} \psi_i(x) \quad (12)$$

with a similar expression for the Schrödinger current.

Again the field  $A_\mu$  is the sum of an external and a selffield parts:

$$A_\mu = A_\mu^{\text{ext}} + A_\mu^{\text{self}} \quad (13)$$

With the choice of gauge  $A_{,\mu}^\mu = 0$  the Maxwell equations become

$$\square A_\mu = j_\mu(x) = \sum_i e_i \bar{\psi}_i(x) \gamma_\mu^{(i)} \psi_i(x) \quad (14)$$

so that the selffield can be expressed in terms of the current as

$$A_\mu(x) = \int dy D(x-y) j_\mu(x) \quad (15)$$

where  $D(x-y)$  is the appropriate Green's function corresponding to initial and boundary conditions. Equation (15) is our generalized Lienard-Wiechert potential. Thus the light emitted by a source depends on the nature and preparation of the current, and also on the nature of the environment determining the Green's function. Furthermore the whole light cone where  $\psi$  is different from zero contributes to the field at the field point and not just a single intersection of the worldline with the light cone, as in the case of a point particle.

Thus the selffield can be eliminated from the coupled Maxwell-Dirac equations. Inserting  $A_\mu$  into the equation of motion we obtain

$$\{\gamma^\mu (i\partial_\mu - e_k A_\mu^{\text{ext}}) - m_k\} \psi_k(x) = e_k \gamma^\mu \psi_k(x) \int dy D(x-y) \sum_i e_i \bar{\psi}_i(y) \gamma_\mu \psi_i(y) \quad (16)$$

Here  $A^{\text{ext}}$  is a fixed external field whose sources are far away and not dynamically relevant. In the next Section we shall treat two or many body systems in which we shall eliminate completely all the fields in favor of the currents. Eq. (16) is a nonlinear integral equation for  $\psi$  analogous to the nonlinear equation of the classical electrodynamics. The corresponding equation for the Schrödinger case is ( $\hbar = 1$ )

$$i \frac{\partial \psi}{\partial t} = \left[ -\frac{1}{2m} \left( \vec{p} - e \vec{A}^{\text{ext}} - e \vec{A}^{\text{self}} \right)^2 + e (A_0^{\text{ext}} + A_0^{\text{self}}) \right] \psi \quad (17)$$

where the selfpotentials are

$$A_0^{\text{self}} = \int dy D(x-y) \sum_k e_k \psi_k^*(y) \psi_k(y), \quad \vec{A}^{\text{self}}(x) = \int dy D(x-y) \sum_k \psi_k^*(y) \frac{\nabla}{i} \psi_k(y) \quad (18)$$

In writing these equations we have assumed that the  $\psi$ -current is an actual material charge current, and not just a probability current. Thus we are inevitably led to contemplate the interpretation and foundations of quantum theory. The foundations of quantum electrodynamics and that of quantum theory must be the same, for quantum mechanics was invented to understand the interactions between light and matter. Not surprisingly, it was Schrödinger who first formulated the selfconsistent coupled Maxwell and matter field equations, i.e., the program of Lorentz, for the new wave mechanics and insisted that for the selfconsistency of the theory the self field of the electron must be included as a nonlinear term. Schrödinger however calculated only the static part of the selfenergy and obtained unacceptable large selfenergies. Subsequently quantum electrodynamics went into a different direction. The selffield was dropped completely. Instead, one introduced a separate quantized radiation field with its own new degrees of freedom and coupled this to the quantized matter field. In the selffield approach the electromagnetic field has no separate degrees of freedom, they are determined by the source's degrees of freedom, but then we must include the full nonlinear selffield term. We shall come to this duality between the two approaches and to the questions of interpretation of quantum theory after the developments of the selffield QED.

### 3.4 Radiative Processes in an External (Coulomb) Field

The basis of selffield quantum electrodynamics is conceptually very simple and is completely expressed by the single equation (16). All QED processes in an external field listed in Table I should be derived from this single equation. To perform actual calculations it is much simpler and more direct to work with the action rather than with the equations of motion. The action  $W$  can, up to an overall  $\delta$ -function, be related to the energies of the system for bound state problems, and to the scattering amplitude for scattering problems.

The action for the system (10) is

$$W = \int dx \left[ \bar{\psi}(x)(\gamma^\mu i\partial_\mu - m)\psi(x) - e\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right] \quad (19)$$

Here we shall express  $A_\mu(x)$  in terms of  $\psi$  using (15). For bound state problems the action of the electromagnetic field can be reexpressed by a partial integration, using (10), as

$$-\frac{1}{4} \int dx F_{\mu\nu}F^{\mu\nu} = +\frac{1}{2} \int dx j_\mu(x)A^\mu(x) \quad (20)$$

Putting all together we have the action underlying our nonlinear equation (16), namely

$$W = \int dx \left[ \bar{\psi}(x) (\gamma^\mu (i\partial_\mu - eA_\mu^{\text{ext}}) - m) \psi(x) - \frac{e^2}{2} \int dy \bar{\psi}(x)\gamma^\mu(x)\psi(x)D(x-y)\bar{\psi}(y)\gamma_\mu\psi(y) \right] \quad (21)$$

We shall consider now the single electron problem in an external field.

We expand the classical field  $\psi$  into a Fourier series

$$\psi(x) = \sum_n \psi_n(\vec{x}) e^{-iE_n t} \quad (22)$$

and shall try to determine the expansion coefficients  $\psi_n(\vec{x})$  and the spectrum  $E_n$ —discrete and continuous. This expansion is quite different than the one used in standard QED and quantumoptics, namely the Coulomb series expansion, for example, in the Coulomb field,

$$\psi(x) = \sum_n c_n(t) \psi_n^c(\vec{x})$$

Here one derives equations for the time-dependent coefficients  $c_n(t)$ . The idea behind is that the system has definite levels and the perturbation will cause transitions between these levels. In our formulation, due to selfenergy, there are no definite (discrete) levels as exact eigenstates of the system to begin with, but the equations will determine the spectrum. In fact it will turn out that only the ground state of the system will be a stable eigenstate followed by a continuum with spectral concentrations around the unperturbed spectrum.

If we insert the Fourier expansion into the action we obtain

$$W = \sum_{nm} \int dx \left\{ \bar{\psi}_n(\vec{x}) e^{iE_n x^0} [\gamma^\mu (i\partial_\mu - eA_\mu^{\text{ext}}) - m] \psi_m(\vec{x}) e^{-iE_m x^0} - \frac{e^2}{2} \int dy \bar{\psi}_n(\vec{x}) \gamma^\mu \psi_m(\vec{x}) e^{i(E_n - E_m)x^0} D(x - y) \bar{\psi}_r(\vec{y}) \gamma_\mu \psi_s(\vec{y}) e^{i(E_r - E_s)y^0} \right\} \quad (23)$$

Time integrations can be performed using

$$D(x - y) = -\frac{1}{(2\pi)^4} \int dk \frac{e^{-ik(x-y)}}{k^2} \quad (24)$$

and we can write the interaction part of the action entirely in terms of the Fourier components of the current

$$W_{\text{int}} = + \frac{e^2}{2} \sum_{nmrs} \delta(E_n - E_m + E_r - E_s) \int \bar{\psi}_n(\vec{x}) \gamma^\mu \psi_m(\vec{x}) \times \frac{e^{i\vec{k} \cdot (\vec{x} - \vec{y})}}{(E_n - E_m)^2 - \vec{k}^2} \bar{\psi}_r(\vec{y}) \gamma_\mu \psi_s(\vec{y}) d\vec{x} d\vec{y} d\vec{k} \quad (25)$$

For the exact solutions of our equations the action  $W$  will vanish identically. We will now solve the system iteratively.

To lowest order of iteration we take the field to be given by the solutions of the external field problem without the selfenergy terms, and the energies to be shifted by a small amount:

$$\begin{aligned}\psi_n(x) &= \psi_n^{\text{ext}}(x) \\ E_n &= E_n^{\text{ext}} + \Delta E_n\end{aligned}\quad (26)$$

The first term in (22) therefore gives simply, using the orthonormality of  $\psi_n$ 's,

$$\begin{aligned}W_0 &= \int d\vec{x} \sum_{nm} \bar{\psi}_n (\gamma^0 E_n^{\text{ext}} - \vec{\gamma} \cdot \vec{p} - m - eA_\mu^{\text{ext}}) \psi_m \delta(E_n - E_m) \\ &\Rightarrow \sum_{nm} \Delta E_n \delta(E_n - E_m) \delta_{nm}\end{aligned}$$

In the second term we separate the terms according to  $E_n = E_m$ ,  $E_r = E_s$  and according to  $E_n = E_s$ ,  $E_m = E_r$ , the two ways of satisfying the overall  $\delta$ -function. And since  $W = 0$  to this order of iteration we can solve for  $\Delta E_n$ . The action and the total energy of the system are related by a  $\delta$ -function. Cancelling this  $\delta$ -function and also the sum over  $n$  to obtain the energy shift of a fixed level  $n$ , we obtain

$$\begin{aligned}\Delta E_n &= \frac{e^2}{2} \int d\vec{x} \bar{\psi}_n(\vec{x}) \gamma_\mu \psi_n(\vec{x}) P \int \frac{d\vec{k}}{(2\pi)^3} \int d\vec{y} \frac{e^{i\vec{k} \cdot (\vec{x} - \vec{y})}}{k^2} \cdot \sum_s \bar{\psi}_s(\vec{y}) \gamma^\mu \psi_s(\vec{y}) \\ &\quad - \frac{e^2}{2} \sum_s \int d\vec{x} d\vec{y} \bar{\psi}_n(\vec{x}) \gamma_\mu \psi_s(\vec{x}) \int \frac{d\vec{k}}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} \bar{\psi}_s(\vec{y}) \gamma^\mu \psi_n(\vec{y}) \cdot \\ &\quad \cdot \left[ \frac{1}{E_s - E_n - k} - \frac{1}{E_s - E_n + k} \right] \\ &\quad - \frac{e^2}{2} \sum_{(s < n)} \int d\vec{x} d\vec{y} \bar{\psi}_n(\vec{x}) \gamma_\mu \psi_s(\vec{x}) \int \frac{d\vec{k}}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} \bar{\psi}_s(\vec{y}) \gamma^\mu \psi_n(\vec{y}) \cdot \\ &\quad \cdot \frac{i\pi}{2k} \delta(E_s - E_n - k)\end{aligned}\quad (27)$$

This can be written in the form

$$\begin{aligned}\Delta E_n &= -\frac{e^2}{2} \sum_n \int \frac{d\vec{k}}{(2\pi)^3} \frac{j_{nn}^\mu(\vec{k}) j_\mu^{mm}(-\vec{k})}{k^2} - \frac{e^2}{2} \sum_{m < n} \int \frac{d\vec{k}}{(2\pi)^3} j_{nm}^\mu(\vec{k}) j_\mu^{mn}(-\vec{k}) \frac{i\pi}{2} \delta(E_m - E_n - k) \\ &\quad - \frac{e^2}{2} \sum_m \int \frac{d\vec{k}}{(2\pi)^3} j_{nm}^\mu(\vec{k}) j_\mu^{mn}(-\vec{k}) \frac{1}{2k} \left[ \frac{1}{E_m - E_n - k} - \frac{1}{E_m - E_n + k} \right]\end{aligned}\quad (28)$$

Thus the energy shifts are entirely expressed in terms of the integrals over the Fourier spectra of currents of all states. The first term corresponds to vacuum polarization, the second to spontaneous emission, and the third term to the Lamb shift proper. In

arriving at these results we have used the causal Green's function and separated the integrals into a principal and a imaginary part according to the formula

$$\frac{1}{x} = P\frac{1}{x} \pm i\pi\delta(x) \quad (29)$$

All the main QED effects are obtained here from a single expression. In fact one can also read off the anomalous magnetic moment ( $g - 2$ ) from this expression as we shall show in Section VI.

The evaluation of these expressions is a rather laborious technical problem. We have to use relativistic Coulomb wave functions for both the discrete and continuous spectrum and integrate the products of such functions and sum over the whole spectrum. We shall indicate some of these calculations and give results in Section VIII. The most important feature of the present formulation is that there are no infrared nor ultraviolet divergences.

The spontaneous emission term in Eq. (27) has been exactly evaluated<sup>12</sup>. We have now complete relativistic spontaneous decay rates for all hydrogenic states<sup>13</sup>. Table II shows some of these results.

TABLE II. Decay rates ( $s^{-1}$ ) in hydrogen and muonium

Transition	Hydrogen	Muonium
$2S_{1/2} \rightarrow 1S_{1/2}$	$2.4964 \times 10^{-6}$	$2.3997 \times 10^{-6}$
$2S_{1/2} \rightarrow 1P_{1/2}$	$5.194 \times 10^{-10}$	$5.172 \times 10^{-10}$
$2P_{1/2} \rightarrow 1S_{1/2}$	$2.0883 \times 10^8$	$2.0794 \times 10^8$
$2P_{3/2} \rightarrow 1S_{1/2}$	$4.1766 \times 10^8$	$4.1587 \times 10^8$
$2P \rightarrow 1S_{1/2}$	$6.2649 \times 10^8$	$6.2382 \times 10^8$

The vacuum polarization term has also been evaluated analytically<sup>14</sup> to lowest order term in  $\alpha(Z\alpha)^4$ . This is the most divergent term in perturbative QED and vanishes in the nonrelativistic limit.

The Lamb shift term which correctly reduces to the standard expressions in the dipole approximation has also been shown to be finite and will be evaluated in closed form<sup>15</sup>.

In all these calculations, since we are using Coulomb wave functions instead of the plane waves, the individual integrals are all finite. The summation over all the discrete and continuous levels are done by means of the relativistic Coulomb Green's functions.

### 3.5 Quantumelectrodynamics of the Relativistic Two-Body System

One of the most important and perhaps unexpected features of the selffield formulation of quantumelectrodynamics turned out to be a nonperturbative treatment of two and many body systems in closed form. It is well known that bound state problems cannot be treated in perturbative QED starting from first principles. Instead

one begins from a Schrödinger or Dirac-like equation obtained from some approximation to the Bethe-Salpeter relations and then calculates the perturbation diagrams to the bound state solutions of these equations. What one really needs is a genuine two-body relativistic equation which includes all the radiative terms as well as all the recoil corrections at once. We shall now discuss the principles of this theory.

In nonrelativistic quantum theory the many body problem is formulated in configuration space by a wave equation with pair potentials  $v_{ij}(x_i - x_j)$  of the form

$$\left( \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \dots V_{12} + V_{13} + V_{23} + \dots \right) \psi(x_1, \dots, x_n; t) = i\hbar \frac{\partial \psi}{\partial t}$$

This a priori not obvious. We may also think that each particle has its own field  $\psi(x)$  and satisfy a wave equation with a potential coming from the charge distribution of the other particles. For two particles, for example, we would have the coupled Hartre-type equations

$$\begin{aligned} i\hbar \frac{\partial \psi_1(\vec{x}_1, t)}{\partial t} &= \left( -\frac{\hbar^2}{2m_1} \Delta + \int \frac{\psi_2^*(x_2, t) \psi_2(x_2, t)}{|\vec{x}_2 - \vec{x}_1|} d\vec{x}_2 \right) \psi_1(\vec{x}_1, t) \\ i\hbar \frac{\partial \psi_2(\vec{x}_2, t)}{\partial t} &= \left( -\frac{\hbar^2}{2m_2} \Delta + \int \frac{\psi_1^*(\vec{x}_1, t) \psi_1(\vec{x}_1, t)}{|\vec{x}_1 - \vec{x}_2|} d\vec{x}_1 \right) \psi_2(\vec{x}_2, t) \end{aligned}$$

These two formulations are closely related but not identical. We shall see that they correspond to two different types of variational principles and actually describe two different types of physical situations. Quantum theory has a separate new postulate for two or more particles, namely that the state space is the tensor product of one particle state spaces. This leads immediately to the first formulation in configuration space. Such combined systems are called in the axiomatic of quantum theory “nonseparated” systems with all the nonlocal properties of quantum theory. But this postulate does not apply universally. There are other systems, namely the “separated” systems, which are described by the second type of equations. For example, for the system hydrogen molecule the two protons are separated, whereas the two electrons are nonseparated. The superposition principle holds for the nonseparated systems only. We shall now see how all this comes about from two different basic variational principles in the relativistic case (the nonrelativistic case is similar).

Consider a number of matter fields  $\psi_1(x), \psi_2(x) \dots$ . The action of these fields interacting via the electromagnetic field is

$$W = \int dx \left\{ \sum_k \bar{\psi}_k (\gamma^\mu i \partial_\mu - m_k) \psi_k - j_\mu(x) A^\mu(x) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\} \quad (30)$$

where the current  $j^\mu$  is the sum of Dirac currents for each field

$$j^\mu(x) = \sum_k e_k \bar{\psi}_k(x) \gamma^\mu \psi_k(x) \quad (31)$$

Again in the gauge  $A^\mu{}_{,\mu} = 0$  we obtain the equations for the electromagnetic field as

$$\square A_\mu = j_\mu = \sum_k j_\mu^{(k)} \quad (32)$$

with the solution

$$A_\mu(x) = \int dy D(x-y) j_\mu(x). \quad (33)$$

If we insert this into the action both in the  $j^\mu \cdot A_\mu$  term as well as in the term  $-(1/4)F_{\mu\nu}F^{\mu\nu}$ , and using the identity (20), we obtain

$$W = \int dx \sum_k \bar{\psi}_k (\gamma^\mu i\partial_\mu - m_k) \psi_k - \sum_{k,\ell} \frac{1}{2} \int dx dy j_\mu^{(k)}(x) D(x-y) j_\mu^{(\ell)}(y) \quad (34)$$

The interaction action is a sum of current-current interactions containing both the mutual interaction terms, e.g.

$$-\frac{e_1 e_2}{2} \int dx dy \bar{\psi}_1(x) \gamma^\mu \psi_1(x) D(x-y) \bar{\psi}_2(y) \gamma_\mu \psi_2(y) - (1 \leftrightarrow 2)$$

and the self interaction terms like

$$-\frac{e_1^2}{2} \int dx dy \bar{\psi}_1(x) \gamma^\mu \psi_1(x) D(x-y) \bar{\psi}_1(y) \gamma_\mu \psi_1(y)$$

If we vary this action with respect to each field  $\psi_k$  separately we obtain coupled nonlinear equations. For example for two particles

$$\begin{aligned} (\gamma^\mu i\partial_\mu - m_1) \psi_1 &= \frac{e_1 e_2}{2} \gamma^\mu \psi_1 \int dy D(x-y) \bar{\psi}_2(y) \gamma_\mu \psi_2(y) \\ &\quad + \frac{e_1^2}{2} \gamma^\mu \psi_1 \int dy D(x-y) \bar{\psi}_1(y) \gamma_\mu \psi_1(y) \\ (\gamma^\mu i\partial_\mu - m_2) \psi_2 &= \frac{e_1 e_2}{2} \gamma^\mu \psi_2 \int dy D(x-y) \bar{\psi}_1(y) \gamma_\mu \psi_1(y) \\ &\quad + \frac{e_2^2}{2} \gamma^\mu \psi_2 \int dy D(x-y) \bar{\psi}_2(y) \gamma_\mu \psi_2(y) \end{aligned} \quad (35)$$

Next let us define a composite field  $\Phi$  by

$$\Phi(x_1, x_2) = \psi_1(x_1) \psi_2(x_2) \quad (36)$$

This is a 16-component spinor field. We can rewrite our action (34) entirely in terms of the composite field. This is straightforward in the mutual interaction terms. In the

kinetic energy and selfinteraction terms we multiply suitable by normalization factors. For example for the first kinetic energy term we get

$$\int dx_1 \bar{\psi}_1(x_1) (\gamma^\mu i \partial_\mu - m_1) \psi_1(x_1) \cdot \int d\sigma_2 \bar{\psi}_2(x_2) \gamma \cdot n \psi_2(x_2)$$

where  $d\sigma_2 n^\mu = d\sigma_2^\mu$  is a 3-dimensional volume element perpendicular to the normal  $n^\mu$ . Similarly for the other kinetic energy term. The selfenergy terms need two such normalization factors. The resultant action in terms of the composite field is then

$$\begin{aligned} W = & \left[ \int dx_1 d\sigma_2 \bar{\Phi}(x_1 x_2) (\gamma^\mu \pi_{1\mu} - m_1) \otimes \gamma \cdot n \Phi(x_1 x_2) \right. \\ & + \int dx_2 d\sigma_1 \bar{\Phi}(x_2 x_1) \gamma \cdot n \otimes (\gamma^\mu \pi_{2\mu} - m_2) \Phi(x_2 x_1) \\ & \left. - e_1 e_2 \int dx_1 dx_2 \bar{\Phi}(x_1 x_2) \gamma^\mu \otimes \gamma_\mu D(x_1 - x_2) \Phi(x_1 x_2) \right] \end{aligned} \quad (37)$$

The generalized canonical momenta  $\pi_{i\mu}$  are given further below. Here and through the rest of the paper we shall write spin matrices in the form of tensor products  $\otimes$ , the first factor always referring to the spin space of particle 1, the second to particle 2. We shall give the selfenergy terms explicitly below.

Now our second variational principle is that the action be stationary not with respect to the variations of the individual fields but with respect to the total composite field only. This is a weaker condition than before and leads to an equation for  $\Phi$  in configuration space. For bound state problems only the symmetric Green's function contributes and it contains a  $\delta(x^2)$ -function which we decompose relative to the space-like surface with normal  $n^\mu$  as follows

$$\delta(r^2) = \frac{\delta[(r \cdot n) - r_\perp] \pm \delta[(r \cdot n) + r_\perp]}{2r_\perp}, \quad r_\perp = [(r \cdot n)^2 - r^2]^{1/2} \quad (38)$$

where  $r_\perp$  is a relativistic three dimensional distance which for  $n = (1000)$  reduces to the ordinary distance  $r$ . All the integrals in the action (37) are 7-dimensional. For covariance purposes it is necessary to have the vector  $n^\mu$ . It tells us how to choose the time axis. The vector  $n$  is also present, in principle, in the one-body Dirac equation but we usually do not write it when discussing the solutions, but automatically choose it to be  $n = (1000)$ , i.e., the rest frame. The final form of our two-body equation is then

$$\left\{ (\gamma^\mu \pi_1^\mu - m_1) \otimes \gamma \cdot n + \gamma \cdot n \otimes (\gamma^\mu \pi_2^\mu - m_2) - e_1 e_2 \frac{\gamma^\mu \otimes \gamma_\mu}{r_\perp} \right\} \Phi(x_1 x_2) = 0 \quad (39)$$

where now the selfpotentials are inside the generalized momenta

$$\pi_i^\mu = p_i^\mu - e_i A_i^{\mu\text{self}} - e_i A_i^{\mu\text{ext}} \quad (40)$$

with

$$\begin{aligned} A_{\mu,1}^{\text{self}}(x) &= \frac{e_1}{2} \int dz d\sigma_u D(x-z) \bar{\Phi}(z,u) \gamma_\mu \otimes \gamma \cdot n \Phi(z,u) \\ A_{\mu,2}^{\text{self}}(x) &= \frac{e_2}{2} \int d\sigma_z du D(x-u) \bar{\Phi}(z,u) \gamma \cdot n \otimes \gamma_\mu \Phi(z,u) \end{aligned} \quad (41)$$

We note that the last term in (39) can also be put into the potential  $A_\mu$ , one half for each particle;  $\frac{1}{2}e_2 \frac{1 \otimes \gamma_\mu}{r}$  and  $\frac{1}{2}e_1 \frac{\gamma_\mu \otimes 1}{r}$ , respectively.

The self potentials are nonlinear integral expressions. The arguments of  $\Phi$  consist of seven variables because  $\Phi(x_1, x_2)$  is different from zero only if  $(x_1 - x_2)$  is lightlike; only then there is a communication between the particles. This means that we have one time-variable and three space variables for each particle. We see this more clearly if we introduce center of mass and relative variables according to

$$\begin{aligned} \Pi &= \pi_1 + \pi_2, & \pi &= \pi_1 - \pi_2 \\ x &= x_1 - x_2, & X &= x_1 + x_2 \end{aligned} \quad (42)$$

Then equation (39), without the selffield terms for simplicity, becomes

$$\left\{ \Gamma^\mu \Pi_\mu + k^\mu \pi_\mu - \frac{e_1 e_2}{r_\perp} \gamma^\mu \otimes \gamma_\mu - m_1 I \otimes \gamma \cdot n - m_2 \gamma \cdot n \otimes I \right\} \Phi = 0 \quad (43)$$

where we have introduced

$$\Gamma^\mu = \frac{1}{2} (\gamma^\mu \otimes \gamma \cdot n + \gamma \cdot n \otimes \gamma^\mu)$$

and

$$k^\mu = \frac{1}{2} (\gamma^\mu \otimes \gamma \cdot n - \gamma \cdot n \otimes \gamma^\mu) \quad (44)$$

We see now that  $k \cdot n = 0$ , i.e., the component of  $k^\mu$  parallel to  $n^\mu$  vanishes which means that the component of the relative momentum  $\pi_\mu$  parallel to  $n_\mu$  drops out of the equation automatically. For  $n = (1000)$ , in particular, we have

$$\left\{ \Gamma^0 \Pi_0 - \vec{\gamma} \cdot \vec{\Pi} - \vec{k} \cdot \vec{\pi} - \frac{e_1 e_2}{r} \gamma^0 \otimes \gamma_0 - m_1 I \otimes \gamma_0 - m_2 \gamma_0 \otimes I \right\} \Phi = 0 \quad (45)$$

Thus we have only one time variable conjugate to the center of mass energy  $\Pi_0$  and three degrees of freedom for the center of mass momentum  $\vec{\Pi}$  and three degrees of freedom for the relative momentum  $\vec{\pi}^0$ ;  $\pi_0$  does not enter, as it should be so on physical grounds. In contrast the Bethe-Salpeter equation has two time coordinates. Since  $\Pi_0$  is the ‘‘Hamiltonian’’ of the system we obtain, by multiplying (45) by  $\Gamma_0^{-1}$  the Hamiltonian form of the two-body equation

$$\Pi_0 \Phi = \left\{ \vec{\alpha} \cdot \vec{\Pi} + (\vec{\alpha}_1 - \vec{\alpha}_2) \cdot \vec{\pi} + \frac{e_1 e_2}{r} (1 - \vec{\alpha}_1 \cdot \vec{\alpha}_2) + m_1 \beta_1 \cdot I + m_2 I \cdot \beta_2 \right\} \Phi$$

where we have defined

$$\vec{\alpha} \equiv \frac{1}{2}(\vec{\alpha}_1 + \vec{\alpha}_2); \quad \vec{\alpha}_i = \gamma_i^0 \vec{\gamma}_i, \beta_i = \gamma_{0i}, \quad i = 1, 2 \quad (46)$$

Our two-body equation has the form of a generalized Dirac equation, now a 16-component wave equation. In fact it reduces to the one-body Dirac equation in the limit when one of the particles is heavy.

The above developments are completely relativistic and covariant. The physical results are independent of the vector  $n$  although a vector  $n$  must appear for manifest covariance. Thus recoil corrections are included to all orders. Further interesting properties of the equation, beside being a one-time relativistic equation, are that relative and center of mass terms in the Hamiltonian are additive, and radial and angular parts of the relative equation are exactly separable. It has also a nonrelativistic limit to the two-body Schrödinger equation. We shall discuss numerical results in Section VIII.

### 3.6 The Interpretation of Negative Energy States

It is often stated that only in second quantized field theory can one have an adequate description of antiparticles and negative energy solutions where one changes the roles of the creation and annihilation operators for the negative energy solutions. We shall now show that there is also a consistent way of dealing with the negative energy solutions and antiparticles in the Dirac equation as a classical field theory and elaborate how we obtain the annihilation potential in positronium, for example.

There are actually not one but two Dirac equations

$$\begin{aligned} (\gamma \cdot p - m)\psi_I &= 0 \\ (\gamma \cdot p + m)\psi_{II} &= 0 \end{aligned} \quad (47)$$

obtained from the factorization of the Klein-Gordon operator, for example. By convention we just peak one and work with the complete set of solutions of this equation. Now the negative energy solutions of  $\psi_I$  coincide with the positive energy solutions of  $\psi_{II}$ . Furthermore in the presence of the electromagnetic field with minimal coupling we have the two equations

$$\begin{aligned} (\gamma \cdot (p - eA) - m)\psi_I &= 0 \\ (\gamma \cdot (p - eA) + m)\psi_{II} &= 0 \end{aligned} \quad (48)$$

and we can easily prove that

$$\psi_I(-p, -e) = \psi_{II}(p, e) \quad (49)$$

that is the negative energy momentum solutions of  $\psi_I$  coincide with the positive energy solutions of  $\psi_{II}$  of opposite charge. Therefore we should consider positive energy solutions of both equations as physical particles. The total number of such physical

solutions is the same as the total number of both positive and negative energy solutions of a single Dirac equation.

With this interpretation we obtain quite naturally the annihilation diagrams and annihilation potentials between particles and antiparticles. Consider our interaction action

$$\int dx dy \bar{\psi}_1(x) \gamma^\mu \psi_1(x) D(x-y) \bar{\psi}_2(y) \gamma_\mu \psi_2(y)$$

Here the classical fields  $\psi_i(x)$  contain all positive and negative energy solutions according to our general expansion (22). Separating positive and negative energy solutions as

$$\psi_n(x) = \psi_{E_N > 0} + \psi_{E_N < 0} \equiv \psi_n^+ + \psi^-$$

and inserting it into the action we get 16 terms. In the limiting case of the lowest order scattering, where we replace the fields by plane wave solutions, we have essentially two distinct types of vertices at each point  $x$  or  $y$ , namely

$$\bar{\psi}^+(x) \gamma_\mu \psi^+(x) \quad \begin{array}{c} \nearrow \\ \searrow \\ x \end{array}$$

and

$$\bar{\psi}^+(x) \gamma_\mu \psi^-(x) \quad \begin{array}{c} \nwarrow \\ \nearrow \\ x \end{array}$$

(50)

In the second case we have used our interpretation of the negative energy solutions as the antiparticles with reversed energy-momentum  $p_\mu$ . The complete interaction action to this order consists of all combinations of these two vertices located at  $x$  and  $y$  for particles 1 and 2 multiplied with the Green's function  $D(x-y)$ . Of these 16 terms some cannot be realized because of the overall energy-momentum conserving  $\delta$ -function,  $\delta(p_1 + p_2 - p_3 - p_4)$ , and we are left with two distinct types of terms

(51)

plus the same terms with particles and antiparticles interchanged. This result agrees with the standard QED. But we shall go a bit further and apply it to bound state problems in Section VII after a discussion of the case of identical particles.

Identical Particles

For two identical particles we use the postulate of the first quantized quantum theory that the field is symmetric or antisymmetric under the interchange of all dynamical variables of identical particles. In our formulation we go back to the original action principle and assume that the current  $j_\mu$  is antisymmetric in the two fields

$$j_\mu = \frac{1}{2} e (\bar{\psi}_1 \gamma_\mu \psi_2 - \bar{\psi}_2 \gamma_\mu \psi_1), \quad e_1 = e_2 = e \quad (52)$$

This implies in the interaction action

$$W_{\text{int}} = \frac{1}{4}e^2 \left[ \int dx dy \bar{\psi}_1(x) \gamma_\mu \psi_2(x) D(x-y) \bar{\psi}_1(y) \gamma^\mu \psi_2(y) - \int dx dy \bar{\psi}_1 \gamma_\mu \psi_2 D(x-y) \bar{\psi}_2 \gamma^\mu \psi_1 + (1 \leftrightarrow 2) \right] \tag{53}$$

and again when the fields are expanded we see that identical particles with exactly the same wave functions i.e., the same quantum numbers or the same state, will not interact and that in the lowest approximation we will get besides the direct interaction also an exchange term as shown in the following diagrams



Finally we combine the two effects, identical particles and particle-antiparticle properties, to discuss systems like electron-positron complex and positronium. According to our discussion this system is just a part of the larger electron-electron system taking into account the interpretation of the negative energy levels and the identity of the particles.

### 3.7 Calculation of the Anomalous Magnetic Moment ( $g - 2$ )

We show now that our basic interaction action also contains besides Lamb shift, spontaneous emission and vacuum polarization also the anomalous magnetic moment in the same single expression. We shall also introduce at this occasion the more general four-dimensional energy-momentum Fourier expansion instead of the energy Fourier expansion (22) which was appropriate for the fixed external field problem.

The interaction action is given by

$$W_{\text{int}} = -\frac{e^2}{2} \int dx dy j^\mu(x) D(x-y) j_\mu(y) \tag{54}$$

We expand the fields as four dimensional Fourier integrals

$$\psi(x) = \int dp e^{-ipx} \psi(p) \tag{55}$$

and insert it into the action

$$W_{\text{int}} = -\frac{e^2}{2} \frac{1}{(2\pi)^4} \int dx dy dk dp dq dr ds \bar{\psi}(p) \gamma^\mu \psi(p) \frac{e^{-ik(x-y)}}{k^2 + i\epsilon} \bar{\psi}(r) \gamma_\mu \psi(s) \times e^{i(p-q)x + (r-s)y}$$

which can be written as

$$W_{\text{int}} = -\frac{e^2}{2}(2\pi)^4 \int dpdqdrds j^\mu(p, q) \frac{1}{(r-s)^2 + i\epsilon} j_\mu(r, s) \delta(p-q+r-s) \quad (56)$$

where  $j^\mu(p, q)$  stand for the double Fourier transform

$$j^\mu(p, q) = (2\pi)^8 \int dx dy \bar{\psi}(x) \gamma^\mu \psi(y) e^{-ipx} e^{iqy} \quad (57)$$

The  $\delta$ -function arises from the  $x, y$  and  $k$ -integrations. We separate the action into two terms to satisfy the  $\delta$ -function

$$\begin{aligned} (i) \quad & p = q, \text{ hence } r = s \\ (ii) \quad & p = s, \text{ hence } q = r \end{aligned} \quad (58)$$

Again as before the first corresponds to vacuum polarization; the second term contains Lamb shift and spontaneous emission as real and imaginary parts of the energy shift  $\Delta E$ . It also contains the anomalous magnetic moment as the coefficient of the magnetic part of the Lamb shift for any external field. For the calculation of  $(g-2)$  it is thus not necessary to solve a problem with an external magnetic field or to solve any external problem for that matter.

The second term with (ii) can be written as

$$W_{\text{int}}^{(ii)} = -\frac{e^2}{2} \int dx dy dp dq D(x-y) j^\mu(q, p) j_\mu(q, p) e^{ipx} e^{i(y-x)q} \quad (59)$$

Here we recognize the c-number electron propagator-function  $S(x-y)$

$$\int dq \psi(q) \bar{\psi}(q) e^{i(y-x)q} \equiv S(x-y) \quad (60)$$

which satisfy the inhomogeneous wave equation

$$[\gamma^\mu(p_\mu - eA_\mu) - m] = i\delta(x-y) \quad (61)$$

Let us also take the Fourier transform of  $S$

$$S(x-y) = \frac{1}{(2\pi)^4} \int dp e^{-ip(x-y)} S(p) \quad (62)$$

Then the action (59) becomes

$$W_{\text{int}}^{(ii)} = -\frac{e^2}{2} i \int dp dP \bar{\psi}(p) \frac{\gamma^\mu S(p) \gamma_\mu}{(p-P)^2 + i\epsilon} \psi(p) \quad (63)$$

It is related to the energy, more precisely to a mass shift by an overall  $\delta$ -function and we can write

$$\Delta E = \frac{W_{\text{int}}^{(ii)}}{(2\pi)^4} = \int dp \bar{\psi}(p) \Delta M(p) \psi(p) \quad (64)$$

where we have introduced an effective *mass matrix* by

$$\Delta M(p) = \frac{e^2}{2} \frac{i}{(2\pi)^4} \int ds \frac{\gamma^\mu S(p-s) \gamma_\mu}{s^2 - i\epsilon} \quad (65)$$

It remains now to evaluate the mass matrix  $\Delta M$ . First we expand the Green's function or propagator in an external field as follows

$$\frac{1}{\gamma^\mu(p_\mu - eA_\mu) - m} = \frac{\not{p} - \epsilon + m}{p^2 - m^2} + 2e \frac{p \cdot A(\not{p} + m)}{(p^2 - m^2)^2} - ie \frac{(\not{p} + m) \gamma^\mu \gamma^\nu F_{\mu\nu}}{(p^2 - m^2)^2} + \dots$$

where

$$\not{p} \equiv \gamma^\mu p_\mu, \not{A} = \gamma^\mu A_\mu, p \cdot A = p^\mu A_\mu. \quad (66)$$

It turns out that only the third term which is gauge invariant gives a nonvanishing contribution to lowest order in  $\alpha$ : the terms containing  $A_\mu$  give vanishing contributions. The mass operator becomes

$$\Delta M(p) = \frac{e^2}{2} \frac{i}{(2\pi)^4} (-ie) \int ds \frac{\gamma^\mu ((p-s) - m) \gamma^\lambda \gamma^\varphi F_{\lambda\varphi} \gamma_\mu}{s^2 ((p-s)^2 - m^2)^2} \quad (67)$$

The integrals can be performed giving<sup>16</sup>

$$\Delta M(p) = \frac{e^2}{2} \frac{i}{(2\pi)^4} (-ie) \frac{2}{m} \int_0^1 dy (1-y) \sigma^{\mu\nu} F_{\mu\nu}$$

and finally

$$\Delta M(p) = -\frac{\alpha}{2\pi} \frac{e}{2m} \sigma_{\mu\nu} F_{\mu\nu} = -\frac{\alpha}{2\pi} \frac{e}{2m} (\vec{\sigma} \cdot \vec{B} + i\vec{\alpha} \cdot \vec{E}) \quad (68)$$

Thus we recognize the anomalous magnetic moment to this order in front of the  $\vec{\sigma} \cdot \vec{B}$  term for any field as we have mentioned.

If we insert the mass operator into the energy shift formula (64) we can evaluate it as the expectation value of the operator  $\vec{\sigma} \cdot \vec{B} + i\vec{\alpha} \cdot \vec{E}$ . For relativistic Coulomb problem for example the magnetic part of the Lamb shift can be analytically evaluated exactly.<sup>17</sup>

This way of calculating the anomalous magnetic moment also shows now how to calculate higher order terms. We must take more terms in the Green's function expansion (66). This may be much simpler than the diagrammatic method of perturbative QED where there are already 891 Feynman diagrams in the order  $(\alpha/\pi)^4$ .

## 3.8 Covariant Analysis of Radiative Processes for Two-Body Systems

In this section we discuss how to treat radiative processes, like Lamb shift, etc., for a system like positronium or muonium beyond the naive reduced mass method. As mentioned above the action formalism is more convenient than the equations of motion.

We go back to our covariant 2-body action (37) and separate center of mass and relative coordinates and momenta according to

$$\begin{aligned}
 r &= x_1 - x_2 & x_1 &= R + \frac{1}{2}r & P &= p_1 + p_2 \\
 R &= \frac{1}{2}(x_1 + x_2) & x_2 &= R - \frac{1}{2}r & p &= \frac{1}{2}(p_1 - p_2) \\
 q &= z - u & z &= Q + \frac{1}{2}q & p_1 &= \frac{1}{2}P + p \\
 Q &= \frac{1}{2}(z + u) & u &= Q - \frac{1}{2}q & p_2 &= \frac{1}{2}P - p
 \end{aligned} \tag{69}$$

All quantities here are four-vectors. Then the action becomes

$$\begin{aligned}
 W &= \int dRdq\bar{\Phi}(R,r) \left\{ \left[ \gamma \cdot \left( \frac{1}{2}P + p \right) - m_1 \right] \otimes \gamma \cdot n + \gamma \cdot n \otimes \left[ \gamma \cdot \left( \frac{1}{2}P - p \right) - m_2 \right] \right. \\
 &\quad - e_1 e_2 D(r) - \frac{e_1^2}{2} \int dQdq\gamma_\mu \otimes \gamma \cdot n D \left( R - Q + \frac{1}{2}(r - q) \right) \bar{\Phi}(Q,q)\gamma^\mu \otimes \gamma \cdot n \Phi(Q,q) \\
 &\quad \left. - \frac{e_2^2}{2} \int dQdq\gamma \cdot n \otimes \gamma_\mu D \left( R - Q - \frac{1}{2}(r - q) \right) \bar{\Phi}(Q,q)\gamma \cdot n \otimes \gamma^\mu \Phi(Q,q) \right\} \Phi(R,r)
 \end{aligned} \tag{70}$$

In the absence of a fixed external field the system is translationally invariant and the generalization of the Fourier expansion (22) is the four dimensional Fourier transform of the composite field  $\Phi(R,r)$  which has actually one time variable,  $\Phi(R,r_\perp)$ , the relative coordinates is a 3-vector  $r_\perp$  perpendicular to  $n$ .

$$\Phi(R,r_\perp) = \int \frac{d^4P}{(2\pi)^4} e^{iPR} \psi(P,r_\perp)$$

We insert this expansion everywhere in our action and obtain

$$\begin{aligned}
W = & \int dR dr_{\perp} \int \frac{dP_n}{(2\pi)^4} \frac{dP_m}{(2\pi)^4} \bar{\psi}(P_n, r_{\perp}) e^{-iP_n R} \left\{ [\Gamma_{\mu} P^{\mu} + \mathcal{L}_{rel}(r_{\perp}, p)] e^{iP_m R} \right. \\
& - \frac{1}{2} \int \frac{dP_r}{(2\pi)^4} \frac{dP_s}{(2\pi)^4} dQ dq_{\perp} dk \left[ \frac{e_1^2}{2} \frac{e^{-ik[R-Q+\frac{1}{2}(r_{\perp}-q_{\perp})]}}{k^2} \gamma^{\mu} \otimes \gamma \cdot n \bar{\psi}(P_r, q_{\perp}) e^{-iP_r Q} \gamma_{\mu} \right. \\
& \quad \otimes \gamma \cdot n e^{iP_s Q} \psi(P_s, q_{\perp}) \\
& \quad \left. + \frac{e_2^2}{2} \frac{e^{-ik[R-Q-\frac{1}{2}(r_{\perp}-q_{\perp})]}}{k^2} \gamma \cdot n \otimes \gamma^{\mu} \bar{\psi}(P_r, q_{\perp}) e^{-iP_r Q} \gamma \cdot n \otimes \gamma_{\mu} e^{iP_s Q} \right. \\
& \quad \left. \left. \times \psi(P_s, q_{\perp}) \right] \right\} \psi(P_m, r_{\perp})
\end{aligned} \tag{71}$$

where  $\mathcal{L}_{rel}$  is the Lagrangian of the relative motion and is given by

$$\mathcal{L}_{rel}(r_{\perp}, p) = (\gamma^{\mu} p_{\mu} - m_1) \otimes \gamma \cdot n + \gamma \cdot n \otimes (-\gamma^{\mu} p_{\mu} - m_2) - \frac{e_1 e_2}{r_{\perp}} \gamma^{\mu} \otimes \gamma_{\mu} \tag{72}$$

and

$$\Gamma_{\mu} = \frac{1}{2} (\gamma_{\mu} \otimes \gamma \cdot n + \gamma \cdot n \otimes \gamma_{\mu}) \tag{73}$$

The result of performing the  $R$  and  $Q$ -integrations, letting

$$k_{\mu} = \gamma_{\mu} \otimes \gamma \cdot n - \gamma \cdot n \otimes \gamma_{\mu} \tag{74}$$

is

$$\begin{aligned}
W = & \int \frac{dP_n}{(2\pi)^4} dP_m dr_{\perp} \bar{\psi}(p_n, r_{\perp}) \left\{ \left[ \Gamma_{\mu} P^{\mu} + k^{\mu} p_{\mu} - m_1 I \otimes \gamma \cdot n - m_2 \gamma \cdot n \otimes I - \frac{e_1 e_2}{r_{\perp}} \gamma^{\mu} \otimes \gamma_{\mu} \right] \right. \\
& \times \delta(P_n - P_m) - \frac{1}{2} \int \frac{dP_r}{(2\pi)^4} dP_s \frac{dk}{(2\pi)^4} dr_{\perp} dq_{\perp} \delta(P_n - P_m - k) \delta(P_r - P_s - k) \left[ e_1^2 e^{-i\frac{1}{2}k(r_{\perp}-q_{\perp})} \right. \\
& \times \gamma^{\mu} \otimes \gamma \cdot n \bar{\psi}(P_r, q_{\perp}) \gamma_{\mu} \otimes \gamma \cdot n + e_2^2 e^{i\frac{1}{2}k(r_{\perp}-q_{\perp})} \gamma \cdot n \otimes \gamma^{\mu} \bar{\psi}(P_r, q_{\perp}) \gamma \cdot n \otimes \gamma_{\mu} \left. \right] \psi(P_s, q_{\perp}) \left. \right\} \\
& \psi(P_m, r_{\perp})
\end{aligned} \tag{75}$$

Introducing the form factors

$$\bar{T}_{nm}^{(1)}(k) \equiv \int dr_{\perp} \bar{\psi}(P_n, r_{\perp}) e^{\frac{1}{2}k r_{\perp}} \gamma^{\mu} \otimes \gamma \cdot n \psi(P_m, r_{\perp})$$

and

(76)

$$\bar{T}_{nm}^{(2)} = \int dr_{\perp} \bar{\psi}(P_n, r_{\perp}) e^{-\frac{i}{2}kr_{\perp}} \gamma \cdot n \otimes \gamma^{\mu} \psi(P_m, r_{\perp})$$

for the two particles, we can write the action in the compact form

$$\begin{aligned} W &= \sum_{nm} \int dr_{\perp} \bar{\psi}(P_n, r_{\perp}) [\Gamma_{\mu} P^{\mu} + \mathcal{L}_{rel}] \psi(P_m, r_{\perp}) \delta(P_n - P_m) \\ &\quad - \frac{1}{2} \sum_{nm} \sum_{rs} \frac{dk}{(2\pi)^4} \frac{1}{k^2} \left[ e_1^2 T_{nm}^{(1)k} T_{rs}^{(1)\mu} + e_2^2 \bar{T}_{nm}^{(2)\mu} \bar{T}_{rs}^{(2)\mu} \right] \delta(P_n - P_m + k) \delta(P_r - P_s - k) \end{aligned} \quad (77)$$

where

$$\mathcal{L}_{rel}(r_{\perp}) = k^{\mu} p_{\mu} - m_1 I \otimes \gamma \cdot n - m_2 \gamma \cdot n \otimes I - \frac{e_1 e_2}{r_{\perp}} \gamma^{\mu} \otimes \gamma_{\mu}$$

Note that  $k^{\mu} n_{\mu} = 0$ .

Now we can perform the  $k^0$ -integration, and without loss of generality set  $n = (1000)$ , and obtain

$$\begin{aligned} W &= \sum_{nm} \delta(E_n - E_m) \delta(\vec{P}_n - \vec{P}_m) d\vec{r} \bar{\psi}_n(\vec{P}_n, \vec{r}) [\Gamma_0 P^0 - \vec{\Gamma} \cdot \vec{P} + \mathcal{L}_{rel}] \psi_m(\vec{P}_m, \vec{r}) \\ &\quad - \frac{1}{2} \sum_{nm} \sum_{rs} \frac{d\vec{k}}{(2\pi)^4} \frac{1}{\omega_{nm}^2 - \vec{k}^2} \left[ e_1^2 \bar{T}_{nm}^{(1)\mu}(\omega_{nm}, \vec{k}) \bar{T}_{rs}^{(1)\mu}(\omega_{rs}, \vec{k}) \right. \\ &\quad \left. + e_2^2 \bar{T}_{nm}^{(2)\mu}(\omega_{nm}, \vec{k}) \bar{T}_{rs}^{(2)\mu}(\omega_{rs}, \vec{k}) \right] \delta(\omega_{nm} + \omega_{rs}) \delta(\vec{P}_n - \vec{P}_m + \vec{k}) \delta(\vec{P}_r - \vec{P}_s - \vec{k}) \end{aligned} \quad (78)$$

Now we look at the selfinteraction terms only and expand the denominator

$$\begin{aligned} W^{\text{self}} &= \frac{1}{2} \sum_{nm} \sum_{rs} \frac{d\vec{k}}{(2\pi)^4} \left[ e_1^2 \bar{T}_{nm}^{(1)\mu}(\omega_{nm}, \vec{k}) \bar{T}_{rs}^{(1)\mu}(-\omega_{nm}, -\vec{k}) + e_2^2 \bar{T}_{nm}^{(2)\mu}(\omega_{nm}, \vec{k}) \bar{T}_{rs}^{(2)\mu}(-\omega_{nm}, -\vec{k}) \right] \\ &\quad \times \delta(\omega_{nm} + \omega_{rs}) \left\{ (\delta(\omega_{rs} - k) + \delta(\omega_{rs} + k)) \frac{2\pi i}{2k} + P \frac{1}{ik} \left( \frac{1}{\omega_{rs} - k} - \frac{1}{\omega_{rs} + k} \right) \right\} \\ &\quad \delta(\vec{P}_n - \vec{P}_m + \vec{k}) \delta(\vec{P}_n - \vec{P}_m + \vec{P}_r - \vec{P}_s) \end{aligned} \quad (79)$$

where  $P$  stands for the principal value of the integral and  $\sum_{\int}$  means a summation over discrete states and an integration over the continuum states. As in the case of the Coulomb problem and the  $(g-2)$ -calculation, we separate the two terms corresponding to

(a)  $n = m$ , hence  $r = s$

and

$$(b) \ n = s, \text{ hence } m = r$$

and dictated by the  $\delta$ -functions and obtain finally

$$\begin{aligned} W^{\text{self}} = & -\frac{1}{2} \sum_{ns} \int \frac{d\vec{k}}{(2\pi)^4} \left\{ e_1^2 T_{nn}^{(1)}(0, \vec{k}) T_{ss}^{(1)} \mu(0, -\vec{k}) + e_2^2 T_{nn}^{(2)}(0, \vec{k}) T_{ss}^{(2)} \mu(0, -\vec{k}) \right\} \\ & \times \left\{ \frac{i\pi}{k} (\delta(k) + \delta(-k)) + \frac{1}{2k} P \left( -\frac{2}{k} \right) \right\} \delta(\vec{P}_n - \vec{P}_m + \vec{k}) \delta(\vec{P}_n - \vec{P}_m + \vec{P}_r - \vec{P}_s) \\ & - \frac{1}{2} \sum_{nm} \int \frac{d\vec{k}}{(2\pi)^4} \left\{ e_1^2 T_{nm}^{(1)}(\omega_{nm}, \vec{k}) T_{rs}^{(1)} \mu(-\omega_{nm}, -\vec{k}) + e_2^2 T_{nm}^{(2)}(\omega_{nm}, \vec{k}) T_{rs}^{(2)} \mu(-\omega_{nm}, -\vec{k}) \right\} \\ & \times \left\{ \delta(\omega_{nm} - k) + \delta(\omega_{nm} + k) \frac{i\pi}{k} + P \frac{1}{2k} \left( \frac{1}{\omega_{nm} - k} - \frac{1}{\omega_{nm} + k} \right) \right\} \end{aligned} \quad (80)$$

We recognize again the following terms:

- (i) Term containing  $\delta(k) + \delta(-k)$ . The contribution of this term to the  $dk$ -integral vanishes.
- (ii) The term  $P1/k$ : This term corresponds to vacuum polarization.
- (iii) The term with  $(i\pi/k)[\delta(\omega_{nm} - k) + \delta(\omega_{nm} + k)]$ . This term gives the spontaneous emission or absorption from level  $n$  to  $m$  or vice versa.
- (iv) The term with  $P \frac{1}{2k} \left( \frac{1}{\omega_{nm} - k} - \frac{1}{\omega_{nm} + k} \right)$  gives the Lamb shift.

These are our formulas for the radiative processes of the two-fermion system<sup>18</sup>. In the limit they go over to the fixed center Coulomb problem on the one hand, and for free particles to perturbative QED results.

For identical particles and particle-antiparticle system like positronium we have to antisymmetrize our currents as discussed in Sec. VI. Thus the mutual interaction action has two terms. The first is the usual direct interaction term

$$\begin{aligned} W_{e^-e^+}^{\text{int}}(1) &= -e^2 \int dx dy \bar{\psi}_I(x) \gamma^\mu \psi_I(x) D(x-y) \bar{\psi}_{II}(x) \gamma_\mu \psi_{II}(y) \\ &= -e^2 \int dx dy \bar{\phi}_I(x, y) \gamma^\mu D(x-y) \gamma_\mu \phi(x, y) \end{aligned} \quad (81)$$

corresponding to the potential

$$V = -e^2 \gamma^\mu \frac{1}{r} \gamma_\mu \quad (82)$$

The second term is

$$\begin{aligned} W_{e^-e^+}^{\text{int}}(2) &= e^2 \int dx dy \bar{\phi}(x, y) \gamma^\mu D(x-y) \gamma_\mu \phi(y, x) \\ &= \frac{e^2}{2\pi} \int d\vec{r} \bar{\phi}_E(p_0, -\vec{r}) \gamma^\mu \frac{e^{i\vec{k} \cdot \vec{r}}}{(p_0 + p'_0)^2 - \vec{k}^2} \gamma_\mu \phi_E(p_0, -\vec{r}) \frac{d\vec{k}}{(2\pi)^3} \end{aligned} \quad (83)$$

where  $p_0$  and  $p'_0$  are the initial and final state energies and  $E$  is total conserved center of mass energy of the whole system. In the positronium the relative momentum is approximately zero so that we can set

$$p_\mu \cong (m, \vec{0})$$

and the action becomes

$$W_{e^-e^+}^{\text{int}}(\text{annihilation}) \cong \frac{2\alpha}{m^2} \int d\vec{r} \bar{\phi}(\vec{r}) \gamma^\mu \delta(\vec{r}) \gamma_\mu \phi(-\vec{r}) \quad (84)$$

Now we show that this term gives correctly the annihilation contribution to the hyperfine splitting in the  $n = 1$  state of positronium, for example. The effective potential above (84), when inserted into our wave equations gives an energy shift only for the levels  $j = l \cong 0$  and for  $j - 1 = l \cong 0$ .

$$\delta E(j = l = 0) \cong \frac{m\alpha^4}{2n^3}$$

and

$$\delta E(j - 1 = l = 0) \cong \frac{m\alpha^4}{4n^3}$$

The difference is the annihilation contribution in the hyperfinesplitting

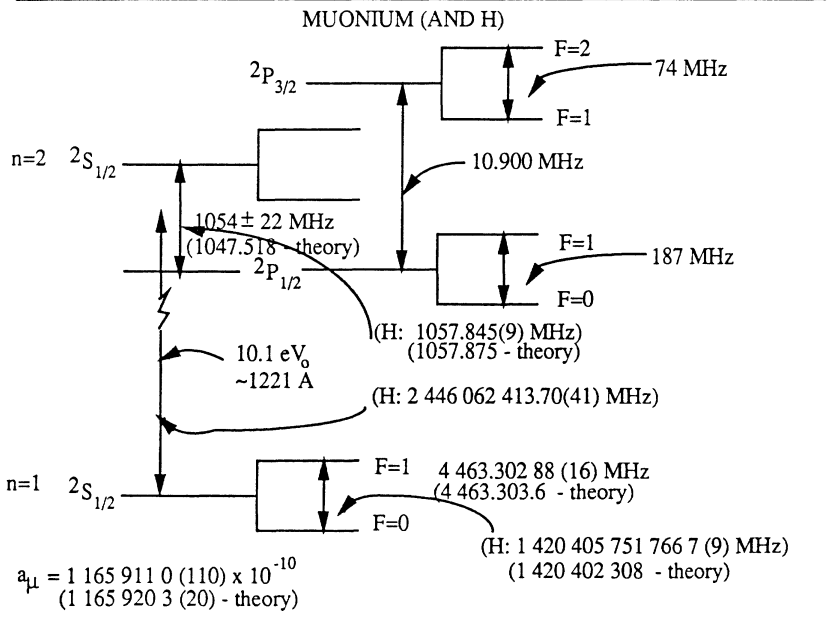
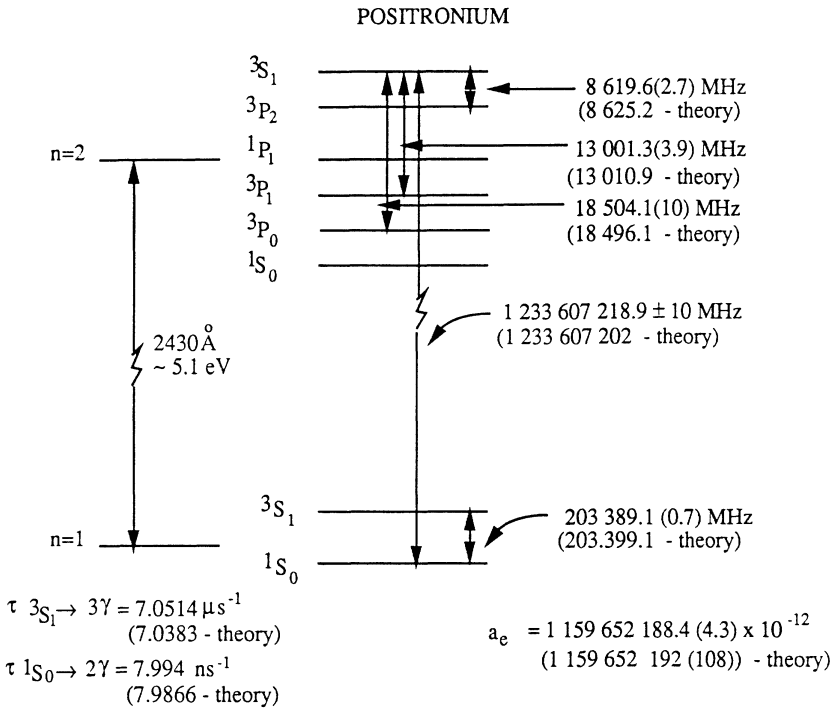
$$\delta E_{Hfs}(\text{annihilation}) \cong \frac{3m\alpha^4}{12n^3} \quad (85)$$

To this order it agrees with perturbative QED. It is however obtained here in first quantized QED with selffields.

### 3.9 Further Results

There are still discrepancies between theory and experiment in almost all tests of QED. The Table III summarizes all measured levels in positronium, muonium and Hydrogen, positronium lifetimes, the anomalous magnetic moments  $a_e$  and  $a_\mu$ , and some theoretical values in parenthesis. In reviewing some of these discrepancies, *W-Lichten*<sup>18</sup> writes "It seems likely that the problem lies in the difficulty of QED calculations which have not been carried out to a high order enough, perhaps a totally new type of calculation is needed". The self field approach to QED provides a new type of calculation. It's important to have a complementary or alternate method to perturbative QED, for a theory is tested not only against experiment but also against other theories in order to clarify the basic assumptions and concepts, specially in view of recent results that perturbative QED might be inconsistent or a trivial theory. Self-field QED modifies our notion of the quantized radiation field and the interpretation of quantum theory. The emphasis is shifted from the field to the source of radiation, an electronic charge distribution which objectively and deterministically evolves as a

TABLE III. BOUND STATE TESTS OF QED



classical field and produces a selffield which acts back on the charge itself. Quantized properties of the light reflect the discrete frequencies of the oscillating charge distribution.

The two-body relativistic equation discussed in Sections V and VII gives us a possibility to make improved calculations for positronium and muonium, in particular. In positronium, the experiments seem to be more accurate than the theory and the perturbative calculations remain incomplete<sup>19</sup>. Considerable analytical work has been done on the study of the two-body equation (39)ff: separation of radial and angular parts and further reduction of the radial equations<sup>20</sup>. It turns out that the two-body equation, when the electromagnetic potentials are kept to order  $\alpha^4$ , is exactly soluble with an energy spectrum

$$E^2 = \frac{M^2 + \Delta m^2}{2} + \frac{M^2 - \Delta m^2}{2} \left[ 1 + \frac{\alpha^2}{(n_r + \ell)^2} \right]^{-1/2} \quad (86)$$

with  $M = m_1 + m_2$ ,  $\Delta m = m_1 - m_2$ , generalizing the Dirac spectrum. We have treated the remaining potentials of order  $\alpha^5$  and higher as perturbations. But having tested the equation in this way, one can now make direct nonperturbative numerical calculations. The treatment of the negative energy states and the covariance of the equation has also been discussed<sup>22</sup> according to the methods outlined in Section VI. We give here some of the results<sup>20</sup>.

- 1) For parapositronium, eq. (86), is exact including terms of the order  $\alpha^4$  since normal and anomalous magnetic moment terms do not contribute to this order. It gives

$$E^{\text{para ps}} = 2m - \frac{m\alpha^2}{4n^2} - \frac{m\alpha^4}{2n^3(2j+1)} + \frac{11}{64} \frac{m\alpha^4}{n^4} + 0(\alpha^6) \quad (87)$$

- 2) Introducing the anomalous magnetic moments  $a_1$ ,  $a_2$ , which are in the selfenergy term, as a Pauli-coupling we obtain the ground-state hyperfine splitting

$$\Delta E^{\text{Hfs}} = \frac{8}{3} \frac{\zeta}{(1+\zeta)^2} m\alpha^4 \left[ (1+a_1)(1+a_2) - \frac{3}{4}\zeta a_2^2 + \frac{3}{4}a_1 \frac{a_2}{(1+\zeta)^2} \right]$$

with  $\zeta = m_1/m_2$ . Numerically this gives 1420.348 MHz for  $H$ , and 4.463.060 MHz for muonium, compared to the experimental values 1420.405752 and 4.463302, respectively.

- 3) Positronium hyperfinesplitting including the annihilation term, eq. (85), gives

$$\Delta E^{\text{Hfs}} = \frac{7}{12} m\alpha^4 + \frac{5}{12} m\alpha^4 \left( \frac{\alpha}{2\pi} \right).$$

This ‘‘Lambshift’’ term  $-\frac{\alpha}{2\pi}(\frac{16}{9} + \ell n 2)m\alpha^4$  has to be added perturbatively, but we hope to calculate these terms and more eventually numerically.

- 4) Positronium ( $n = 2, n = 1$ ) splitting, including annihilation and anomalous magnetic moment contributions

$$\Delta E_{21} = \frac{3}{8} Ry - 0.468098\alpha^2 Ry - \frac{\alpha^2 Ry}{2\pi} \frac{35}{96}.$$

- 5) Positronium fine structure

$$\begin{aligned} \Delta E(2^3 S_1 - 2^3 P_2) &= -\frac{1}{12}\alpha^2 Ry + \frac{7}{48}\alpha^2 Ry - \frac{7}{480}\alpha^2 Ry + 0(\alpha^5) \\ &\quad \text{(recoil)} \quad \text{(annihilation)} \quad \text{(normal magnetic moment)} \\ &= \frac{23}{480}\alpha^2 Ry + 0(\alpha^5) \end{aligned}$$

- 6)  $H$  or muonium ( $n = 2, n = 1$ ) splitting

$$\begin{aligned} \Delta E_{21} &= \frac{3}{8}\mu\alpha^2 + \frac{8}{128}\mu\alpha^4 + \frac{15}{128}\mu\alpha^4 \frac{\zeta}{1+\zeta} \\ &\quad - \frac{7}{16}\mu\alpha^4 \left( 1 + \frac{2(a_1 + \zeta^2 a_2)}{(1+\zeta)^2} - 2a_1 a_2 \frac{\zeta}{1+\zeta} \right) \\ &\quad - \frac{7}{12} \frac{\mu^2}{M} \alpha^4 (1 + a_1 + a_2 + a_1 a_2) \end{aligned}$$

where  $M = m_1 + m_2$ ,  $\zeta = m_1/m_2$ ,  $\mu = \frac{m_1 m_2}{M}$ .

For other details and applications of self-field QED we refer to the literature listed in the Appendix.

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