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ABSTRACT

It is shown that QED radiative processes of an atom, like Lamb shift and spontaneous emission, are modified in the presence of a one-dimensional "mirror" at a distance R by an amount proportional to $1/R^2$, compared to a 2-dimensional mirror where it is proportional to $1/R^4$.

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I. Introduction

The modification of the QED radiative processes (spontaneous emission, Lamb-shift, $g-2$, vacuum polarization) in cavities of different geometries is being actively studied at present, both theoretically and experimentally [1]. In addition to these the Casimir force between two neutral atoms (or between an atom and a conductor) have been treated as manifestations of field quantization (zero point fluctuations) [2]. We can further add to this list, the effect of the microwave background radiation on the QED-processes [3], and the change of the properties of the atom due to the accelaration of the atom (e.g. the Unruh "effect") [4].

All these effects have also been discussed recently in a unified form from an alternate point of view, namely from the modification of the Green's function, hence of self-energy, of the wave equation by the boundary conditions. In the general formulas for the radiative effects due to self energy all we have to do is to replace the Green's function $D(x-y)$ of the free space by the Green's function of the cavity, $D^{\text{cavity}}(x-y)$,or by the Green's function of the temperature bath, or that of an accelerated detector, etc. This is the method we shall adopt in this work.

We shall consider here even a simpler nontrival geometry than any considered before. This is a multiply connected space which is obtained by excluding the x_3-axis. Such a space can be realized by placing an "impenetrable conducting wire", or an "Aharonov-Bohm solenoid" parallel to the x_3-axis. When two inpenetrable parallel lines are placed in the space, they attract each other by a Casimir force per unit length which is given by $F/L = -n_1^2 n_2^2/2\pi^2 R^3$ with n_1, n_2 are the quantized fluxes confined to the lines and R is the distance between them [5]. This force can be compared to the inverse fourth power Casimir force between two parallel plates. It is an interesting question to study the power law with respect to the distance R of the Casimir or Casimir-Polder forces as a function of the dimensions of the conductors (wires, plates, balls, ...).

In the present work we calculate the change occuring in the radiative processes of the H-atom in the above mentioned space. This study is also important for the experimental investigation of the Casimir force between parallel lines, since it is in general easier to measure the Casimir forces indirectly, that is, by observing the shifts in the atomic energy levels caused by the presence of the boundaries.

II. Green's Functions in Non-trivial Geometries

We use the self-field approach to QED which turns out to be the most direct method to deal with the boundary conditions of the surrounding space of the physical system. The system we are considering is a bound or trapped electron near one (or more) one-dimensional conductors. We start from the (general) Maxwell-Dirac action

$$W = \int dx \left[\bar{\Psi} \left(\gamma^\mu \left(i\partial_\mu - eA_\mu^{ext} \right) - m \right) \Psi - e\bar{\Psi}\gamma^\mu \Psi A_\mu - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \right] \tag{1}$$

where A_μ^{ext} is a fixed external field to bind the electron. We eliminate A_μ using one of the field equations, namely $\Box A_\mu = e\bar{\Psi}\gamma_\mu\Psi$ in the covariant gauge $A^\mu{}_{,\mu} = 0$, by

$$A_\mu(x) = \int dx' D(x - x') e\bar{\Psi}(x')\gamma_\mu \Psi(x') \tag{2}$$

where $D(x - x')$ is the symmetric (Causal) Green's function of space-time in the presence of the boundaries or cavity. Boundary effects in QED as compared to free space QED comes here only in the choice of the appropriate Green's function. The modification of the atomic Ψ itself due to boundaries is very small and will be neglected.

If we insert eq. (2) into eq. (1), and because the surface terms do not contribute for bound state problems, we obtain a new action written in terms of the Ψ-field alone [6]

$$\begin{aligned} W = \int dx \Big[& \bar{\Psi} \left(\gamma^\mu \left(i\partial_\mu - eA_\mu^{ext} \right) - m \right) \Psi \\ & - \frac{e^2}{2} \bar{\Psi}(x)\gamma^\mu\Psi(x) \int dx' D(x - x')\bar{\Psi}(x')\gamma_\mu\Psi(x') \Big] \end{aligned} \tag{3}$$

The interaction action

$$W_{\text{int.}} = -\frac{e^2}{2} \int dx dx' \bar{\Psi}(x)\gamma^\mu \Psi(x) D(x - x') \bar{\Psi}(x')\gamma_\mu \Psi(x') \tag{4}$$

has been shown to contain all the radiative processes. One can read from this term spontaneous emission [7], the Lamb shift [8], vacuum polarizaton [9], $(g-2)$ [10], as well as the modification of all these results in the presence of cavities [11] and Unruh effect [12].

To study the radiation reactions in the multiply connected space of one-dimensional lines we first review the calculation of the appropriate Green's function [5]. For that purpose it is convenient to realize first the multiply connected character of the space by an Aharanov-Bohm solenoid. We place an infinitely long and tightly wound solenoid parallel to x_3 axis with its center at $x_1 = -R$, $x_2 = 0$. We choose the center of mass of the H-atom to be the origin of our coordinate system and assume the distance R between the atom and the solenoid to be fixed. The potential of the solenoid in the outside region is given by

$$\vec{A} = \frac{\bar{\Phi}}{2\pi} \frac{1}{\zeta} \left(\sin\beta \hat{x}_1 - \cos\beta \hat{x}_2 \right) \tag{5}$$

where "the flux centered polar coordinates" ζ and β are given by

$$\zeta^2 = (x_1 + R)^2 + x_2^2, \quad tg\beta = \frac{x_2}{x_1 + R} \tag{6}$$

The symmetric Green's function $D(x, x')$ can be obtained from the propagator $G(x, x')$ of a massless charged relativistic particle moving in the presence of the external field $\vec{A}$:

$$D(x, x') = \frac{1}{2} \left(G(x, x') + G(x', x) \right) \tag{7}$$

and the propagator can be represented as a covariant path-integral

$$G(x, x') = \int_0^\infty dW \int D^4 x exp \left[\frac{i}{4} \int_0^W dw \left(-\dot{t}^2 + \dot{\vec{x}}^2 + 4eA \cdot \dot{x} \right) \right] \tag{8}$$

Here overdots stand for the derivative with respect to the invariant time parameter w. We do not need in the following the explicit form of the above path integral which can be found in the literature [13].

Inserting the potential (5) into (8) we observe that the interaction term can be integrated explicitly

$$\frac{i}{4}\int_0^W dw(4eA\cdot\dot{x}) = -i\int_0^W dwe\frac{\Phi}{2\pi}\dot{\beta} = -ie\frac{\Phi}{2\pi}(\beta-\beta'\pm 2\pi\Lambda) \tag{9}$$

with $\beta(\omega = 0) = \beta'$, $\beta(\omega = W) = \beta$. The integer Λ is the winding number which distinguishes the different homotopy classes. In order to take into account all the paths connecting x to x' we have to sum over all winding numbers Λ, which by virtue of the Poisson formula

$$\sum_{\Lambda=-\infty}^{\infty} e^{2\pi i\Lambda\frac{e\Phi}{2\pi}} = \sum_{\Lambda=-\infty}^{\infty}\delta\left(\Lambda - e\frac{\Phi}{2\pi}\right)$$

implies the flux quantization:

$$\Phi = \frac{2\pi}{e}\Lambda, \tag{10}$$

Thus the interaction part of the propagator factorizes from the free part and we obtain the final form of the Green's function

$$D_\Lambda(x,x') = \cos\Lambda(\beta-\beta')\ D(x,x') \tag{11}$$

where $D(x,x')$ is the free space Green's function

$$D(x,x') = \frac{1}{2\pi^2\left[-(t-t')^2+(\vec{x}-\vec{x}')^2\right]} = \frac{1}{2\pi^2}(x-x')^2$$

or

$$= \int_0^\infty dW\int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}e^{-ik_0(t-t')+i\vec{k}\cdot(\vec{x}-\vec{x}')+i(k_0^2-\vec{k}^2)W}$$

$$= \int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik_0(t-t')+i\vec{k}\cdot(\vec{x}-\vec{x}')}}{k_0^2-\vec{k}^2+i\varepsilon} \tag{12}$$

The result in (11) can easily be generalized to more than one solenoids. Equation (11) will also apply to a conducting wire in the absence of currents instead of a solenoid. This is because the potential $\vec{A}$ is a way to characterize the field free space except the one-dimensional cut of space ($\vec{E}=0$, $\vec{B}=0$, except the line $x_3=0$). In this case Φ expresses a property of the conductor and the integer Λ again the winding number of paths.

III. QED - Effects

Having obtained the appropriate Green's function we go back to our electromagnetic interactional to calculate the interaction action of eq. (4): We expand the field $\Psi(x)$ into a Fourier series

$$\Psi(x) = \sum_n \Psi_n(\vec{x})e^{-iE_n t} \tag{13}$$

where $\{\Psi_n(\vec{x})\}$ is a set to be determined indexed by the oscillatory t-dependence. Note that this is distinct from the Coulomb series expansion $\Psi(x) = \sum_n c_n(t)\Psi_n^c(\vec{x})$, where $\Psi_n^c(\vec{x})$ is the complete set of solutions of the external field (Coulomb) problem. Inserting (13) and the Fourier transform (12) into (4) we obtain after t, t' and k_0-integrations

$$\begin{aligned} W_{int} = &-\frac{e^2}{2}\sum_{nmrs}\int d\vec{x}\ d\vec{x}'\ d\vec{k}\ \delta(E_n-E_m+E_r-E_s) \\ &\times\bar{\Psi}_n(\vec{x})\gamma^\mu\Psi_m(\vec{x})\quad\cos\Lambda(\beta-\beta') \\ &\times\frac{e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}}{(E_n-E_m)^2-\vec{k}^2+i\varepsilon}\quad\bar{\Psi}_r(\vec{x}')\gamma_\mu\Psi_s(\vec{x}') \end{aligned} \tag{14}$$

Here each of the numbers n, m, ... stand for the set of discrete and continuous quantum numbers. The standard form factors

$$T^\mu_{nm}(\vec{k}) = \int d\vec{x}\ e^{-i\vec{k}\cdot\vec{x}}\bar{\Psi}_n(\vec{x})\gamma^\mu\Psi_m(\vec{x})$$

are now modified by the presence of the $\cos\Lambda(\beta-\beta')$ factor due to the one-dimensional cavity.

Since, as we have mentioned, all the QED-radiative processes are contained in (14), and have been evaluated before in free space and in the presence of boundaries, it is sufficient to calculate the change of the products of two form factors in Eq. (14), i.e. the integral

$$I(\vec{k}) = \int d\,\vec{x}\; d\,\vec{x}'\bar{\Psi}_n(\vec{x})\gamma^\mu\Psi_m(\vec{x})e^{-i\vec{k}\cdot\vec{x}} \bar{\Psi}_r(\vec{x}')\gamma_\mu\Psi_s(\vec{x}')e^{i\vec{k}\cdot\vec{x}'}\cos\Lambda(\beta-\beta') \tag{15}$$

This expression can be evaluated in principle for Dirac wave functions in full generality. But to see the essence of the effect and its magnitude, it is sufficient to calculate the nonrelativistic limit of $I(\vec{k})$ in the dipole approximation, i.e. $e^{i\vec{k}\cdot\vec{k}} \simeq 1$. The important term in the nonrelativistic limit (NR) in eq. (15) for the H-atom is

$$\begin{aligned} m^2 I^{NR}_{nlm}(\vec{k}) = \sum\!\!\!\!\!\!\int_{n'l'm'} & d\,\vec{x}\; d\,\vec{x}'\Psi^*_{nlm}(\vec{x})\frac{\vec{\nabla}}{i}\Psi_{n'l'm'}(\vec{x}) \\ & \times\cos\Lambda(\beta-\beta') \\ & \times\Psi^*_{n'l'm'}(\vec{x}')\frac{\vec{\nabla}}{i}\Psi_{nlm}(\vec{x}') \end{aligned} \tag{16}$$

Here nlm are the usual quantum numbers of the H-atom and m is the reduced mass. We shall evaluate this integral in the limit $R \gg |\,\vec{x}\,|$ or $|\,\vec{x}'\,|$, i.e. when the distance of the atom from the solenoid is large compared to the atomic Bohr radius. In this limit (with $\vec{x} = (r,\theta,\varphi)$) we have

$$tg\beta \simeq \beta \simeq \frac{x_2}{R} = \frac{r\cos\theta\sin\varphi}{R} \tag{17}$$

Hence

$$\begin{aligned} \cos\Lambda(\beta-\beta') &\simeq 1 - \frac{\Lambda^2}{2}\left(\beta^2 - 2\beta\beta' + \beta'^2\right) \\ &\simeq 1 - \frac{\Lambda^2}{2R^2}\left(r^2\cos^2\theta\sin^2\varphi\right. \\ &\qquad - 2rr'\cos\theta\cos\theta'\sin\varphi\sin\varphi' \\ &\qquad \left. + r'^2\cos^2\theta'\sin^2\varphi'\right) \end{aligned} \tag{18}$$

For bound states we employ the usual wave functions in polar coordinates

$$\begin{aligned} \Psi_{nlm}(\vec{x}) = & \sqrt{\left(\frac{2Me^2}{n}\right)^3 \frac{2l+1}{\delta\pi n}\frac{(l-|m|)!}{(l+|m|)!}\frac{(n-l-1)!}{[(n+l)!]^3}} \\ & e^{im\varphi}P^m_l(\cos\theta)\left(\frac{2Me^2}{n}r\right)^l e^{-\frac{Me^2}{n}r}L^{2l+1}_{n-l-1}\left(\frac{2Me^2}{n}r\right) \end{aligned} \tag{19}$$

while for continuum levels we use the wave function in parabolic coordinates

$$\begin{aligned} \Psi_{i\lambda,l,m}(\vec{x}) = & \frac{(-)^{l+m}}{\sqrt{2p_0 r}}\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}\;\frac{\Gamma(l+i\lambda+1)}{(2l+1)!} \\ & e^{im\varphi}P^m_l(\cos\theta)e^{\pi\lambda/2}\;M_{-i\lambda,l+1/2}(2p_0 r) \end{aligned} \tag{20}$$

with $p_0 = Me^2/i\lambda$ where $M_{-i\lambda,l+1/2}$ are the Whittaker functions.

When the expressions of Eqs. (18), (19) and (20) are inserted into (16), and because the φ-dependence of the wave functions are always in the form of $e^{im\varphi}$ we have the integrals

$$\begin{aligned} &\int_0^{2\pi} d\varphi e^{-im\varphi}e^{im'\varphi}\sin^2\varphi = \pi\left[\delta_{mm'} - 1/2\left(\delta_{m,m'-2} + \delta_{m,m'+2}\right)\right] \\ &\int_0^{2\pi} d\varphi e^{-im\varphi}e^{im'\varphi}\sin\varphi\int_0^{2\pi} d\varphi' e^{im'\varphi}e^{-im\varphi}\sin\varphi' = \end{aligned} \tag{21}$$

$$\begin{aligned} &= -\frac{1}{4}(2\pi)^2\left(\delta_{m,m'+1} + \delta_{m,m'-1}\right)\left(\delta_{m,m'+1} + \delta_{m,m'-1}\right) \\ &= -\frac{1}{4}(2\pi)^2\left(\delta_{m,m'+1} + \delta_{m,m'-1}\right) \end{aligned} \tag{22}$$

Inspecting the remaining integrals over $d\theta, dr$ we observe that their values are almost equal for $\delta_{m,m'}$, $\delta_{m,m'-2}$ or $\delta_{m,m'+2}$ [14]. Thus the contributions coming from the terms involving integrals of type (21) are negligible compared to the ones involving (22); i.e., the largest contribution comes from the third term of Eq. (18). Thus the energy shift of the atom due to self energy in the presence of the solenoid is

$$\Delta E_n \simeq \int \frac{d\,\vec{k}}{(E-E)^2 - \vec{k}^2}\Delta^{NR}_{nlm}I(\vec{k}) \tag{23}$$

Here

$$\Delta^{NR}_{nlm}I(\vec{k}) \simeq -\frac{\pi^2\Lambda^2}{m^2R^2}\sum\!\!\!\!\!\!\int_{n'l'm'}\left(\delta_{m,m'+1} + \delta_{m,m'-1}\right)\vec{\zeta}^{\,nlm}_{\,n'l'm'}\cdot\vec{\zeta}^{\,n'l'm'}_{\,nlm} \tag{24}$$

and

$$\vec{\zeta}_{\substack{nlm \\ n'l'm'}} = \int_0^\infty dr r^3 \int_{-1}^{+1} d(\cos\theta) \cos\theta N_{nlm} N'_{n'l'm'} P_l^m(\cos\theta) R_{nl}(r) \frac{\vec{\nabla}}{i} \times P_{l'}^{m'}(\cos\theta) R_{nl}(r'), \tag{25}$$

Where $R_{nl}(r)$ is the radial part of the wave function and N is the normalization constant given by eqs. (19) and (20). After integrating over $d\,\vec{k}$, eq. (23) gives the change to the Lambshift due to "cavity"

$$\Delta(\Delta E_n) \cong \frac{e^2 \pi^3 \Lambda^2}{M^2 R^2} \sum_{n'} (\delta_{m,m'+1} + \delta_{m,m'-1}) \omega_{nn'} \vec{\zeta}_{nn'} \cdot \vec{\zeta}'_{nn'} \tag{26}$$

where n, n' stand for (nlm) and $(n'l'm')$, and $\omega_{nn'} = E_n - E'_n$.

Equation has to be compared to the energy change in the vicinity of a conducting plate at a distance R from the atom. There the factor $\pi^2\Lambda^2/R^2$ is replaced by $1/8\pi R^4$ (for $R \gg$ Bohr radius)[11]. Thus for a typical distance of $R \sim 5 \times 10^{-5}$ cm, the shift due to solenoid, taking $\Lambda = 1$, is about 10^7 times smaller than the parallel mirror case.

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