# PATH INTEGRAL FORMULATION OF QUANTUM ELECTRODYNAMICS FROM CLASSICAL PARTICLE TRAJECTORIES


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*Contents*



*Abstract*

The classical particle interactions based on a covariant action principle directly lead, after path integration to the perturbation expansion of QED and to Feynman graphs without the intermediary of quantized fields Both scalar electrodynamics and spinor electrodynamics are treated The latter is based on a covariant classical model of the Dirac electron which already provides the notion of antiparticles and zitterbewegung so that vacuum polarization and pair production can also be discussed We give a procedure on how classical spin leads to quantum spin in successive time-sliced integrations

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*Dedicated to R P. Feynman (1918–1988)*
*on the occasion of the 40th Anniversary of Path Integration*

## Introduction

There is an advantage in looking at familiar things from new points of view. Different formulations of a theory have always been very enlightening in physics. They usually have different starting points and a different set of ideas. Perturbative quantum electrodynamics and the ensuing rules for Feynman graph expansions can be based on at least three different approaches: (1) The standard operator product expansions in quantum field theory [1]; (2) the Green's function expansions [2] and (3) the $S$-matrix unitarity expansions [3]. In addition, there are various non-perturbative approaches [4, 5]. There is still another approach to perturbative QED, namely via the path integral quantization of classical particle trajectories. This is the subject matter of the present work, and, to our knowledge, has not been treated before. Of course, path integral methods have been applied to classical Maxwell and Dirac fields to perform second quantization with functional integrals $\int \mathscr{D}(\psi)\, \mathscr{D}(\bar{\psi})\, \mathscr{D}(A_\mu)$. But what we have in mind here is not to start from Dirac fields but from classical particle trajectories and quantize only once, and only the particles, with functional integrals $\int \mathscr{D}(x)\, \mathscr{D}(p)$, and by eliminating the electromagnetic field, to arrive at the Feynman rules.

The development of the idea of path integral formulation of quantum mechanics has been reviewed by Feynman [6], where he also traces his attempts to apply this method to the Dirac electron with spin and his solution of the one-dimensional case with zig-zag paths of a charge with velocity $c$ [7] (one-dimensional zitterbewegung). He also recounts his original idea of quantizing classical electrodynamics starting from a classical action-at-a-distance Lagrangian between the particles. These two problems have remained unsolved; in the meantime Feynman, Schwinger and Tomonaga have developed quantum electrodynamics from other points of view, based on the second quantization of fields. Remembering that "perhaps a thing is simple if you can describe it fully in several different ways without immediately knowing that you are describing the same thing" [6], we address ourselves to these two problems: path integral quantization of a spinning electron in 3 + 1 dimensions and path integral formulation of both scalar and spinor quantum electrodynamics starting from an interparticle classical Lagrangian

Up to now there is no derivation of quantum electrodynamics directly from a classical particle action To quote Feynman and Hibbs [7] "for a relativistic particle with spin the amplitude cannot be described by a simple path integral based on any reasonable action", and "... and because spin takes discrete values it has been difficult to suggest for it continuous paths subsequently to be summed over so as to obtain the quantum propagator" [8]. We can now present a solution to these problems thanks to a better covariant classical model for the electron, one which contains spin and the notion of antiparticles already in the classical domain, and is in a one-to-one correspondence with the Dirac theory in canonical quantization (i.e. Poisson brackets replaced by commutators) [9]. If there is a one-to-one correspondence in the symplectic formulation there should also be a one-to-one correspondence between classical paths and the Dirac propagator. In fact the Dirac propagator $K$ is a matrix, and each matrix element $K_{\alpha\beta}$ refers to a propagator with initial and final spin indices being fixed, in addition to fixed initial and final positions. We shall give a prescription of how the spin indices change and are summed along the path. The application of this new classical model of the Dirac electron to the path integral of the free electron has already been initiated [10], as well as a preliminary application to perturbative theory [11].

In section 2 we give a general perturbation theory of the path integral formulation for a system of $N$ particles with arbitrary free Lagrangian $L_0$ and interactions of the type $j_\mu A^\mu$ via any vector fields $A_\mu$ The general expansion formulas for the propagator are obtained in terms of free propagators The theory is then applied in section 3 to scalar particles which have no spin complications but additional $A^2$-interactions, which are separately treated, and then in section 4 to spinor particles, where we discuss in detail the treatment of the spin indices along the paths [12], we also treat processes involving vacuum polarization and pair production Finally we discuss the external field problems

This paper is intended and so written that a reader with a knowledge of classical mechanics and of the notion of path integration only can completely reconstruct perturbative quantum electrodynamics

## 1. General theory

### *1 1 The action of classical electrodynamics*

In order to derive the covariant Feynman rules by path integration it is most convenient to start from an action integral over an invariant time parameter $\tau$ For $N$ particles interacting via the electromagnetic field $A_\mu$ the action will be taken to be

$$W = \sum_{k=1}^{N} \left( \int \mathrm{d}\tau_k \, L_k^0 - \int \mathrm{d}x \; j_k^\mu(x) A_\mu(x) \right) - \frac{1}{4} \int \mathrm{d}x \; F^{\mu\nu} F_{\mu\nu} \,, \tag{1}$$

where $L_k^0$ is the free Lagrangian of the $k$th particle and $j_k^\mu(x)$ its current, both of which we shall specify later for scalar and spinor particles We use units with $c = \hbar = 1$

Using the resultant Maxwell equations and assuming that the fields vanish at infinity we can write by partial integration

$$-\frac{1}{4} \int \mathrm{d}x \; F_{\mu\nu} F^{\mu\nu} = +\frac{1}{2} \int \mathrm{d}x \; A_\mu(x) \sum_{k=1}^{N} j_k^\mu(x) \,, \tag{2}$$

because $F^{\mu\nu}{}_{,\nu} = -j^\mu_{\mathrm{tot}}(x)$ Hence the action (1) can be written in a simpler form

$$W = \sum_{k=1}^{N} \left( \int \mathrm{d}\tau_k \, L_k^0 - \frac{1}{2} \int \mathrm{d}x \; j_k^\mu(x) A_\mu(x) \right) \tag{3}$$

The reduced Lagrangian (3) can no longer be used, however, to derive the equation of motion

Next we express the field $A_\mu(x)$ in (3) in terms of the current itself by solving the Maxwell equations in the gauge $A^\mu{}_{,\mu} = 0$, that is

$$\Box A_\mu = \sum_{k=1}^{N} j_k^\mu \,, \tag{4}$$

as

$$A_\mu(x) = \int \mathrm{d}y \; D(x-y) \sum_k j_k^\mu(x) \,, \tag{5}$$

where the Green function $D(x-y)$ will be specified explicitly This leads to the third form of the action,

$$W=\sum_{k=1}^{N}\left(\int \mathrm{d}\tau_k\, L_k^0-\frac{1}{2}\sum_{m=1}^{N}\int \mathrm{d}x\, j_k^{\mu}(x)D(x-y)j_{\mu}^{m}(y)\,\mathrm{d}y\right)=W_0+W_{\mathrm{int}} \tag{6}$$

If there is, in addition, a fixed, non-dynamical external field $A_{\mu}^{\mathrm{ext}}$ (whose sources are far away, or are non-dynamical), we have one more term in the action,

$$W_{\mathrm{ext}}=-\int \mathrm{d}x\, A_{\mu}^{\mathrm{ext}}(x)\sum_{k=1}^{N} j_k^{\mu}(x) \tag{7}$$

Equation (6) parallels in field theory the action of $N$ interacting Dirac fields $\psi_j(x)$, when $A_{\mu}(x)$ is eliminated [13]. Note further that here self-interactions are included ($k=m$ terms), in contrast to Feynman–Wheeler absorber theory, because the boundary conditions are different.

### 1 2 *Path integrals for propagators of N classical particles*

We propose to determine the propagator $K$ for a system of $N$ particles with action (6) by the following path integral in phase space:

$$K(a,b)=\prod_{r=1}^{N}\int_0^{\infty}\mathrm{d}S_r\,\mathrm{e}^{-\mathrm{i}f(m)S_r}\int \mathscr{D}(r)\exp\left(\mathrm{i}\int_0^{S_r}\mathrm{d}\tau_r\, L_r^0\right)\mathrm{e}^{\mathrm{i}W_{\mathrm{int}}} \tag{8}$$

Here $(a,b)$ stands for $(x_{1a},x_{2a},\ldots,x_{Na})$ and $(x_{1b},x_{2b},\ldots,x_{Nb})$, the co-ordinates of the fixed initial and final points $a$ and $b$. Each particle $r$ has an arrival "time" $S_r$ between $a$ and $b$, and then we integrate over all possible arrival times $S_r$ from 0 to $\infty$ with a measure of integration $\exp[-\mathrm{i}f(m)S_r]$ for these different arrival times. This factor can also be put as a mass term into $L_r^0$. The function $f(m)$ shall later be specified for scalar and spinor particles. The measures $\mathscr{D}(1),\mathscr{D}(2),\ldots,\mathscr{D}(N)$ stand for the actual path measure for each particle defined by the time-sliced integral elements in phase space,

$$\mathscr{D}(\cdot)=\prod_{j=1}^{n+1}\frac{\mathrm{d}^4p_j}{(2\pi)^4}\prod_{j=1}^{n}\mathrm{d}^4x_j\prod_{j=1}^{n+1}\mathrm{d}\xi_j\,,\quad n\to\infty,\ \varepsilon(n+1)=S_r\,, \tag{9}$$

where $(\mathrm{d}\xi_j)$ stands for the other possible internal co-ordinates, again to be specified later in the case of spinning particles. In the case of spinning particles in fact $K(a,b)$ will have spinor indices which are for the moment suppressed.

The formulation of QED based on (8) is, we believe, new. We shall explain, as we go along, how this equation is interpreted and applied to explicit processes.

Next we write the term $\exp(\mathrm{i}W_{\mathrm{int}})$

### 1 3. *The expansion of the interaction $W_{\mathrm{int}}$*

We first use the Fourier expansion of the $D$ function,

$$D(x-y)=-\frac{1}{(2\pi)^4}\int\frac{\mathrm{d}k}{k^2}\,\mathrm{e}^{-\mathrm{i}k(x-y)}\,, \tag{10}$$

where the contour of integration in $k^0$-space (corresponding to the boundary conditions) will be specified later, to express $W_{\rm int}$ of eq. (6) as

$$W_{\rm int} = \frac{1}{2}\sum_{l\,m}\frac{1}{(2\pi)^4}\int \mathrm{d}x\,\mathrm{d}y\; j_l^\mu(x)\,\mathrm{e}^{-ikx} j_\mu^m(y)\,\mathrm{e}^{iky}\,\frac{\mathrm{d}k}{k^2}$$
$$= \frac{1}{2}\sum_{l,m}\frac{1}{(2\pi)^4}\int \frac{\mathrm{d}k}{k^2}\, j_l^\mu(k) j_\mu^m(-k)\,, \tag{11}$$

where $j^\mu(k)$ is the Fourier transform of the current $j^\mu(x)$

In classical mechanics the currents have $\delta$-functions along the world line of the particles, i e ,

$$j^\mu(x) = e\int \mathrm{d}\tau\,\delta(x-x(\tau))\,j^\mu(\tau)\,, \tag{12}$$

so that we evaluate the Fourier integrals over $\mathrm{d}x$, $\mathrm{d}y$ immediately and obtain a purely "action-at-a-distance" interaction in terms of the particle co-ordinates alone,

$$W_{\rm int} = \frac{1}{2}\sum_{l\,m}\frac{e_l e_m}{(2\pi)^4}\int \mathrm{d}\tau'\,\mathrm{d}\tau''\,\frac{\mathrm{d}k}{k^2}\; j_l^\mu(\tau')\,\mathrm{e}^{-ikx_l(\tau')} j_\mu^m(\tau'')\,\mathrm{e}^{ikx_m(\tau'')}$$
$$= \frac{1}{2}\sum_{l\,m}\frac{e_l e_m}{(2\pi)^4}\int \mathrm{d}\tau'\,\mathrm{d}\tau''\,\frac{\mathrm{d}k}{k^2}\; M_l(k,\tau')\;\; M_m(-k,\tau'')$$
$$= \int \mathrm{d}\tau'\,\mathrm{d}\tau''\,L_{\rm int} = \int \mathrm{d}\tau_l\,\mathrm{d}\tau_m \sum_{l\,m} e_l e_m L(x_l(\tau_l), x_m(\tau_m))\,, \tag{13}$$

where we have introduced the quantities

$$M_l^\mu(k,\tau) \equiv j_l^\mu(\tau)\,\mathrm{e}^{-ikx_l(\tau)}\,, \tag{14}$$

and the dot product $(M\cdot M)$ means the usual four-vector product

Now in the time-sliced path integrals we have the exponential phase factors

$$\exp\Big(i\int \mathrm{d}\tau'\,\mathrm{d}\tau''\,L_{\rm int}\Big) = \exp\Big(i\int \mathrm{d}\tau'\,\mathrm{d}\tau''\sum_{l\,m} e_l e_m L(x_l(\tau'), x_m(\tau''))\Big)\,, \tag{15}$$

where $L(l, m)$ represents the interaction between particles $l$ and $m$ (including the self-interaction $l = m$) as defined in (13) For a perturbative treatment of the path integral we now expand the phase factor (15) in powers of $(e_l e_m)$,

$$\mathrm{e}^{iW_{\rm int}} = \prod_{l,m}\sum_{r_{lm}=0}^{\infty}(e_l e_m)^{r_{lm}}\,\frac{i^{r_{lm}}}{r_{lm}!}\Big(\Big(\int \mathrm{d}\tau_l\,\mathrm{d}\tau_m\,L(x_l(\tau_l), x_m(\tau_m))\Big)\Big)^{r_{lm}} \tag{16}$$

Thus we have a product of sums of two-body interactions (including self-interaction) of increasing order A typical term in the expansion of order $r$ in the coupling parameter is of the form

$$(e_1 e_2)^r\,\frac{i^r}{r!}\Big(\Big(\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,L(1,2)\Big)\Big)^r = \frac{(e_1 e_2)^r}{(2\pi)^4 r!}\,i^r\Big(\Big(\int \mathrm{d}\tau'\,\mathrm{d}\tau''\,\frac{\mathrm{d}k}{k^2}\,M(k,\tau')\cdot M(-k,\tau'')\Big)\Big)^r \tag{17}$$

This expansion inserted into the propagator (8) allows us to write $K(a, b)$ also as a sum of propagators,

$$K(a, b) = \sum_{\{r_{lm}\}=0}^{\infty} K_{\{r_{lm}\}}(a, b) , \tag{18}$$

where a summand

$$K_{\{r_{lm}\}}(a, b) = K_{r_{11}, r_{12}, r_{22}, r_{13}, \ldots}(a, b)$$

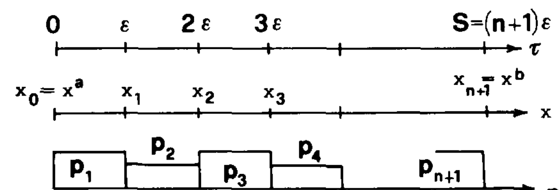
characterized by a set of integers $r_{11}, r_{12}, \ldots$ corresponds to a diagram in which there are $r_{11}$ interactions between particle 1 with itself (self-energy), $r_{12}$ interactions between particle 1 and particle 2, etc. *The whole of the combinatorial rules of perturbation theory is embedded in the expansion formulae* (16) *or* (18). This is more direct and simple than in quantum field theory. The complete expression for each term in (18) is given by

$$\begin{aligned} K_{r_{11}, r_{12}, r_{22}, \ldots}(a, b) = {} & \prod_{r=1}^{N} \int_0^\infty \mathrm{d}S_r\, \mathrm{e}^{-\mathrm{i}f(m_r)S_r} \mathscr{D}(r) \exp\left(\mathrm{i} \int_0^{S_r} \mathrm{d}\tau_r\, L_r^0\right) \\ & \times (e_1 e_1)^{r_{11}} \frac{\mathrm{i}^{r_{11}}}{r_{11}!} \left(\int \mathrm{d}\tau'\, \mathrm{d}\tau''\, L(1,1)\right)^{r_{11}} (e_1 e_2)^{r_{12}} \frac{\mathrm{i}^{r_{12}}}{r_{12}!} \left(\int \mathrm{d}\tau'\, \mathrm{d}\tau''\, L(1,2)\right)^{r_{12}} \cdots \end{aligned} \tag{19}$$

This term is of order $(e_1 e_1)^{r_{11}} (e_1 e_2)^{r_{12}} \cdots$ in the coupling constants

## 1 4 *Evaluation of path integrals*

For each particle the time interval and the corresponding position and momentum intervals are sliced as follows:




We first consider all paths of a given duration $S_r$ for each particle $r$, and then integrate $S_r$ from zero to infinity as indicated in (19). There are $n$ $x_j$-integrations (the end points $x_a$, $x_b$ being fixed) but $(n+1)$ $p$-integrations

With this time slicing the free particle phase factor in (19) is written as usual as

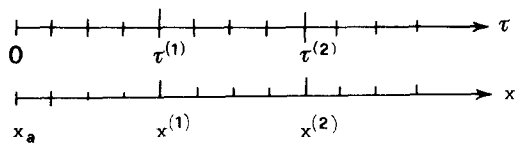
$$\exp\left(\mathrm{i} \sum_{j=1}^{n+1} \Delta\tau_r^{(j)}\, L_r^{0(j)}\right) ,$$

and $\mathscr{D}(r)$ as given in eq. (9). If there were no interactions then we would perform the $\mathrm{d}p_j$, $\mathrm{d}x_j$ integrations at the discrete points indicated above. We see from (19) that a particle $s$ which does not interact in a given order with other particles ($r_{sl}=0$) contributes a free propagator as a factor to the total $K$. In a graph it will be represented by an unconnected straight line. We also see that each interaction occurs at two time points $\tau_l'$ and $\tau_m''$ between a pair of particles $l$ and $m$ (including the interaction of a particle with itself), and this $r_{lm}$ times. Graphically this will be represented by a dotted line connecting the particles $l$ and $m$ at times $\tau_l'$ and $\tau_m''$, respectively. This situation is particularly suitable for the path integral formulation as defined by time slicing of the whole time interval $S$ into $n$ intervals of length $\varepsilon$. It means that we have free particle path integrals everywhere except at the points $\tau_l^{(1)}, \tau_l^{(2)}, \tau_l^{(3)}, \ldots$ Since we know the free propagators, the path integrals in perturbation theory are reduced to just *ordinary integrals* at the interaction points.

An interaction term in (19) gives a factor

$$(e_l e_m)^r \frac{1^r}{r!}\left(\int_0^{S_l} \mathrm{d}\tau_l^{(1)} \int_0^{S} \mathrm{d}\tau_m^{(1)}\, L(x_l(\tau_l^{(1)}), x_m(\tau_m^{(1)}))\right)^r \exp\left(i\sum_j \left(\Delta\tau_l^{(j)} L_l^{0(j)} + \Delta\tau_m^{(j)} L_m^{0(j)}\right)\right)$$

We now identify $\mathrm{d}\tau_l^{(1)}, \mathrm{d}\tau_m^{(1)}$ with any one of the time slicings so that $L(l,m)$ depends on $x_l$ and $p_l$ at these time points. This means that the particle $l$ moves freely except at definite points $\tau_l^{(1)}, \tau_l^{(2)}, \ldots$ Thus perturbation theory means that we should use $L_l^{0(j)}$ at all time points except at certain interaction points where we have interaction terms.




We denote the interaction points by a superscript. Thus we select interaction times $\tau^{(1)}, \tau^{(2)}, \ldots$, use free path integrals *between* these times, and then use *ordinary integrals* at the interaction points.

For the free motion of a particle between any two times $\tau^{(1)}$ and $\tau^{(2)}$ we introduce *the propagator kernels F* by

$$\int \mathscr{D}(\cdot) \exp\left(i\int_{\tau^{(1)}}^{\tau^{(2)}} \mathrm{d}\tau\, L^0\right) \equiv \int \frac{\mathrm{d}p}{(2\pi)^4} F(p, x^{(2)}-x^{(1)}, \tau^{(2)}-\tau^{(1)})\, \mathrm{d}x^{(2)}$$

$$\equiv \int \mathrm{d}x^{(2)}\, F(x^{(2)}-x^{(1)}, \tau^{(2)}-\tau^{(1)}) \tag{20}$$

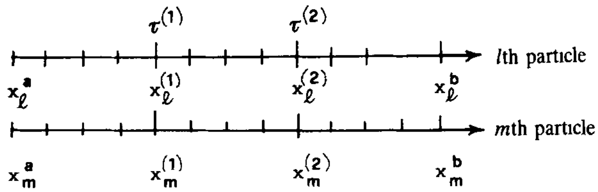
We shall give the explicit expression for these free propagator kernels $F$ for scalar and spinor particles later. In terms of these intermediate functions $F$, eq. (19) takes the following form

$$K_{r_{11}\, r_{12}\, r_{13}\, \ldots}(a,b) = \prod_{r=1}^{N} \int_0^\infty \mathrm{d}S_r\, \mathrm{e}^{-if(m_r)S_r} \prod_{l\, m} \frac{1^{r_{lm}}}{r_{lm}!}$$

$$\times \int F_l(p_l^{(1)}, x_l^{(1)}-x_l^a, \tau_l^{(1)}-\tau_l^{(0)}) \frac{\mathrm{d}p_l^{(1)}}{(2\pi)^4}\, \mathrm{d}x_l^{(1)}\, F_m(p_m^{(1)}, x_m^{(1)}-x_m^a, \tau_m^{(1)}-\tau_m^{(0)}) \frac{\mathrm{d}p_m^{(1)}}{(2\pi)^4}\, \mathrm{d}x_m^{(1)}$$

$$
\begin{aligned}
&\times (e_l e_m) L(x_l^{(1)}(\tau_l^{(1)}), x_m^{(1)}(\tau_m^{(1)})\, \mathrm{d}\tau_l^{(1)}\, \mathrm{d}\tau_m^{(1)} \\
&\times \int F_l(p_l^{(2)}, x_l^{(2)} - x_l^{(1)}, \tau_l^{(2)} - \tau_l^{(1)}) \frac{\mathrm{d}p_l^{(2)}}{(2\pi)^4}\, \mathrm{d}x_l^{(2)}\, F_m(p_m^{(2)}, x_m^{(2)} - x_m^{(1)}, \tau_m^{(2)} - \tau_m^{(1)}) \frac{\mathrm{d}p_m^{(2)}}{(2\pi)^4}\, \mathrm{d}x_m^{(2)} \\
&\times (e_l e_m) L(x_l^{(2)}, x_m^{(2)})\, \mathrm{d}\tau_l^{(2)}\, \mathrm{d}\tau_m^{(2)} \cdots\cdots \\
&\qquad\qquad\qquad\qquad (r_{lm} \text{ times}) \\
&\times \int F_l(p_l, x_l^b - x_l^{(r_{lm})}, S_l - \tau_l^{(r_{lm})}) \frac{\mathrm{d}p_l}{(2\pi)^4}\, F_m(p_m, x_m^b - x_m^{(r_{lm})}, S_m - \tau_m^{(r_{lm})}) \frac{\mathrm{d}p_m}{(2\pi)^4}
\end{aligned}
\tag{21}
$$

Graphically, times and co-ordinates of the $l$th and $m$th particles are divided as follows:




Between the interaction points we have the corresponding $F$ functions and we integrate at each interaction point over $\mathrm{d}\tau^{(i)}$, $\mathrm{d}x^{(i)}$ and $\mathrm{d}p^{(i)}$.

It is best to write some lowest-order terms explicitly and evaluate the propagators, then the general expression (21) and its evaluation will be quite clear. We shall give the lowest-order self-interaction of particle 1 with itself ($r_{11} = 1$, all other $r_{lm} = 0$), and the lowest-order interaction between particles 1 and 2 ($r_{12} = 1$, all other $r_{lm} = 0$)

### 1 5 Examples

*Example* 1 Self-interaction to lowest order of particle 1. Equation (21) now simplifies to (with $e_m = e_l = e$, $x_m^{(1)} \to x_1^{(2)}$, $p_m^{(1)} \to p_1^{(2)}$, etc.)

$$
\begin{aligned}
K_{r_{11}=1}(a,b) = K_0^{(2)} K_0^{(3)} \cdots K_0^{(N)} \int_0^\infty &\mathrm{d}S_1\, \mathrm{e}^{-\mathrm{i}f(m_1)S_1} \\
&\times \int \mathrm{d}\tau_1^{(1)}\, \mathrm{d}\tau_1^{(2)}\, F_1(p_1^{(1)}, x_1^{(1)} - x_1^a, \tau_1^{(1)} - \tau_1^{(0)}) \frac{\mathrm{d}p_1^{(1)}}{(2\pi)^4}\, \mathrm{d}x_1^{(1)} \\
&\times \mathrm{i}e^2 L(x_1^{(1)}, x_1^{(2)}) F_1(p_1^{(2)}, x_1^{(2)} - x_1^{(1)}, \tau_1^{(2)} - \tau_1^{(1)}) \frac{\mathrm{d}p_1^{(2)}}{(2\pi)^4}\, \mathrm{d}x_1^{(2)} \\
&\times F_1(p, x_1^b - x_1^{(2)}, S_1 - \tau_1^{(2)}) \frac{\mathrm{d}p}{(2\pi)^4}\,,
\end{aligned}
\tag{22}
$$

with

$$
L(x_1^{(1)}, x_1^{(2)}) = \frac{1}{2}\, \frac{1}{(2\pi)^4} \int \frac{\mathrm{d}k}{k^2}\, j_1^\mu(\tau_1^{(1)})\, \mathrm{e}^{-\mathrm{i}kx_1(\tau_1^{(1)})} j_\mu^1(\tau_1^{(2)})\, \mathrm{e}^{\mathrm{i}kx_1(\tau_1^{(2)})}\,. \tag{23}
$$

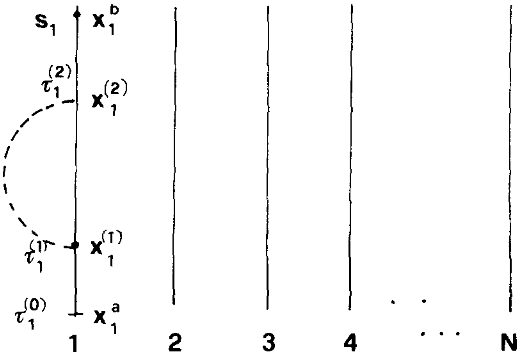



Fig 1

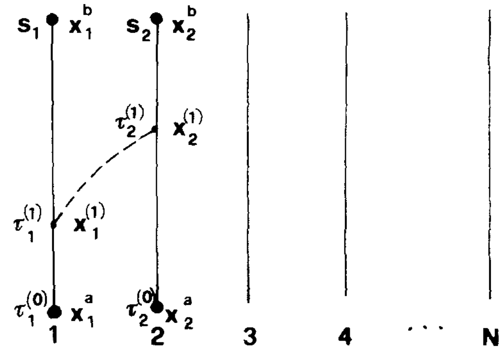



Fig 2

The graphical representation of (22) is immediate (fig 1) All particles propagate freely with their free propagators $K_0^{(i)}$, except particle 1, which undergoes interactions at two points $x_1^{(1)}$ and $x_2^{(2)}$ at times $\tau_1^{(1)}$ and $\tau_1^{(2)}$, respectively, with itself and we integrate over these times and over these intermediate interaction phase space points

*Example* 2 Lowest-order interaction between particles 1 and 2 We have from (21) immediately

$$K_{r_{12}=1}(a,b) = K_0^{(3)} K_0^{(4)} \cdots K_0^{(N)} \int_0^\infty \mathrm{d}S_1\, \mathrm{d}S_2\, \mathrm{e}^{-\mathrm{i}f(m_1)S_1}\, \mathrm{e}^{-\mathrm{i}f(m_2)S_2}$$

$$\times \int \mathrm{d}\tau_1^{(1)}\, \mathrm{d}\tau_2^{(1)}\, F_1(p_1^{(1)}, x_1^{(1)} - x_1^a, \tau_1^{(1)} - \tau_1^0)\, \frac{\mathrm{d}p_1^{(1)}}{(2\pi)^4}\, \mathrm{d}x_1^{(1)}\, F_2(p_2^{(1)}, x_2^{(1)} - x_2^a, \tau_2^{(1)} - \tau_2^0)\, \frac{\mathrm{d}p_2^{(1)}}{(2\pi)^4}\, \mathrm{d}x_2^{(1)}$$

$$\times \mathrm{i} e_1 e_2 L(x_1^{(1)}(\tau_1^{(1)}), x_2^{(1)}(\tau_2^{(1)}))$$

$$\times F_1(p_1, x_1^b - x_1^{(1)}, S_1 - \tau_1^{(1)})\, \frac{\mathrm{d}p_1}{(2\pi)^4}\, F_2(p_2, x_2^b - x_2^{(1)}, S_2 - \tau_2^{(1)})\, \frac{\mathrm{d}p_2}{(2\pi)^4}\,, \qquad (24)$$

or graphically, as shown in fig 2

## 1 6 *Further development of the general formula Time integrations*

We shall further develop the general expression (21) into a composition formula in terms of free propagators $K_0$ The $F$ function introduced in eq (20), which is related to the free propagator, will be assumed to be of the following form

$$F(p, \Delta x, \Delta\tau) = \mathrm{e}^{\mathrm{i}p\,\Delta x}\, \mathrm{e}^{-\mathrm{i}\tilde{f}(p)\,\Delta\tau} \qquad (25)$$

We shall see indeed that for relativistic scalar and spinor particles this is the case, and shall determine the function $\tilde{f}(p)$, which goes together with the function $f(m)$ in the measure in the $\int \mathrm{d}S$ integration [see eq (21)]. Now we can perform the $\mathrm{d}\tau_l^{(1)}\, \mathrm{d}\tau_m^{(1)}$ integrations

In the example of the self-energy, eq (22), to begin with, we use the following well-known rule for the change of the limits of the $\tau$ integrations

$$\int_0^{S_1}\int_0^{S_1} \mathrm{d}\tau_1^{(1)}\,\mathrm{d}\tau_1^{(2)} = \frac{1}{2!}\int_0^{S_1}\mathrm{d}\tau_1^{(2)}\int_0^{\tau_1^{(2)}}\mathrm{d}\tau_1^{(1)} \tag{26}$$

We drop the subscript 1 everywhere from now on for simplicity of writing since there are no other subscripts in this expression. Consequently eq. (22) becomes

$$\begin{aligned}K_{r_{11}=1}(a,b) = {}& K_0^{(2)}K_0^{(3)}\cdots K_0^{(N)}\int_0^\infty \mathrm{d}S\,\mathrm{e}^{-\mathrm{i}f(m)S}\\ &\times\frac{1}{2}\int_0^S \mathrm{d}\tau^{(2)}\int_0^{\tau^{(2)}}\mathrm{d}\tau^{(1)}\,\frac{\mathrm{d}p^{(1)}}{(2\pi)^4}\,\mathrm{d}x^{(1)}\exp[-\mathrm{i}\tilde{f}(p^{(1)})(\tau^{(1)}-\tau^0)]\exp[\mathrm{i}p^{(1)}(x^{(1)}-x^a)]\\ &\times \mathrm{i}e^2L(x^{(1)},x^{(2)})\exp[-\mathrm{i}\tilde{f}(p^{(2)})(\tau^{(2)}-\tau^{(1)})]\exp[\mathrm{i}p^{(2)}(x^{(2)}-x^{(1)})]\,\frac{\mathrm{d}p^{(2)}}{(2\pi)^4}\\ &\times \mathrm{d}x^{(2)}\exp[-\mathrm{i}\tilde{f}(p)(S-\tau^{(2)})]\exp[\mathrm{i}p(x^b-x^{(2)})]\,\frac{\mathrm{d}p}{(2\pi)^4}\,.\end{aligned} \tag{27}$$

(a) $\mathrm{d}\tau^{(1)}$ *integration*:

$$I^{(1)} = \int_0^{\tau^{(2)}}\mathrm{d}\tau^{(1)}\exp\{-\mathrm{i}\tau^{(1)}[\tilde{f}(p^{(1)})-\tilde{f}(p^{(2)})]\} = \frac{\exp\{-\mathrm{i}\tau^{(2)}[\tilde{f}(p^{(1)})-\tilde{f}(p^{(2)})]\}-1}{-\mathrm{i}[\tilde{f}(p^{(1)})-\tilde{f}(p^{(2)})]}\,.$$

(b) $\mathrm{d}\tau^{(2)}$ *integration*:

$$\begin{aligned}I^{(2)} &= \int_0^S \mathrm{d}\tau^{(2)}\,\frac{\exp\{-\mathrm{i}\tau^{(2)}[\tilde{f}(p^{(1)})-\tilde{f}(p^{(2)})]\}-1}{-\mathrm{i}[\tilde{f}(p^{(1)})-\tilde{f}(p^{(2)})]}\exp\{-\mathrm{i}\tau^{(2)}[\tilde{f}(p^{(2)})-\tilde{f}(p)]\}\\ &= \frac{1}{-\mathrm{i}[\tilde{f}(p^{(1)})-\tilde{f}(p^{(2)})]}\left(\frac{\exp\{-\mathrm{i}S[\tilde{f}(p^{(1)})-\tilde{f}(p)]\}-1}{-\mathrm{i}[\tilde{f}(p^{(1)})-\tilde{f}(p)]} - \frac{\exp\{-\mathrm{i}S[\tilde{f}(p^{(2)})-\tilde{f}(p)]\}-1}{-\mathrm{i}[\tilde{f}(p^{(2)})-\tilde{f}(p)]}\right).\end{aligned}$$

(c) Now we can perform the $\mathrm{d}S$ integration:

$$\begin{aligned}\int_0^\infty \mathrm{d}S\,\mathrm{e}^{-\mathrm{i}f(m)S}\,\mathrm{e}^{-\mathrm{i}\tilde{f}(p)S}\,I^{(2)} = {}& \frac{1}{-[\tilde{f}(p^{(1)})-\tilde{f}(p^{(2)})][\tilde{f}(p^{(1)})-\tilde{f}(p)]}\\ &\times\int_0^\infty \mathrm{d}S\,\{\exp\{-\mathrm{i}S[f(m)+\tilde{f}(p^{(1)})]\}-\exp\{-\mathrm{i}S[f(m)+\tilde{f}(p)]\}\}\\ &+\frac{1}{[\tilde{f}(p^{(1)})-\tilde{f}(p^{(2)})][\tilde{f}(p^{(2)})-\tilde{f}(p)]}\\ &\times\int_0^\infty \mathrm{d}S\,\{\exp\{-\mathrm{i}S[f(m)+\tilde{f}(p^{(2)})]\}-\exp\{-\mathrm{i}S[f(m)+\tilde{f}(p)]\}\}\end{aligned}$$

$$= -\frac{1}{[\tilde{f}(p^{(1)})-\tilde{f}(p^{(2)})][\tilde{f}(p^{(1)})-\tilde{f}(p)]}\left(\frac{1}{-i[f(m)+\tilde{f}(p^{(1)})]}-\frac{1}{-i[f(m)+\tilde{f}(p)]}\right)$$
$$+\frac{1}{[\tilde{f}(p^{(1)})-\tilde{f}(p^{(2)})][\tilde{f}(p^{(2)})-\tilde{f}(p)]}\left(\frac{1}{-i[f(m)+\tilde{f}(p^{(2)})]}-\frac{1}{-i[f(m)+\tilde{f}(p)]}\right)$$

These four terms algebraically add up simply to

$$i\{[f(m)+\tilde{f}(p)][f(m)+\tilde{f}(p^{(1)})][f(m)+\tilde{f}(p^{(2)})]\}^{-1}$$

Inserting these integrations in (27) we have

$$K_{r_{11}=1}(a,b)=K_0^{(2)}K_0^{(3)}\cdots K_0^{(N)}\int \mathrm{d}x^{(1)}\,\frac{\mathrm{d}p^{(1)}}{(2\pi)^4}\,\mathrm{d}x^{(2)}\,\frac{\mathrm{d}p^{(2)}}{(2\pi)^4}\,\frac{\mathrm{d}p}{(2\pi)^4}$$
$$\times \tfrac{1}{2}ie^2L(x^{(1)},x^{(2)})i\,\frac{\exp[ip^{(1)}(x^{(1)}-x^a)]\exp[ip^{(2)}(x^{(2)}-x^{(1)})]\exp[ip(x^b-x^{(2)})]}{[f(m)+\tilde{f}(p)][f(m)+\tilde{f}(p^{(1)})][f(m)+\tilde{f}(p^{(2)})]} \tag{28}$$

The three $p$-integrations nicely factorize and we recognize the three free-particle propagators We introduce the expressions

$$K_0(\Delta x)\equiv\int\frac{\mathrm{d}p}{(2\pi)^4}\,\frac{e^{ip\,\Delta x}}{f(m)+\tilde{f}(p)}\,, \tag{29}$$

whence we can write (28) in the final form

$$K_{r_{11}=1}(a,b)=K_0^{(2)}K_0^{(3)}\cdot\quad K_0^{(N)}\int \mathrm{d}x_1^{(1)}\,\mathrm{d}x_1^{(2)}\,K_0^{(1)}(x_1^{(1)}-x_1^a)$$
$$\times\tfrac{1}{2}(ie_1^2)iL(x_1^{(1)},x_1^{(2)})K_0^{(1)}(x_1^{(2)}-x_1^{(1)})K_0^{(1)}(x_1^b-x_1^{(2)})\,, \tag{30}$$

with $L(x_1^{(1)},x_1^{(2)})$ as given in eq. (23) This now establishes the rule for the diagram of fig 1 in terms of three free propagators and one pair of interaction points $x_1^{(1)}$ and $x_1^{(2)}$

In the example of fig 2, eq (24), we use again eq (25) and perform time integrations,

$$I^{(1)}=\int_0^{S_1}\mathrm{d}\tau_1^{(1)}\exp\{-i\tau_1^{(1)}[\tilde{f}(p_1^{(1)}-\tilde{f}(p_1)]\}=\frac{\exp\{-iS_1[\tilde{f}(p_1^{(1)})-\tilde{f}(p_1)]\}-1}{-i[\tilde{f}(p_1^{(1)})-\tilde{f}(p_1)]}$$

Exactly the same expression holds for the $\mathrm{d}\tau_2^{(1)}$ integration, with subscripts 2, but superscripts the same For the $S_1$ integration we have

$$\int_0^\infty \mathrm{d}S_1 \exp\{-\mathrm{i}S_1[f(m_1)+\tilde{f}(p_1)]\}\, I^{(1)}$$
$$= \frac{1}{-\mathrm{i}[\tilde{f}(p_1^{(1)})-\tilde{f}(p_1)]}\left(\frac{1}{-\mathrm{i}[f(m_1)+\tilde{f}(p_1^{(1)})]} - \frac{1}{-\mathrm{i}[f(m_1)+\tilde{f}(p_1)]}\right)$$
$$= \{[f(m_1)+\tilde{f}(p_1^{(1)})][f(m_1)+\tilde{f}(p_1)]\}^{-1}\,.$$

Similarly for $\int \mathrm{d}S_2$. Hence eq. (24) becomes

$$K_{r_{12}=1}(a,b) = K_0^{(3)}K_0^{(4)}\cdots K_0^{(N)} \int \mathrm{d}x_1^{(1)}\,\mathrm{d}x_2^{(1)}\, \frac{\exp[\mathrm{i}p_1^{(1)}(x_1^{(1)}-x_1^a)]}{f(m_1)+\tilde{f}(p_1^{(1)})}\,\frac{\mathrm{d}p_1^{(1)}}{(2\pi)^4}$$
$$\times \frac{\exp[\mathrm{i}p_2^{(1)}(x_2^{(1)}-x_2^a)]}{f(m_2)+\tilde{f}(p_2)}\,\frac{\mathrm{d}p_2^{(1)}}{(2\pi)^4}\,\tfrac{1}{2}\mathrm{i}e_1e_2L(x_1^{(1)},x_2^{(1)})$$
$$\times \frac{\exp[\mathrm{i}p_1(x_1^b-x_1^{(1)})]}{f(m_1)+\tilde{f}(p_1)}\,\frac{\mathrm{d}p_1}{(2\pi)^4}\,\frac{\exp[\mathrm{i}p_2(x_2^b-x_2^{(1)})]}{f(m_2)+\tilde{f}(p_2)}\,\frac{\mathrm{d}p_2}{(2\pi)^4}\,,$$

which is, using (29),

$$K_{r_{12}=1}(a,b) = K_0^{(3)}K_0^{(4)}\cdots K_0^{(N)}\int \mathrm{d}x_1^{(1)}\,\mathrm{d}x_2^{(1)}\,K_0^{(1)}(x_1^{(1)}-x_1^a)$$
$$\times K_0^{(2)}(x_2^{(1)}-x_2^a)\,\tfrac{1}{2}\mathrm{i}e_1e_2L(x_1^{(1)},x_2^{(1)})K_0^{(1)}(x_1^b-x_1^{(1)})K_0^{(2)}(x_2^b-x_2^{(1)}) \tag{31}$$

It is now clear how to interpret and evaluate the general expression (21) for more complicated diagrams. The perturbation theory developed here so far is applicable to any particle theory in which the total action is written as $W = W_0 + W_{\text{int}}$, and where the free propagator is known at least in form, as in eq. (20), e.g. interacting oscillators, etc The more specific forms, eqs. (30) and (31), use the expression of the free-particle propagators in the form (25); but the function $\tilde{f}(p)$ in (25) is still quite general. In the next two sections we shall discuss more specific particle theories.

## 2. Scalar electrodynamics

### 2.1. The action of scalar electrodynamics

We shall now specify for a covariant scalar particle the free Lagrangian $L^0$, the current $j_\mu(\tau)$ and the interaction term $W_{\text{int}}$ used in the previous formulation. The action in the configuration space with respect to an invariant parameter $\tau$ is

$$W = -\int \mathrm{d}\tau\,(\tfrac{1}{2}\dot{x}^2 + eA\cdot\dot{x}) - \frac{1}{4}\int \mathrm{d}x\,F_{\mu\nu}F^{\mu\nu}\,, \tag{32}$$

which leads to the equations of motion $\ddot{x}_\mu = eF_{\mu\nu}\dot{x}^\nu$ and $F^{\mu\nu}{}_{,\nu} = -j_\mu$. It can be written as

$$W = -\int \mathrm{d}\tau\,[\tfrac{1}{2}(\dot{x}+eA)^2 - \tfrac{1}{2}e^2A^2] - \frac{1}{4}\int \mathrm{d}x\,F_{\mu\nu}F^{\mu\nu}\,. \tag{32'}$$

We shall rather use the phase space action

$$W = -\int \mathrm{d}\tau \, (p \cdot x - \lambda p^2 + 2e\lambda p \cdot A - \lambda e^2 A^2) - \frac{1}{4}\int \mathrm{d}x \, F_{\mu\nu}F^{\mu\nu} \tag{33}$$

($\lambda$ is a Lagrange multiplier which will drop out), which in the phase space formulation of the path integrals is equivalent to the previous forms and also gives the correct equations of motion $p_\mu = eA_{\nu,\mu}x^\nu$ and $x_\mu = p_\mu - eA_\mu$ ($\lambda = \frac{1}{2}$) In appendices A and B we obtain the action (33) as a limit from the action of the spinning particle

Now due to the $A^2$ term, the interaction is not simply of the form $j^\mu A_\mu$, or if it is written in this form, $j^\mu$ would be itself $A_\mu$ dependent. This is of course the well-known peculiarity of scalar electrodynamics and will lead, as we shall see, to two-photon, two-fermion vertices (seagull diagrams) Since we are using perturbation theory, that is, the particle is free between two interaction points, the best procedure is to take the current to be the usual,

$$j_\mu(\tau) = e\bar{p}_\mu(\tau) = e\,\tfrac{1}{2}[p_\mu(\tau_-) + p_\mu(\tau_+)]\,, \tag{34}$$

where $p_\mu(\tau_-)$ and $p_\mu(\tau_+)$ are momenta in the time slice before and after the interaction point $\tau$, and then to evaluate the $A^2$ interaction separately According to eqs (5) and (10)

$$A_\mu(x) = -\frac{1}{(2\pi)^4}\int \frac{\mathrm{d}k}{k^2} \sum_j e_j \int \mathrm{d}\tau_j \, \bar{p}^j_\mu(\tau_j) \exp[-\mathrm{i}k(x - x_j(\tau_j))] \tag{35}$$

Using this we can derive an expression for $A^2_\mu(x)$,

$$A^2_\mu(x) = \sum_{j,m} \frac{e_j e_m}{(2\pi)^8} \int \frac{\mathrm{d}k}{k^2}\frac{\mathrm{d}q}{q^2}\,\mathrm{d}\tau_j\,\mathrm{d}\tau_m \times \exp[-\mathrm{i}k(x - x_j(\tau_j))]\,\bar{p}^\mu_j(\tau_j)\bar{p}_{\mu m}(\tau_m)\exp[-\mathrm{i}q(x - x_m(\tau_m))] \tag{36}$$

Care must be taken in evaluating the product of two distributions at the limiting point $\tau_j = \tau_m$ Consider the $A^2$ term for a single particle,

$$A^2_\mu(x) = e^2 \int \mathrm{d}y\,\mathrm{d}z\,\mathrm{d}\tau'\,\mathrm{d}\tau''\, D(x-y)D(x-z)\bar{p}^\mu(\tau')\bar{p}_\mu(\tau'')\delta(y - x(\tau'))\delta(z - x(\tau''))$$

For $\tau' \neq \tau''$, the $y$- and $z$-integrations can be performed giving

$$A^2_\mu(x) = e^2 \int \mathrm{d}\tau'\,\mathrm{d}\tau''\, D(x - x(\tau'))D(x - x(\tau''))\bar{p}(\tau')\cdot\bar{p}(\tau'')$$

For $\tau' \to \tau''$ we have the product of two distributions In this case we evaluate the integrals for $\tau' \neq \tau''$ and then take the limit

$$\tau' \to \tau'' \quad \text{such that} \quad \bar{p}^\mu(\tau'_+)\bar{p}_\mu(\tau''_-) \to \infty\,, \tag{37}$$

i e the intermediate momentum between $\tau'$ and $\tau''$, because only at infinite momentum can the two particles be at the same point. This procedure gives correctly the seagull diagrams of scalar electrodynamics as we show in the next section.

The general formulas (21) and (22), (24) or (30), (31) hold for the $j \cdot A$ interaction. We have just to insert for (23)

$$L(x_l^{(i)}, x_m^{(j)}) = \frac{1}{2} \frac{1}{(2\pi)^4} \int \frac{\mathrm{d}k}{k^2} \bar{p}_l^{\mu}(\tau^{(i)})\, \mathrm{e}^{-\mathrm{i}kx_l(\tau^{(i)})}\, \bar{p}_{\mu m}(\tau^{(j)})\, \mathrm{e}^{\mathrm{i}kx_m(\tau^{(j)})} \tag{38}$$

Furthermore, the measure in the $S$ integration in (8) is now

$$\mathrm{e}^{-\mathrm{i}f(m)S} = \mathrm{e}^{-\mathrm{i}m^2 S} \tag{39}$$

[a mass term $m^2$ could be added into the Lagrangian instead of (30), as we have noted, i.e., $L = p\dot{x} - \lambda(p^2 - m^2)$]. The free scalar propagator kernel function $F$ in (20), i.e.,

$$F(p, \Delta\tau, \Delta x) = \int \mathscr{D}(1) \exp\left(\mathrm{i} \int_{\tau_1}^{\tau_2} \mathrm{d}\tau\, (p \cdot \dot{x} - \tfrac{1}{2} p^2)\right), \tag{40}$$

can be evaluated to give (see appendix C)

$$F(p, \Delta\tau, \Delta x) = \mathrm{e}^{-(\mathrm{i}/2)p^2 \Delta\tau}\, \mathrm{e}^{\mathrm{i}p\,\Delta x}, \tag{41}$$

so that eq. (29),

$$K_0 = \int \frac{\mathrm{d}p}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}p\,\Delta x}}{f(m) + \tilde{f}(p)},$$

becomes the standard propagator

$$K_0(\Delta x) = \int \frac{\mathrm{d}p}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}p\,\Delta x}}{p^2 + m^2}. \tag{42}$$

Thus the diagram of figs. 1 and 2 are now given explicitly from (30) and (31) as

$$\begin{aligned} K_{r_{11}=1}(a, b) = K_0^{(2)} \cdots K_0^{(N)} \int \mathrm{d}x^{(1)}\, \mathrm{d}x^{(2)} \frac{\mathrm{d}p^{(1)}}{(2\pi)^4} \frac{\exp[\mathrm{i}p^{(1)}(x^{(1)} - x^a)]}{(p^{(1)})^2 + m^2}\, \tfrac{1}{2}(\mathrm{i}e^2)\mathrm{i} \\ \times \frac{1}{2} \frac{1}{(2\pi)^4} \frac{\mathrm{d}k}{k^2}\, \mathrm{e}^{-\mathrm{i}kx^{(1)}} \bar{p}^{(1)} \cdot \bar{p}^{(2)}\, \mathrm{e}^{\mathrm{i}kx^{(2)}} \\ \times \frac{\mathrm{d}p^{(2)}}{(2\pi)^4} \frac{\exp[\mathrm{i}p^{(2)}(x^{(2)} - x^{(1)})]}{(p^{(2)})^2 + m^2} \frac{\mathrm{d}p}{(2\pi)^4} \frac{\exp[\mathrm{i}p(x^b - x^{(2)})]}{p^2 + m^2}, \end{aligned}$$

or, performing the $x$ integrations and the $p^{(1)}$ and $p^{(2)}$ integrations,

$$K_{r_{11}=1}(a,b)=\prod_{j=2}^{N}K_0^{(j)}\,\tfrac{1}{4}e^2\int\frac{\mathrm{d}k}{(2\pi)^4}\,\frac{\mathrm{d}p}{(2\pi)^4}\,\frac{\mathrm{e}^{\mathrm{i}p(x^b-x^a)}}{(p^2+m^2)^2[(p-k)^2+m^2]}\,\frac{(p-\frac{1}{2}k)^2}{k^2}\,, \tag{43}$$

and, similarly,

$$\begin{aligned}K_{r_{12}=1}(a,b)&=\prod_{j=3}^{N}K_0^{(j)}\int\mathrm{d}x_1^{(1)}\,\mathrm{d}x_2^{(1)}\,\frac{\exp[\mathrm{i}p_1^{(1)}(x_1^{(1)}-x_1^a)]}{(p_1^{(1)})^2+m_1^2}\,\frac{\mathrm{d}p_1^{(1)}}{(2\pi)^4}\,\frac{\exp[\mathrm{i}p_2^{(1)}(x_2^{(1)}-x_2^a)]}{(p_2^{(2)})^2+m_2^2}\,\frac{\mathrm{d}p_2^{(1)}}{(2\pi)^4}\\&\quad\times\frac{1}{2}(\mathrm{i}e_1e_2)\,\frac{1}{2}\,\frac{\mathrm{d}k}{(2\pi)^4}\,\mathrm{e}^{-\mathrm{i}kx_1^{(1)}}\,\frac{\bar p_1^{(1)}\cdot\bar p_2^{(1)}}{k^2}\,\mathrm{e}^{\mathrm{i}kx_2^{(1)}}\\&\quad\times\frac{\exp[\mathrm{i}p_1(x_1^b-x_1^{(1)})]}{p_1^2+m_1^2}\,\frac{\mathrm{d}p_1}{(2\pi)^4}\,\frac{\exp[\mathrm{i}p_2(x_2^b-x_2^{(1)})]}{p_2^2+m_2^2}\,\frac{\mathrm{d}p_2}{(2\pi)^4}\\&=\tfrac{1}{4}(\mathrm{i}e_1e_2)\int\frac{\mathrm{d}k}{(2\pi)^4}\,\frac{\mathrm{d}p_1}{(2\pi)^4}\,\frac{\mathrm{d}p_2}{(2\pi)^4}\,\frac{\exp[\mathrm{i}p_1(x_1^b-x_1^a)]}{p_1^2+m_1^2}\,\frac{\exp[\mathrm{i}p_2(x_2^b-x_2^a)]}{p_2^2+m_2^2}\prod_{j=3}^{N}K_0^{(j)}\\&\quad\times\frac{\exp[\mathrm{i}k(x_2^a-x_1^a)]}{k^2}\,\frac{(p_1+\frac{1}{2}k)\cdot(p_2-\frac{1}{2}k)}{[(p_1+k)^2+m_1^2][(p_2-k)^2+m_2^2]}\,,\end{aligned} \tag{44}$$

which can also be written alternatively in terms of the incoming momenta $p_1^{(1)}$ and $p_2^{(1)}$,

$$\begin{aligned}K_{r_{12}=1}(a,b)&=\prod_{j=3}^{N}K_0^{(j)}\,\tfrac{1}{4}(\mathrm{i}e_1e_2)\int\frac{\mathrm{d}k}{(2\pi)^4}\,\frac{\mathrm{d}p_1^{(1)}}{(2\pi)^4}\,\frac{\mathrm{d}p_2^{(1)}}{(2\pi)^4}\,\frac{\exp[\mathrm{i}p_1^{(1)}(x_1^b-x_1^a)]}{(p_1^{(1)})^2+m_1^2}\,\frac{\exp[\mathrm{i}p_2^{(1)}(x_2^b-x_2^a)]}{(p_2^{(1)})^2+m_2^2}\\&\quad\times\frac{\exp[\mathrm{i}k(x_2^b-x_1^b)]}{k^2}\,\frac{(p_1^{(1)}-\frac{1}{2}k)\cdot(p_2^{(1)}+\frac{1}{2}k)}{[(p_1^{(1)}-k)^2+m_1^2][(p_2^{(1)}+k)^2+m_2^2]}\end{aligned} \tag{45}$$

### *2 2 $A^2$ interactions in scalar electrodynamics*

The $A^2$ interaction, according to (33), is

$$W_{\mathrm{int}}=-\frac{1}{2}\sum_j e_j^2\int\mathrm{d}\tau\,A_j^2(x_j)\,, \tag{46}$$

with $A^2(x_j)$ given by (36) We thus have the phase factor

$$\exp\Big(-\tfrac{1}{2}\mathrm{i}\int\mathrm{d}\tau\sum_i e_i\sum_{j,m}\frac{e_je_m}{(2\pi)^8}\int\frac{\mathrm{d}k}{k^2}\,\frac{\mathrm{d}q}{q^2}\,\mathrm{d}\tau_j\,\mathrm{d}\tau_m$$
$$\times\exp[-\mathrm{i}(k+q)\cdot x_i(\tau)]\;\bar p_j^{\mu}(\tau_j)\exp[\mathrm{i}k\cdot x_j(\tau_j)]\;\bar p_{\mu m}(\tau_m)\exp[\mathrm{i}q\cdot x_m(\tau_m)]\Big) \tag{47}$$

Expanding this, we reduce the path integral (8) to free propagators multiplied with ordinary integrals at the interaction points as in eq (21) It is best again to begin with lowest-order terms We shall show that the lowest-order terms from (47) reproduce the graphs shown in fig. 3, and in the limits the graphs of fig 4 For fig. 3a we take in the expansion of (47) the first non-trivial term with $i=1$ (particle 1) and

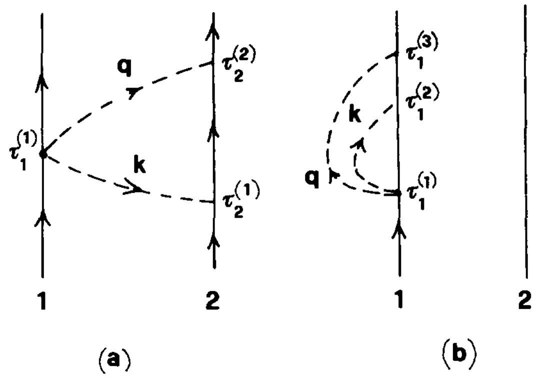



Fig 3

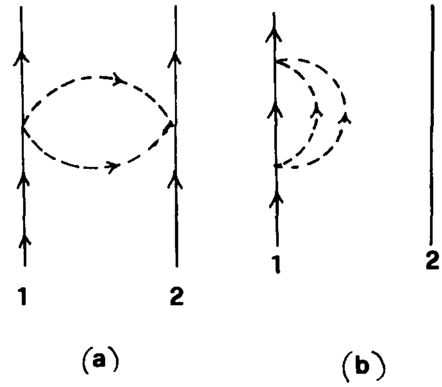



Fig 4

$j=2$, $m=2$ (particle 2) The propagator according to the general formula (21) is then, using (41),

$$
\begin{aligned}
K(a,b) = \int_0^\infty\!\!\int & \mathrm{d}S_1\,\mathrm{d}S_2\,\mathrm{e}^{-\mathrm{i}m_1^2S_1}\,\mathrm{e}^{-\mathrm{i}m_2^2S_2}\,\mathrm{d}\tau_1^{(1)}\,\mathrm{d}\tau_2^{(1)}\,\mathrm{d}\tau_2^{(2)} \\
& \times \exp[\mathrm{i}p_1^{(1)}(x_1^{(1)}-x_1^a)-\mathrm{i}(p_1^{(1)})^2(\tau_1^{(1)}-\tau_1^0)]\,\frac{\mathrm{d}p_1^{(1)}}{(2\pi)^4}\,\mathrm{d}x_1^{(1)} \\
& \times \exp[\mathrm{i}p_1(x_1^b-x_1^{(1)})-\mathrm{i}p_1^2(S_1-\tau_1^{(1)})]\,\frac{\mathrm{d}p_1}{(2\pi)^4} \\
& \times (-\tfrac{1}{2}\mathrm{i})(e_1^2e_2^2)\,\frac{\mathrm{d}k}{(2\pi)^4}\,\frac{\mathrm{d}q}{(2\pi)^4}\,\frac{\mathrm{e}^{-\mathrm{i}(k+q)x_1^{(1)}}}{k^2q^2}\,\mathrm{e}^{\mathrm{i}kx_2^{(1)}}\,\mathrm{e}^{\mathrm{i}qx_2^{(2)}}\,\bar{p}_2^{(1)}\cdot\bar{p}_2^{(2)} \\
& \times \exp[\mathrm{i}p_2^{(1)}(x_2^{(1)}-x_2^a)-\mathrm{i}(p_2^{(1)})^2(\tau_2^{(1)}-\tau_2^0)]\,\frac{\mathrm{d}p_2^{(1)}}{(2\pi)^4}\,\mathrm{d}x_2^{(1)} \\
& \times \exp[\mathrm{i}p_2^{(2)}(x_2^{(2)}-x_2^{(1)})-\mathrm{i}(p_2^{(2)})^2(\tau_2^{(2)}-\tau_2^{(1)})]\,\frac{\mathrm{d}p_2^{(2)}}{(2\pi)^4}\,\mathrm{d}x_2^{(2)} \\
& \times \exp[\mathrm{i}p_2(x_2^b-x_2^{(2)})-\mathrm{i}p_2^2(S_2-\tau_2^{(2)})]\,\frac{\mathrm{d}p_2}{(2\pi)^4}\,.
\end{aligned}
$$

The time integrations are performed first and then the $\mathrm{d}x^{(i)}$ integrations, finally the $\mathrm{d}p^{(i)}$ integrations (see appendix D), and we are left with ($\tau_1^{(0)}=\tau_2^{(0)}=0$)

$$
\begin{aligned}
K(a,b) = (-\tfrac{1}{2}\mathrm{i})(-\tfrac{1}{2}\mathrm{i})(e_1^2e_2^2)\int & \frac{\mathrm{d}p_1}{(2\pi)^4}\,\frac{\exp[\mathrm{i}p_1(x_1^b-x_1^a)]}{p_1^2+m_1^2}\,\frac{\mathrm{d}k}{(2\pi)^4}\,\frac{\mathrm{d}q}{(2\pi)^4}\,\frac{(p_2-\frac{1}{2}q)\cdot(p_2-q-\frac{1}{2}k)}{k^2q^2} \\
& \times\frac{\exp[\mathrm{i}p_2(x_2^b-x_2^a)]}{p_2^2+m_2^2}\,\frac{\mathrm{d}p_2}{(2\pi)^4}\,\frac{\exp[\mathrm{i}(k+q)(x_2^a-x_1^a)]}{[(p_1+k+q)^2+m_1^2][(p_2-q)^2+m_2^2][(p_2-q-k)^2+m_2^2]}\,.
\end{aligned}
\tag{48}
$$

Similarly, for the self-energy diagram, fig 3b, we have $i = 1$, as well as $j = m = 1$ (all particle 1) Hence from (47) (dropping the subscript 1, since only one particle is involved)

$$K(a, b) = \int_0^\infty dS\, e^{-im^2 S}\, d\tau^{(1)}\, (\tfrac{1}{2}i)\, \frac{e^2 \cdot e \cdot e}{(2\pi)^8}\, \frac{dk}{k^2}\, \frac{dq}{q^2}\, d\tau^{(2)}\, d\tau^{(3)}$$
$$\times \exp[ip^{(1)}(x^{(1)} - x^a) - i(p^{(1)})^2(\tau^{(1)} - \tau^0)]\, \frac{dp^{(1)}}{(2\pi)^4}\, dx^{(1)}$$
$$\times \exp[ip^{(2)}(x^{(2)} - x^{(1)} - i(p^{(2)})^2(\tau^{(2)} - \tau^{(1)})]\, \frac{dp^{(2)}}{(2\pi)^4}\, dx^{(2)}$$
$$\times e^{-i(k+q)x^{(1)}}\, \bar{p}^{(2)} \cdot \bar{p}^{(3)}\, e^{ikx^{(2)}}\, e^{iqx^{(3)}}$$
$$\times \exp[ip^{(3)}(x^{(3)} - x^{(2)}) - i(p^{(3)})^2(\tau^{(3)} - \tau^{(2)})]\, \frac{dp^{(3)}}{(2\pi)^4}\, dx^{(3)}$$
$$\times \exp[ip(x^b - x^{(3)}) - ip^2(S - \tau^{(3)})]\, \frac{dp}{(2\pi)^4}\,,$$

giving, after lengthy, but by now straightforward integrations

$$K = (\tfrac{1}{2}i)e^4 \int \frac{dk}{(2\pi)^4}\, \frac{dq}{(2\pi)^4}\, \frac{1}{k^2 q^2}\, \frac{e^{ip(x^b - x^a)}}{(p^2 + m^2)^2}\, \frac{dp}{(2\pi)^4}\, \frac{(p - q - \frac{1}{2}k)\ (p - \frac{1}{2}q)}{[(p - k - q)^2 + m^2][(p - q)^2 + m^2]} \tag{49}$$

The results (43), (44) or (45), (48) and (49) agree with those of standard quantum electrodynamics (cf Bjorken and Drell, refs [1, 2])

## 2 3 Seagull terms

The limiting "seagull" diagrams of figs 4a and 4b are obtained from (48) and (49), by letting the two points $\tau_2^{(1)}$ and $\tau_2^{(2)}$ and $\tau_1^{(2)}$, $\tau_1^{(3)}$, respectively, approach each other In (48) the limit is

$$(p_2 - q)^2 \to \infty \quad \text{or} \quad \frac{(p_2 - q + \frac{1}{2}q) \cdot (p_2 - q - \frac{1}{2}k)}{(p_2 - q)^2} \to 1$$

In (49),

$$(p - q)^2 \to \infty \quad \text{or} \quad \frac{(p - q - \frac{1}{2}k)\ (p - q + \frac{1}{2}q)}{(p - q)^2} \to 1$$

Then

$$K = -\tfrac{1}{4}e_1^2 e_2^2 \int \frac{dp_1}{(2\pi)^4}\, \frac{dp_2}{(2\pi)^4}\, \frac{dk}{(2\pi)^4}\, \frac{dq}{(2\pi)^4}\, \frac{\exp[ip_1(x_1^b - x_1^a)]}{p_1^2 + m_1^2}\, \frac{\exp[ip_2(x_2^b - x_2^a)]}{p_2^2 + m_2^2}$$
$$\times \frac{\exp[i(k + q)(x_2^a - x_1^a)]}{k^2 q^2[(p_1 + k + q)^2 + m_1^2][(p_2 - k - q)^2 + m_2^2]}\,, \tag{50}$$

and

$$K=\tfrac{1}{8}\mathrm{i}e^4\int\frac{\mathrm{d}p}{(2\pi)^4}\,\frac{\mathrm{d}k}{(2\pi)^4}\,\frac{\mathrm{d}q}{(2\pi)^4}\,\frac{\mathrm{e}^{\mathrm{i}p(x^b-x^a)}}{(p^2+m^2)^2}\,\frac{1}{k^2q^2[(p-k-q)^2+m^2]}\,. \tag{51}$$

## 3. Spinor electrodynamics

### 3.1 The action for classical spinning particles

We have to introduce spin variables for the electron into the classical action principle. The action is given by our starting equation (1). We have to specify now the free phase space action $L_k^0$ and the current $j_k^\mu$ for the $k$th particle. They are

$$L_k^0=-\frac{\lambda_k}{2\mathrm{i}}(\dot{\bar{z}}_k z_k-\bar{z}_k\dot{z}_k)+p_\mu^k(\dot{x}_k^\mu-\bar{z}_k\gamma^\mu z_k)\,, \tag{52}$$

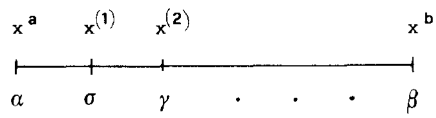
and

$$j_k^\mu(x)=e_k\bar{z}_k\gamma^\mu z_k\delta(x^\mu-x_k^\mu(\tau))\,.$$

Here $(x_\mu, p_\mu)$ are the conjugate co-ordinate and momentum variables, and $(z, \mathrm{i}\bar{z})$ another pair of conjugate (internal) spin variables; $z(\tau)$ is a c-number four-component spinor and $\bar{z}=z^+\gamma^0$; $\lambda_k$ is a constant of the dimension of an action. Thus the whole phase space consists of $\Gamma=(x_\mu, p_\mu; z, \bar{z})$, and $M\otimes\mathbb{C}_4$ is the configuration space $z\in\mathbb{C}_4$. In the above action $p_\mu$ can also be viewed as Lagrange multipliers for the constraints $\dot{x}_\mu=\bar{z}\gamma_\mu z$ The Lagrangian (52) describes a symplectic system; the Hamiltonian, Poisson brackets, equations of motion, and other group properties of this particle have been given in ref. [9]. The variables $\bar{z}, z$ can be substituted by spin variables so that another set of dynamical variables are $\Gamma'=(x_\mu, p_\mu, v_\mu, s_{\mu\nu})$ and the velocities $\dot{x}_\mu=v_\mu$ are independent of the momenta $p_\mu$, a property of the spinning electron. The theory has also been generalized to curved space [14], to Kaluza-Klein higher-dimensional form [15], and the world line has also been generalized to strings and membranes [16]. The canonical quantization leads precisely to Dirac equations [9], so does a Schrodinger picture quantization [17]. The path integral quantization of the free particle has been initiated [10, 11]; we shall elaborate it here in more detail in connection with the perturbation theory

### 3.2. Path integrals with spin

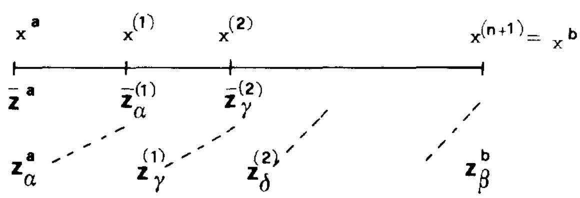
The general theory of section 1 applies, in particular, the basic formula (21) and all other examples. All we have to do is to specify the $F$ function (20) defined in the free propagator and expressed as eq. (25). In the definition of path integrals for spinless particles one simply integrates at time-sliced intermediate points over $\mathrm{d}x^{(j)}$ and $\mathrm{d}p^{(j)}$ with the measure given by the classical action, eq. (8). Now we have also spin variables, $\bar{z}_\alpha$, $z_\beta$. We will integrate over these, but must account also for the spin indices $\alpha, \beta, \ldots$. We shall obtain a procedure to adjust the spin indices at adjacent intermediate points and thereby also obtain an interpretation for the matrix indices $\alpha$, $\beta$ of the quantum propagator $K_{\alpha\beta}(a, b)$. It represents the propagator for fixed initial spin $\alpha$ at the initial point $x^a$ and fixed final spin $\beta$ at the final point $x^b$ Starting from $x^a$ with $d\bar{z}_\alpha$ we take the action which takes us to the spin component $\sigma$, then starting with $\sigma$ at $x^{(1)}$ we go to spin component $\gamma$, etc




But we must add up the amplitudes over all intermediate $\sigma$, $\gamma$, values, keeping $\alpha$ and $\beta$ fixed This much for free propagators At the interaction points we further have to insert, according to eq (21), the interaction terms $L(x^{(i)}, x^{(j)})$, which by (23) and (52) are

$$L(x_l^{(i)}, x_m^{(j)}) = \frac{1}{2} \int \frac{dk}{(2\pi)^4} \frac{1}{k^2} e_l \bar{z}_l \gamma^\mu z_l \, e^{-ikx_l^{(i)}} e_m \bar{z}_m \gamma_\mu z_m \, e^{ikx_m^{(j)}} \tag{53}$$

We now write this procedure explicitly in formulas Consider a section of free propagator between two interaction points,




At each intermediate point $x^{(j)}$ we have one $d\bar{z}_{\alpha_j}^{(i)}$ and one $dz_{\beta_j}^{(j)}$ integration, the indices are as shown above This is the meaning of the propagator matrix element $K_{\alpha\beta}$ We have thus, with eq (52)

$$\begin{aligned} K_{\alpha_0\beta_{n+1}}(a, b) = \int \prod_{j=1}^{n+1} \frac{dp^{(j)}}{(2\pi)^4} \prod_{j=1}^{n} dx^{(j)} \prod_{j=1}^{n+1} \frac{d\bar{z}_{\alpha_j}^{(j)}}{2\pi} \prod_{j=1}^{n} (i\lambda \, dz_{\beta_j}^{(j)}) \\ \times \exp\{ip^{(j)}(x^{(j)} - x^{(j-1)}) - i\bar{z}_{\alpha_j}^{(j)}[(i\lambda I + \varepsilon p^{(j)} \gamma)_{\alpha_j\beta_j} z_{\beta_j}^{(j)} - i\lambda z_{\beta_{j-1}}^{(j-1)}]\} \end{aligned} \tag{54}$$

In the last step we have used the identity

$$\tfrac{1}{2} i\lambda(\bar{z}z - \bar{z}z) = \tfrac{1}{2} i\lambda \frac{d}{d\tau}(\bar{z}z) - i\lambda\bar{z}z \tag{55}$$

The total derivative $(d/d\tau)(\bar{z}z)$ can always be absorbed into an overall normalization or measure of the path integral, since it depends only on the end points [10] Classically $(\bar{z}z)$ is a constant of the motion if equations of motion are used [9].

The $x^{(j)}$ and $p^{(j)}$ integrations can be done as before, giving

$$p^{(1)} = p^{(2)} = \cdots = p^{n+1} = p \,, \qquad \int \frac{dp}{(2\pi)^4} e^{ip(x^b - x^a)}$$

Now one of the factors in the $z$, $\bar{z}$ integrations has the form

$$I=\int \frac{\mathrm{d}\bar{z}_\alpha^{(j)}}{2\pi}\, \mathrm{i}\lambda\, \mathrm{d}z_\beta^{(j)} \exp[-\mathrm{i}\bar{z}_\alpha^{(j)}(\mathrm{i}\lambda A_{\alpha\beta} z_\beta^{(j)} - B_{\alpha\beta'} z_{\beta'}^{(j-1)})]\,,$$

where $A_{\alpha\beta} = (I + (\varepsilon/\mathrm{i}\lambda)p\cdot\gamma)_{\alpha\beta}$, $B_{\alpha\beta'} = \mathrm{i}\lambda\delta_{\alpha\beta'}$, and can be integrated first over $\mathrm{d}\bar{z}_\alpha$ and then over $\mathrm{d}z_\beta$ as follows.

$$\begin{aligned}
I &= \mathrm{i}\lambda \int \mathrm{d}z_\beta^{(j)}\, \delta[\mathrm{i}\lambda A_{\alpha\beta} z_\beta^{(j)} - B_{\alpha\beta'} z_{\beta'}^{(j-1)}] \\
&= \int \mathrm{d}z_\beta^{(j)}\, \frac{1}{A_{\alpha\beta}}\, \delta\Big(z_\beta^{(j)} - \frac{B_{\alpha\beta'}}{A_{\alpha\beta}}\, z_{\beta'}^{(j-1)}\Big) \\
&= \frac{1}{A_{\alpha\beta}} = \frac{1}{[I + (\varepsilon/\mathrm{i}\lambda)p\cdot\gamma]_{\alpha\beta}} = \Big(I - \frac{\varepsilon}{\mathrm{i}\lambda}\, p\cdot\gamma\Big)_{\alpha\beta} \Big(1 + \frac{\varepsilon^2}{\lambda^2}\, p^2 + \cdots\Big) \\
&= (A^{-1})_{\alpha\beta}[1 + \mathrm{O}(\varepsilon^2)]
\end{aligned} \tag{56}$$

We shall neglect terms of the order of $\varepsilon^2$, because *in time-sliced path integration an action must be used which is correct to order* $\varepsilon$

At the successive points we will similarly get the factors

$$\Big(I - \frac{\varepsilon}{\mathrm{i}\lambda}\, p\cdot\gamma\Big)_{\alpha\rho} \Big(I - \frac{\varepsilon}{\mathrm{i}\lambda}\, p\cdot\gamma\Big)_{\rho\sigma} \cdots \Big(I - \frac{\varepsilon}{\mathrm{i}\lambda}\, p\cdot\gamma\Big)_{\sigma\beta}\,,$$

hence matrix multiplications, when the amplitudes are summed over intermediate spin indices $\rho, \sigma, \delta, \ldots,$

$$(\underbrace{A^{-1}\, A^{-1}\, A^{-1} \cdots A^{-1}}_{(n+1)\ \text{times}})_{\alpha\beta}\,.$$

Now we can evaluate the $F$ function in eq. (20), then evaluate eq (25) and finally the free propagator $K_0$. Choosing

$$f(m) = m/\lambda\,, \tag{57}$$

we have

$$\begin{aligned}
&\lim_{\substack{n\to\infty \\ \varepsilon\to 0 \\ (n+1)\varepsilon = S}} \int_0^\infty \Big(-\frac{\mathrm{i}}{\lambda}\Big)\, \mathrm{d}S\, \mathrm{e}^{-\mathrm{i}(m/\lambda)S} \Big[\Big(I - \frac{\varepsilon}{\mathrm{i}\lambda}\, p\cdot\gamma\Big)^{n+1}\Big]_{\alpha\beta} \\
&= -\frac{\mathrm{i}}{\lambda} \int_0^\infty \mathrm{d}S\, \mathrm{e}^{-\mathrm{i}(m/\lambda)S}\, \mathrm{e}^{-(S/\mathrm{i}\lambda)p\cdot\gamma} = -\mathrm{i} \int_0^\infty \frac{\mathrm{d}S}{\lambda}\, (\mathrm{e}^{-(S/\mathrm{i}\lambda)(p\cdot\gamma - m)})_{\alpha\beta} = \Big(\frac{1}{p\cdot\gamma - m}\Big)_{\alpha\beta}\,,
\end{aligned}$$

where we have used

$$\lim_{\substack{n\to\infty \\ \varepsilon\to 0 \\ (n+1)\varepsilon = S}} \Big(1 - \frac{\varepsilon}{\mathrm{i}\lambda}\, p\cdot\gamma\Big)^{n+1}_{\alpha\beta} = \lim\Big(1 - \frac{S}{\mathrm{i}\lambda(n+1)}\, p\cdot\gamma\Big)^{n+1}_{\alpha\beta} = (\mathrm{e}^{-(S/\mathrm{i}\lambda)p\cdot\gamma})_{\alpha\beta}\,.$$

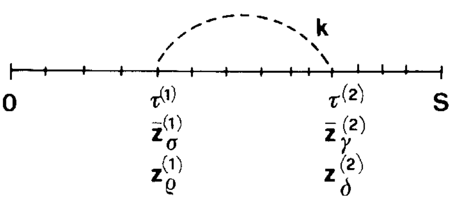
Consequently, in eq (25), $\tilde{f}(p)$ is now

$$\tilde{f}(p) = -\frac{1}{\lambda}\, p\cdot\gamma \tag{58}$$

and from (25) and (29)

$$F(p, \Delta\tau, \Delta x) = \mathrm{e}^{ip\cdot\Delta x + i\,\Delta\tau\,\gamma\cdot p}\,, \qquad K_0(\Delta x) = \int \frac{\mathrm{d}p}{(2\pi)^4}\,\frac{\mathrm{e}^{ip\cdot\Delta x}}{\gamma\cdot p - m}\,, \tag{58'}$$

It remains now to perform the ordinary integrals at the interaction points Let $\tau^{(1)}$, $\tau^{(2)}$ be two such points,




Now according to (21) and using (53) we have the following expression in the propagator

$$\begin{aligned}
&F_{\alpha\sigma}(p^{(1)}, \tau^{(1)} - \tau^0, x^{(1)} - x^a)\,\frac{\mathrm{d}p^{(1)}}{(2\pi)^4}\,\mathrm{d}x^{(1)}\,\mathrm{d}\bar{z}_\sigma^{(1)}\,\mathrm{d}z_\rho^{(1)}\,\mathrm{d}\tau^{(1)}\,\mathrm{d}\tau^{(2)} \\
&\times \bar{z}_\sigma^{(1)}\gamma^\mu_{\sigma\rho} z_\rho^{(1)}\,\mathrm{e}^{-ikx^{(1)}}\,\frac{\mathrm{d}k}{k^2}\,F_{\rho\gamma}(p^{(2)}, \tau^{(2)} - \tau^{(1)}, x^{(2)} - x^{(1)})\,\frac{\mathrm{d}p^{(2)}}{(2\pi)^4}\,\mathrm{d}x^{(2)}\,\mathrm{d}\bar{z}_\gamma^{(2)}\,\mathrm{d}z_\delta^{(2)} \\
&\times \bar{z}_\gamma^{(2)}\gamma_{\mu\gamma\delta} z_\delta^{(2)}\,\mathrm{e}^{ikx^{(2)}} F_{\delta\beta}(p, S - \tau^{(2)}, x^b - x^{(2)})\,\frac{\mathrm{d}p}{(2\pi)^4}
\end{aligned} \tag{59}$$

The interaction now multiplies the change of the spin index by a factor $\bar{z}_\sigma^{(1)}\gamma^\mu z_\rho^{(1)}$, without the interaction this factor would be one with the same $\mathrm{d}\bar{z}_\sigma^{(1)}\,\mathrm{d}z_\rho^{(1)}$ integrations. All the general formulas apply as before Since the interaction depends only on the $z$ co-ordinates, $\bar{z}\gamma^\mu z$, it is sufficient to look at the intermediate $z$-integrations; the intermediate $\mathrm{d}x^{(1)}$, $\mathrm{d}p^{(1)}$ integrations are as in free propagators So we have, for example,

$$\mathscr{I}^\mu_{\sigma\rho} \equiv i\lambda \int \frac{\mathrm{d}\bar{z}_\sigma^{(1)}}{2\pi}\,\mathrm{d}z_\rho^{(1)}\,\bar{z}_\sigma^{(1)}\gamma^\mu_{\sigma\rho} z_\rho^{(1)}\,\exp\{-i\bar{z}_\sigma^{(1)}[(i\lambda I + \varepsilon p^{(1)}\cdot\gamma)_{\sigma\rho} z_\rho^{(1)} - i\lambda z^{(0)}]\}\,, \tag{60}$$

which we write as the derivative of an exponential integral

$$\begin{aligned}
\mathscr{I}^\mu_{\sigma\rho} &= \gamma^\mu_{\sigma\rho}\,\frac{-1}{i\varepsilon}\,\frac{\mathrm{d}}{\mathrm{d}(p^{(1)}\cdot\gamma)_{\sigma\rho}} \\
&\times \int \frac{\mathrm{d}\bar{z}_\sigma^{(1)}}{2\pi}\, i\lambda\,\mathrm{d}z_\rho^{(1)}\,\exp\{-i\bar{z}_\sigma^{(1)}[(i\lambda I + \varepsilon p^{(1)}\cdot\gamma)_{\sigma\rho} z_\rho^{(1)} - i\lambda z^{(0)}]\}
\end{aligned}$$

The integrals have been done in (56). Hence

$$\mathcal{J}^{\mu}_{\sigma\rho} = \gamma^{\mu}_{\sigma\rho} \frac{-\lambda}{\varepsilon} \frac{\mathrm{d}}{\mathrm{d}(p^{(1)}\cdot\gamma)_{\sigma\rho}} \left(1 - \frac{\varepsilon}{\mathrm{i}\lambda} p^{(1)}\cdot\gamma\right)_{\sigma\rho} [1 + \mathrm{O}(\varepsilon^2)] ,$$

which in the limit $\varepsilon \to 0$ gives simply

$$\mathcal{J}^{\mu}_{\sigma\rho} = \gamma^{\mu}_{\sigma\rho} \tag{60'}$$

Compared to scalar electrodynamics the interaction vertex $(\bar{p}^{(1)}_{\mu} \cdot \bar{p}^{(2)\mu})$ is replaced by $(\gamma^{\mu}\cdot\gamma_{\mu})$; the former by the way is the correct non-relativistic limit of the latter. Formulas (30) and (31) now become explicitly, with the correct ordering of spin summations, and referring to figs. 1 and 2,

$$\begin{aligned} K_{r_{11}=1}(a,b) = \prod_{j=2}^{N} K_0^{(j)} \int \mathrm{d}x^{(1)}\,\mathrm{d}x^{(2)}\, K_0(p^{(1)}, x^{(1)} - x^{a}) \\ \times \frac{\mathrm{d}k}{k^2} \frac{1}{2} \frac{e^2}{(2\pi)^4} \gamma^{\mu}\, \mathrm{e}^{-\mathrm{i}kx^{(1)}} K_0(p^{(2)}, x^{(2)} - x^{(1)})\gamma_{\mu}\, \mathrm{e}^{\mathrm{i}kx^{(2)}} K_0(p, x^{b} - x^{(2)}) , \end{aligned} \tag{61}$$

with the matrix

$$K_0(p, \Delta x) = \frac{\mathrm{d}p}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}p\,\Delta x}}{\gamma\cdot p - m} , \tag{62}$$

and

$$\begin{aligned} K_{r_{12}=1}(a,b) = \prod_{j=3}^{N} K_0^{(j)} \int \mathrm{d}x_1^{(1)}\,\mathrm{d}x_2^{(1)}\, K_0(x_1^{(1)} - x_1^{a}) \\ \times \frac{1}{2} \frac{1}{(2\pi)^4} \frac{\mathrm{d}k}{k^2} e_1\gamma_{\mu}\, \mathrm{e}^{-\mathrm{i}kx_1^{(1)}} K_0(x_1^{b} - x_1^{(1)}) \\ \times K_0(x_2^{(1)} - x_2^{a}) e_2 \gamma^{\mu}\, \mathrm{e}^{\mathrm{i}kx_2^{(1)}} K_0(x_2^{b} - x_2^{(1)}) , \end{aligned}$$

$$K_0(\Delta x) = \int \frac{\mathrm{d}p}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}p\,\Delta x}}{\gamma\cdot p - m} . \tag{63}$$

With the help of (62) the intermediate $x^{(i)}$ integration can now be performed in (61) and (62). Introducing a function $S(p)$ by

$$K_0(\Delta x) = \int \frac{\mathrm{d}p}{(2\pi)^4} S(p)\, \mathrm{e}^{\mathrm{i}p\,\Delta x} , \tag{64}$$

we obtain

$$K_{r_{11}=1}(a,b) = \prod_{j=2}^{N} K_0^{(j)} \int \frac{\mathrm{d}p}{(2\pi)^4} \mathrm{e}^{\mathrm{i}px^{b}} S(p) \frac{\mathrm{d}k}{(2\pi)^4} \frac{e^2}{k^2} \gamma_{\mu} S(p-k)\gamma^{\mu} S(p)\, \mathrm{e}^{-\mathrm{i}px^{a}} , \tag{65}$$

and, similarly,

$$K_{r_{12}=1}(a,b)=\prod_{j=3}^{N}K_0^{(j)}\int\frac{dp_1}{(2\pi)^4}\,e^{ip_1x_1^b}S(p_1-k)\,\frac{dk}{(2\pi)^4}\,e^{-ik(x_2^a-x_1^a)}\,\frac{e_1e_2}{k^2}\,\gamma_\mu S(p_1)\,e^{-ipx_1^a}$$
$$\times\frac{dp_2}{(2\pi)^4}\,e^{ip_2x_2^b}S(p_2+k)\gamma^\mu S(p_2)\,e^{-ip_2x_2^a} \tag{66}$$

Note the three propagator functions that were introduced in (20), (29), (62) and (64), namely, $F(p,\Delta x,\Delta t)$, $K(p,\Delta x)$, $S(p)$, with their meaning indicated by their arguments propagators for a length $\Delta x$ in a time $\Delta\tau$ with momentum $p$, or for any subsets of these The remaining $p$ integrals in (65) and (66) say that if we ask for the propagation between fixed points $x^a$ and $x^b$, the particles can still have any momenta and we must integrate over all momenta

Finally, we pass to the scattering amplitude using eq. (E.6) of appendix E For (66), for example, the scattering amplitude for two spin-$\frac{1}{2}$ particles in the Born approximation for fixed initial and final momenta is

$$S(p_1^{\rm f},p_1^{\rm i},p_2^{\rm f},p_2^{\rm i})=\int dx_1^a\,dx_1^b\,\bar u_1(p_1^{\rm f})\,e^{-ip_1^{\rm f}x_1^b}\bar u_2(p_2^{\rm f})\,e^{-ip_2^{\rm f}x_2^b}$$
$$\times dx_2^a\,dx_2^b\,K_{12}(x^a,x^b)u_1(p_1^{\rm i})\,e^{ip_1^{\rm i}x_1^a}u_2(p_2^{\rm i})\,e^{ip_2^{\rm i}x_2^a} \tag{67}$$

Inserting $K(a,b)$ from (66) we get four $\delta$-functions from the four $x$ integrations, three of which we use to perform the $dp_1$, $dp_2$ and $dk$ integrations, and we are left with one overall $\delta$-function We then obtain $p_1=p_1^{\rm f}$, $p_2=p_2^{\rm f}$, $k=p_1^{\rm f}-p_1^{\rm i}$, and

$$S(p_1^{\rm f},p_1^{\rm i},p_2^{\rm f},p_2^{\rm i})=e_1e_2\bar u_1(p_1^{\rm f})S(p_1^{\rm f})\gamma_\mu S(p_1^{\rm i})u(p_1^{\rm i})\,\frac{1}{(p_1^{\rm f}-p_1^{\rm i})^2}$$
$$\times\bar u_2(p_2^{\rm f})S(p_2^{\rm f})\gamma^\mu S(p_2^{\rm i})u(p_2^{\rm i})(2\pi)^4\delta(p_2^{\rm f}-p_2^{\rm i}+p_1^{\rm f}-p_1^{\rm i})$$

The free-particle propagator functions $S(p)$ acting on $u(p)$ reproduce back $u(p)$, because of the reproducing kernel property (E 7), hence we have finally

$$S(p_1^{\rm f},p_1^{\rm i},p_2^{\rm f},p_2^{\rm i})=e_1e_2\,\frac{\bar u_1(p_1^{\rm f})\gamma^\mu u_1(p_1^{\rm i})\bar u_2(p_2^{\rm f})\gamma_\mu u_2(p_2^{\rm i})}{(p_1^{\rm f}-p_1^{\rm i})^2} \tag{68}$$

We remark that in explicitly evaluating the $x$-space form of $K_0(\Delta x)$ from the Fourier expansion (64), if needed (it is not needed in $S$-matrix calculations), one has to choose, as usual, in the $p^0$ integration a contour which goes below the negative-energy pole at $p_0=-E=-(|p|^2+m^2)^{1/2}$, and above the positive-energy pole at $p_0=+E$. This has the effect that positive-energy states propagate forward in time, and negative-energy states backward in time and one obtains

$$K(x^a,x^b)=-i\theta(t'-t)\int d\boldsymbol{p}\sum_{+E}\psi_p^{(+)}(x^b)\bar\psi_p^{(+)}(x^a)+i\theta(t-t')\int d\boldsymbol{p}\sum_{-E}\psi_p^{(-)}(x^b)\bar\psi_p^{(-)}(x^a)\,,$$

which can be accomplished by the $+i\varepsilon$ rule,

$$K(x^a,x^b)=\lim_{\varepsilon\to0^+}\int\frac{dp}{(2\pi)^4}\,\frac{e^{-ip(x^b-x^a)}}{\gamma\cdot p-m+i\varepsilon} \tag{69}$$

### 3.3. Vacuum polarization and pair production

In order to discuss pair production and loop diagrams for vacuum polarization we must consider – at least – three interacting particles. The basic processes of a single-pair production, single-pair annihilation and one-loop vacuum polarization diagrams are shown in fig. 5 Figure 5a shows a lowest-order mutual interaction term in three-body interaction, which can now be immediately written down according to the procedure discussed in the previous sections Figures 5b, 5c and 5d show analytic continuation of the third particle's trajectory to pair production, pair annihilation and vacuum polarization, respectively. In other words, the sign of the proper time $\tau$ is changed appropriately for the antiparticles, or equivalently, the sign of the four-momentum $p_\mu$. Also in classical spinor particle theory the change of sign of $p_\mu$ takes us to antiparticle solutions. We now discuss these steps explicitly.

The propagators for figs. 5b and 5c can be written out immediately because only the portion of the propagator for particle 3 between $x_{3a}$ and $x_3^{(1)}$, or between $x_{3b}$ and $x_3^{(2)}$, respectively, has to be continued analytically with $p_\mu \to -p_\mu$.

For the loop diagram in fig. 5d we start from the full propagator of diagram a and then identify the two end points $x_{3a}$ and $x_{3b}$ of the path of particle 3 and integrate over that end point, and identify and sum over its end point spin indices (for simplicity we take equal masses for all three particles),

$$\begin{aligned} K = \sum_{\alpha_3,\beta_3} \int_{-\infty}^{\infty} (\mathrm{i}\gamma^0\, \mathrm{d}^3x_{3a})\, \mathrm{d}^4x_{3b}\, \delta_{\alpha_3\beta_3} \delta^4(x_{3a}-x_{3b}) \\ \times (-\mathrm{i}/\lambda)^3 \int_0^\infty \mathrm{d}S_1\, \mathrm{d}S_2\, \mathrm{d}S_3\, \mathrm{e}^{-\mathrm{i}(m/\lambda)(S_1+S_2+S_3)} \\ \times \int \mathscr{D}(1)\, \mathscr{D}(2)\, \mathscr{D}(3) \exp\Big(\mathrm{i}\int_0^{S_1} \mathrm{d}\tau_1\, L_1^0 + \mathrm{i}\int_0^{S_2} \mathrm{d}\tau_2\, L_2^0 + \mathrm{i}\int_0^{S_3} \mathrm{d}\tau_3\, L_3^0\Big) \\ \times \exp\{\mathrm{i}[W(1,1)+W(1,2)+W(1,3)+W(2,2)+W(2,3)+W(3,3)]\}\,. \end{aligned} \tag{70}$$

We expand the $\mathrm{e}^{\mathrm{i}W}$ terms as before, eq. (16). In this expansion there will be terms corresponding to an isolated loop of particle 3, and loops coupled to one of the particles 1 and 2 (i.e. tadpole-type diagrams). We shall discuss here only the term corresponding to fig. 5d.

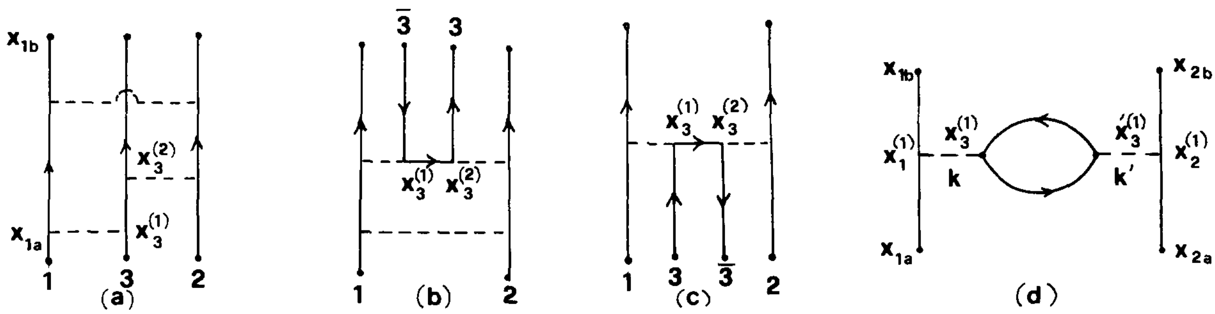



Fig 5

The reason for the three-dimensional $d^3x_{3a}$ integration in eq. (70) is that there is a $\delta$-function in the propagator $K_0(x, y)$. Writing $K_0$ in the form

$$K_0(x, y) = -\frac{1}{\lambda}\int_0^\infty K_0(S)\,dS = -\frac{1}{\lambda}\int_0^\infty dS\int \frac{dp}{(2\pi)^4}\, e^{ip(x-y)}\, e^{i(S/\lambda)(p\cdot\gamma - m)}\,,$$

and performing now the $d^4p$ integration first, we get

$$K_0(x, y) = -\frac{1}{\lambda}\int_0^\infty dS\, e^{-i(m/\lambda)S}\delta^4(x_\mu - y_\mu + (S/\lambda)\gamma_\mu)$$

We then integrate over $S$ with $\delta(x^0 - y^0 + (S/\lambda)\gamma^0)$,

$$K_0(x, y) = -i\gamma^0\, e^{im(x_0-y_0)\gamma^0}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y} + (x_0 - y_0)\gamma^0\boldsymbol{\gamma}) \tag{71}$$

This Green's function is a *reproducing kernel* with the scalar product $i\gamma_0\, d^3x$ [7, p 264], as it should be, i e ,

$$K_0(x, y) = \int d^3x'\, K_0(x, x')i\gamma^0 K_0(x', y) \tag{72}$$

Indeed, with (71) we get

$$\begin{aligned} K_0(x, y) &= \int d^3x'\,(-i\gamma^0)\, e^{im(x_0-x_0')\gamma^0}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}' + (x_0 - x_0')\gamma^0\boldsymbol{\gamma}) \\ &\quad\times (i\gamma^0)(-i\gamma^0)\, e^{im(x_0'-y_0)\gamma^0}\delta^{(3)}(\boldsymbol{x}' - \boldsymbol{y} + (x_0' - y_0')\gamma^0\boldsymbol{\gamma}) \\ &= -i\gamma^0\, e^{im(x_0-y_0)\gamma^0}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y} + (x_0 - y_0)\gamma^0\boldsymbol{\gamma}) \end{aligned}$$

The relevant lowest-order expansion terms of $e^{iW}$ corresponding to fig. 5d give

$$\begin{aligned} K &= \sum_{\alpha_3\beta_3} \delta_{\alpha_3\beta_3}\int d^4x_{3b}\,\delta^{(4)}(x_{3a} - x_{3b})i\gamma^0\, d^3x_{3a} \\ &\quad\times(-i/\lambda)^3\int_0^\infty dS_1\, dS_2\, dS_3\, e^{-i(m/\lambda)(S_1+S_2+S_3)}\left(\frac{e^2}{(2\pi)^4}\right)^2 \frac{dk}{k^2}\,\frac{dk'}{k'^2} \\ &\quad\times\int \mathscr{D}(1)\exp\left(i\int_0^{S_1} d\tau_1\, L_1^0\right)\int_0^{S_1} d\tau_1^{(1)}\, M_1(\tau_1^{(1)}, k) \\ &\quad\times\int \mathscr{D}(3)\exp\left(i\int_0^{S_3} d\tau_3\, L_3^0\right)\int_0^{S_3} d\tau_3^{(1)}\, M_3(\tau_3^{(1)}, k)\int_0^{S_3} d\tau_3'^{(1)}\, M_3(\tau_3'^{(1)}, -k') \\ &\quad\times\int \mathscr{D}(2)\exp\left(i\int_0^{S_2} d\tau_2\, L_2^0\right)\int_0^{S_2} d\tau_2^{(1)}\, M_2(\tau_2^{(1)}, k)\,, \end{aligned}$$

where the functions $M_i(\tau; k)$ were defined in eq. (14) and the four interaction times $\tau_1^{(1)}, \tau_3^{(1)}, \tau_3'^{(1)}, \tau_2^{(1)}$ are shown in fig. 5d. With the methods already developed we obtain immediately after the trace operation $\Sigma_{\alpha_3\beta_3} \delta_{\alpha_3\beta_3}$ and $\delta$-function integration

$$
\begin{aligned}
K = e^4 \int & \mathrm{d}x_1(\tau^{(1)})\, \mathrm{d}x_3(\tau_3^{(1)})\, \mathrm{d}x_3(\tau_3^{(1)})\, \mathrm{d}x_2(\tau_2^{(1)}) \\
& \times K_0(x_1^a, x_1^{(1)}) K_0(x_1^{(1)}, x_1^b) D(x_1^{(1)}, x_3^{(1)}) \\
& \times \int (\mathrm{i}\gamma^0\, \mathrm{d}^3x_3^a)\, \mathrm{Tr}[K_0(x_3^a, x_3^{(1)}) K_0(x_3^{(1)}, x_3'^{(1)}) K_0(x_3'^{(1)}, x_3^a)] \\
& \times D(x_3'^{(1)}, x_2^{(1)}) K_0(x_2^a, x_2^{(1)}) K_0(x_2^{(1)}, x_2^b)\,.
\end{aligned}
$$

Using $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$ and using the property (71) we have finally

$$
\begin{aligned}
K = e^4 \int & \mathrm{d}x_1^{(1)}\, \mathrm{d}x_3^{(1)}\, \mathrm{d}x_3'^{(1)}\, \mathrm{d}x_2^{(1)}\, [K_0(x_1^a, x_1^{(1)}) K_0(x_1^{(1)}, x_1^b)] D(x_1^{(1)}, x_3^{(1)}) \\
& \times \mathrm{Tr}[K_0(x_3^{(1)}, x_3'^{(1)}) K_0(x_3^{(1)\prime}, x_3^{(1)})] D(x_3^{(1)}, x_2^{(1)}) K_0(x_2^a, x_2^{(1)}) K_0(x_2^{(1)}, x_2^b)\,,
\end{aligned}
\tag{73}
$$

which is the standard expression for diagram fig. 5d.

### 3.4. *External field problems*

In this section we consider finally the interaction of particles in a given external field $A_\mu^{\mathrm{ext}}$ The interaction action is

$$
W_{\mathrm{int}} = -\int \mathrm{d}x\, A_\mu^{\mathrm{ext}}(x) j^\mu(x) = -\int \mathrm{d}\tau \sum_k e_k \bar{z}_k(\tau) \gamma^\mu z_k(\tau) A_\mu^{\mathrm{ext}}(x_k(\tau))
$$

From our general theory it is now a simple matter to write down the propagator. At each interaction point $x_k^{(j)}$ the factor $\gamma^\mu$ is now replaced by $\gamma^\mu A_\mu^{\mathrm{ext}}(x_k^{(j)})$. To lowest order we get immediately for each particle

$$
K_{(1)}(a, b) = \mathrm{i}(\mathrm{i}e) \int \mathrm{d}x^{(1)}\, K_0(x^{(1)} - x^a) \gamma \cdot A(x^{(1)}) K_0(x^b - x^{(1)})
\tag{74}
$$

For a Coulomb field, for example, $\gamma \cdot A = \gamma^0 A_0 = \gamma^0 Ze/r$. Now the $\mathrm{d}x^{(1)}$ integration must be done with the external field $A^r(x^{(1)})$. But we can pass to the $S$-matrix using (E.6) and (E.7) and obtain

$$
S(p^{\mathrm{f}}, p^{\mathrm{i}}) = e \int \mathrm{d}x^{(1)}\, \bar{u}(p^{\mathrm{f}}) \gamma \cdot A(x^{(1)}) u(p^{\mathrm{i}})\,,
\tag{75}
$$

with properly normalized $u$ and $\bar{u}$.

## 4. Conclusions

We have derived the rules of perturbative quantum electrodynamics and the ensuing Feynman graphs directly from classical particle trajectories by the method of path integration. Field equations and field quantization have not been used explicitly For covariant results the corresponding particle equations must also be covariant, i e in proper time In the case of spin-$\frac{1}{2}$ particles we have used a new classical model of the Dirac particle which on the classical level exhibits spin, zitterbewegung and the motion of antiparticles Pair production and vacuum polarization can therefore be treated as well Thus we have a new and more direct and intuitive approach to quantum field theory

## Appendix A. Classical equations of motion

Our Lagrangian is always of the form

$$L = L_{\text{part}} + L_{\text{int}} + L_{\text{field}} , \tag{A 1}$$

as given in eqs (1), (33) or eq (52) And we obtain correctly both the equations of particles and fields For the spinor case the equations of motion are

$$\begin{aligned} &x_\mu = \bar z \gamma_\mu z , \qquad p_\mu = e\bar z \gamma^\nu z A_{\nu,\mu} , \\ &\bar z = i\bar z \gamma^\mu p_\mu , \qquad z = -i p_\mu \gamma^\mu z , \end{aligned} \tag{A 2}$$

and the Maxwell equation

$$F^{\mu\nu}{}_{,\nu} = e\bar z \gamma^\mu z \tag{A 3}$$

From (A 2)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(p_\mu - eA_\mu) = eF_{\mu\nu}x^\nu \tag{A 4}$$

Our method was to eliminate $A_\mu$, and therefore eliminate the term $-\frac{1}{4}\int \mathrm{d}x\, F_{\mu\nu}F^{\mu\nu}$ in favour of $+\frac{1}{2}\int \mathrm{d}x\, j_\mu A^\mu$ We remark again that if this is done one can no longer use the reduced Lagrangian to derive the equations of motion

## Appendix B. Scalar limit of the classical spinor theory

The scalar limit of the spinor equation amounts to suppressing the internal co-ordinates $z$, $\bar z$ We now take the following limit in eq. (52).

$$\bar z\gamma_\mu z \rightarrow (p_\mu - eA_\mu)\bar z z , \tag{B 1}$$

which replaces the velocity $x_\mu$ correctly by the kinetic momentum (or velocity) of the scalar particle, and $\bar z z$ is a constant of the motion $[(\mathrm{d}/\mathrm{d}\tau)(\bar z z) = 0]$

Furthermore, neglecting the free kinetic energy of the internal variables, we obtain

$$L \to p_\mu \dot{x}^\mu - p_\mu(p^\mu - eA^\mu)\bar{z}z + eA_\mu(p^\mu - eA^\mu)\bar{z}z = p_\mu \dot{x}^\mu - \tfrac{1}{2}p^2 + ep_\mu A^\mu - \tfrac{1}{2}e^2A^2 , \tag{B.2}$$

with $\bar{z}z = \frac{1}{2}$, which is precisely the action (33).

In this form of the Lagrangian the current is

$$j^\mu = e\dot{x}^\mu = e(p^\mu - eA^\mu) , \tag{B.3}$$

and the equations of motion

$$\begin{aligned} &F^{\mu\nu}{}_{,\nu} = j^\mu , \\ &\dot{p}_\mu = eA_{\nu,\mu}\dot{x}^\nu \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}\tau}(p_\mu - eA_\mu) = eF_{\mu\nu}\dot{x}^\nu \end{aligned} \tag{B.4}$$

## Appendix C. Propagator for spinless relativistic particles

For completeness we give the path integral evaluation of the propagator for a spinless relativistic particle,

$$F = \int \prod_{j=1}^{n+1} \frac{\mathrm{d}p^{(j)}}{(2\pi)^4} \prod_{j=1}^{n} \mathrm{d}x^{(j)} \exp\left(\mathrm{i}\int_{\tau_1}^{\tau_2} \mathrm{d}\tau\,(p\cdot\dot{x} - \lambda p^2)\right) , \tag{C.1}$$

where $\lambda$ is a Lagrange multiplier, $\lambda = \frac{1}{2}$ if $\tau$ is the proper time. The exponent

$$E = \mathrm{i}\sum_{j=1}^{n+1} [p^{(j)}\cdot(x^{(j)} - x^{(j-1)}) - \lambda(p^{(j)})^2\varepsilon] \quad \text{with } \Delta\tau^{(j)} = \varepsilon ,$$

is

$$\begin{aligned} E = \mathrm{i}(&p^{(n+1)}x^b - p^{(1)}x^a + x^{(1)}(p^{(1)} - p^{(2)}) + x^{(2)}(p^{(2)} - p^{(3)}) + \cdots + x^{(n)}(p^{(n)} - p^{(n+1)}) \\ &- \lambda\varepsilon \sum_{j=1}^{n+1} (p^{(j)})^2 . \end{aligned}$$

From the $x^j$ integrations we get $\delta(p^{(1)} - p^{(2)}), \delta(p^{(2)} - p^{(3)}), \ldots, \delta(p^{(n)} - p^{(n+1)})$. Hence we can perform $n$ $p$-integrations, leaving $p^{n+1} = p$. Hence

$$F = \int \frac{\mathrm{d}p}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}p(x^b - x^a)}\, \mathrm{e}^{-\mathrm{i}(\varepsilon/2)(n+1)p^2} . \tag{C 2}$$

But $(n+1)\varepsilon = \tau_2 - \tau_1$, so that

$$F(\Delta x, \Delta\tau) = \int \frac{\mathrm{d}p}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}p\,\Delta x}\, \mathrm{e}^{-(\mathrm{i}/2)\,\Delta\tau\, p^2} .$$

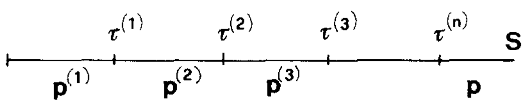
Now we integrate over all times $\Delta\tau$,

$$K = \int \mathrm{d}(\Delta\tau)\, \mathrm{e}^{-(1/2)m^2\,\Delta\tau-(1/2)p^2\,\Delta\tau}\, \frac{\mathrm{d}p}{(2\pi)^4}\, \mathrm{e}^{ip\,\Delta x}$$

$$= \int \frac{\mathrm{d}p}{(2\pi)^4}\, \frac{\mathrm{e}^{ip\,\Delta x}}{p^2+m^2} \tag{C 3}$$

## Appendix D. A lemma on time integrations

For each particle we encounter a series of time integrations,




of the form

$$I = \int_0^\infty \mathrm{d}S\, \mathrm{e}^{-i[f(p)+f(m)]} \int_0^S \mathrm{d}\tau^{(n)} \cdots \int_0^{\tau^{(4)}} \mathrm{d}\tau^{(3)} \exp\{-i\tau^{(3)}[f(p^{(3)})-f(p^{(4)})]\}$$

$$\times \int_0^{\tau^{(3)}} \mathrm{d}\tau^{(2)} \exp\{-i\tau^{(2)}[f(p^{(2)})-f(p^{(3)})]\} \int_0^{\tau^{(2)}} \mathrm{d}\tau^{(1)} \exp\{-i\tau^{(1)}[f(p^{(1)})-f(p^{(2)})]\} \tag{D 1}$$

We can prove by direct computation that

$$I = \frac{(-i)^{n+1}}{[f(p)+f(m)][f(p^{(n)})+f(m)]\cdots[f(p^{(1)}+f(m)]} \tag{D 2}$$

## Appendix E. *S*-matrix from the propagator

Again for completeness we give here the formulas for the action of the propagator $K(a, b)$ on the states and for recovering the $S$-matrix elements or amplitudes from $K(a, b)$

Let the initial many-body state be

$$\psi^{\mathrm{i}}_\alpha(x^a) \equiv \langle x^a, \alpha | \psi^{\mathrm{i}} \rangle\,, \tag{E 1}$$

where $x^a = (x^a_1, x^a_2, \ldots, x^a_N)$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ being the spin indices, then the final state is obtained by

$$\psi^{\mathrm{f}}_\beta(x^b) = \sum_\alpha \int \mathrm{d}x^a\, K_{\alpha\beta}(x^a, x^b)\psi^{\mathrm{i}}_\alpha(x^a)\,, \tag{E 2}$$

which can be written in the bracket notation as

$$\langle x^b, \beta|\psi^f\rangle = \sum\!\!\!\!\!\!\int_\alpha \langle x^b, \beta|x^a, \alpha\rangle\langle x^a, \alpha|\psi^i\rangle . \tag{E.2'}$$

Thus a completeness relation is implied,

$$\sum\!\!\!\!\!\!\int_\alpha \mathrm{d}x^a\, |x^a, \alpha\rangle\langle x^a, \alpha| = I . \tag{E 3}$$

The $S$-matrix is defined by

$$S^{fi} = \langle \psi^f|\psi^i\rangle , \tag{E.4}$$

which we can now expand as follows, using (E.3) twice,

$$\begin{aligned} S^{fi} &= \sum\!\!\!\!\!\!\iint_{\alpha\beta} \langle \psi^f|x^b, \beta\rangle\langle x^b, \beta|x^a, \alpha\rangle\langle x^a, \alpha|\psi^i\rangle\, \mathrm{d}x^a\, \mathrm{d}x^b \\ &= \sum_{\alpha\beta} \int\int \mathrm{d}x^a\, \mathrm{d}x^b\, \langle \psi^f|x^b, \beta\rangle K_{\alpha\beta}(x^a, x^b)\langle x^a, \alpha|\psi^i\rangle . \end{aligned} \tag{E.5}$$

If $|\psi^i\rangle$ and $|\psi^f\rangle$ are momentum and spin eigenstates $(p, s)$, for example, then

$$S(p^f, s^f; p^i, s^i) = \sum_{\alpha\beta} \int\int \mathrm{d}x^a\, \mathrm{d}x^b\, \langle p^f, s^f|x^b, \beta\rangle K_{\alpha\beta}(x^a, x^b)\langle x^a, \alpha|p^i, s^i\rangle . \tag{E.6}$$

Here $K(x^a, x^b)$ is the amplitude with only the end points in space–time fixed, all other quantum numbers are free; the initial and final wave functions then select the desired quantum numbers. But in the $S$-matrix we are not interested in the end points $x^a, x^b$, over which we then integrate.

In eq. (E.2), if we take the connected parts of the propagator $K$ and require that $\psi^i \equiv \psi^f$, we can regard this equation as an integral equation for the stationary or bound state $\psi$. This is then essentially the Bethe–Salpeter "equation"; $K$ then is a reproducing kernel. For free stationary particles $K_0$ is already the reproducing kernel and (E.6) reduces to

$$S_0(p^f, s^f; p^i, s^i) = \delta(p^f - p^i)\bar{u}^{s^f}_\alpha S(p)_{\alpha\beta} u^{s^i}_\beta = \delta(p^f - p^i)\delta_{s^f s^i} . \tag{E.7}$$

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