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Relativistic theory of spontaneous emission

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We derive a formula for the relativistic decay rates in atoms in a formulation of quantum electrodynamics based upon the electron's self-energy. Relativistic Coulomb wave functions are used, the full spin calculation is carried out, and the dipole approximation is not employed. The formula has the correct nonrelativistic limit and is used for calculating the decay rates in hydrogen and muonium for the transitions $2P \rightarrow 1S_{1/2}$ and $2S_{1/2} \rightarrow 1S_{1/2}$. The results for hydrogen are $\Gamma(2P \rightarrow 1S_{1/2}) = 6.2649 \times 10^8\ \mathrm{s}^{-1}$ and $\Gamma(2S_{1/2} \rightarrow 1S_{1/2}) = 2.4946 \times 10^{-6}\ \mathrm{s}^{-1}$. Our result for the $2P \rightarrow 1S$ transition rate is in perfect agreement with the best nonrelativistic calculations as well as with the results obtained from the best known radiative decay lifetime measurements. As for the hydrogen $2S_{1/2} \rightarrow 1S_{1/2}$ decay rate, the result obtained here is also in good agreement with the best known magnetic dipole calculations. For muonium we get $\Gamma(2P \rightarrow 1S_{1/2}) = 6.2382 \times 10^8\ \mathrm{s}^{-1}$ and $\Gamma(2S_{1/2} \rightarrow 1S_{1/2}) = 2.3997 \times 10^{-6}\ \mathrm{s}^{-1}$.

I. INTRODUCTION

At present, the rate of spontaneous emission (or partial-decay lifetimes) in atoms is not among the list of precision tests of quantum electrodynamics. The 2γ and 3γ decay rates of the ${}^1S_0$ and ${}^3S_1$ states of positronium, respectively, are part of that list. In positronium one tests the annihilation rates of the e^+e^- pair, albeit in a bound state. However, in hydrogen or muonium there is no annihilation and we are talking about the rates of atomic transitions in, say, the $\mathrm{H}^* \rightarrow \mathrm{H} + \gamma$ transition.

The reason for excluding the rates of spontaneous emission from the list of precision tests of QED is partly due to the absence of very accurate theoretical calculations because the decay rates are usually calculated in the dipole approximation, with nonrelativistic wave functions. Also, accurate experiments may not be easy to perform, but with the new techniques of trapped and cooled atoms it might now be possible to make accurate lifetime observations in hydrogen and muonium if correspondingly accurate theoretical numbers existed.

With this goal in mind, we have calculated all spontaneous decay rates in the relativistic Coulomb problem using full Dirac-Coulomb wave functions and without making the dipole approximation. The results are thus to all orders in $Z\alpha$. The full spin calculation is rather cumbersome and to our knowledge has not been carried out before.

In Sec. II we give a new derivation of a general spontaneous-emission formula in which the decay rate, $\Gamma_n/2$, appears as the imaginary part of a complex energy shift ΔE_n, the real part being the Lamb shift and the vacuum polarization.[1-3] Section III contains the full spin and angular integrations as well as the radial integrations with some of the details collected in the appendixes. Finally, in Sec. IV we present a number of numerical results and compare them with the available nonrelativistic data.

II. RELATIVISTIC THEORY OF SPONTANEOUS EMISSION

A general formula for spontaneous emission from an electron in an arbitrary external field A_μ^{ext} can be derived in a very simple way directly from the action of QED ($\hbar = c = 1$ and $dx \equiv d^4x$),

$$W = \int dx [\overline{\Psi}(\gamma^\mu i\partial_\mu - m)\Psi + J^\mu A_\mu - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}] \ , \quad (1)$$

where $J^\mu = -e\overline{\Psi}\gamma^\mu\Psi$ is the electron current and A_μ is the total electromagnetic field, $A_\mu = A_\mu^e + A_\mu^s$, with the superscripts e and s standing for external and self, respectively. Here A_μ^e is treated as a given nondynamical function. On the other hand, $F_{\mu\nu} = A^s_{\nu,\mu} - A^s_{\mu,\nu}$ satisfies the Maxwell equations $F^{\mu\nu}{}_{,\nu} = J^\mu$ which can be used to put Eq. (1), after a single integration by parts has been performed on the last term, into the following form:

$$W = \int dx \{\overline{\Psi}[\gamma^\mu(i\partial_\mu - eA_\mu^e) - m]\Psi + \tfrac{1}{2}J^\mu A_\mu^s\} \ . \quad (2)$$

Next, we complete the elimination of A_μ^s from the action by inserting into (2) the solution of the wave equation,[1-3]

$$\Box A_\mu^s = J_\mu = -e\overline{\Psi}\gamma_\mu\Psi \ ,$$

namely,

$$A_\mu^s(x) = A_\mu^{s(H)} - e\int dy\, D_{\mu\nu}(x-y)\overline{\Psi}(y)\gamma^\nu\Psi(y) \ .$$

Here $A_\mu^{s(H)}$ is a solution to the homogeneous wave equation. Since our system is supposed to be completely isolated, this one can safely be dropped, because it depends upon the boundary conditions at infinity. Also, $D_{\mu\nu}(x-y)$ is the causal Green's function in the covariant gauge $A^\mu{}_{,\mu} = 0$, which we take as

$$D_{\mu\nu}(x-y) = -g_{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^2} \ . \quad (3)$$

Thus Eq. (2) now becomes

$$W=\int dx\,\overline{\Psi}(x)[\gamma^{\mu}(i\partial_{\mu}-eA_{\mu}^{e})-m]\Psi(x)$$
$$-\frac{e^2}{2}\int dx\,dy\,\overline{\Psi}(x)\gamma^{\mu}\Psi(x)\times\int\frac{d^4k}{(2\pi)^4}\frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^2}\overline{\Psi}(y)\gamma_{\mu}\Psi(y)$$
$$=W_0+W_1\ . \tag{4}$$

When the Fourier expansion of the matter field Ψ in the time variable, namely,

$$\Psi(x)=\sum\!\!\!\!\!\!\int\ \psi_n(\mathbf{x})e^{-iE_nx_0}\ , \tag{5}$$

where the Fourier coefficients are yet to be determined, is substituted into (3) and after the time integrations over k_0, y_0, and x_0 have been performed, in this order for convenience, we get

$$W_0=2\pi\sum\!\!\!\!\!\!\int_n\ \int d^3x\,\overline{\psi}_n(\mathbf{x})(\gamma^0E_n-\boldsymbol{\gamma}\cdot\mathbf{p}-eA^e-m)\psi_n(\mathbf{x}) \tag{6a}$$

and

$$W_1=-2\pi\frac{e^2}{2}\sum\!\!\!\!\!\!\int_{n,m,r,s}\ \delta(E_n-E_m+E_r-E_s)\int d^3x\,\overline{\psi}_n(\mathbf{x})\gamma^{\mu}\psi_m(\mathbf{x})\times\int d^3y\,\overline{\psi}_r(\mathbf{y})\gamma_{\mu}\psi_s(\mathbf{y})$$
$$\times\int\frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{2}\left[\frac{i\pi}{k}\left[\delta(E_r-E_s+k)+\delta(E_r-E_s-k)\right]+\frac{\mathrm{P}}{2k}\left[\frac{1}{E_r-E_s-k}-\frac{1}{E_r-E_s+k}\right]\right]\ . \tag{6b}$$

Here P stands for the principal-value integral and $\sum\!\!\!\!\!\!\int$ implies a sum over the discrete part and an integration over the continuum part of the system's spectrum. In carrying out the k_0 integration, the contour is closed in the upper half plane for $y_0>x_0$ where it encloses the simple pole at $k_0=-k$ ($k\equiv|\mathbf{k}|$), and in the lower half plane for the case $y_0<x_0$ where it encloses the pole at $k_0=+k$. θ functions are used in order to distinguish between the two cases. The y_0 integrations turn out to be simply Fourier transforms of the θ functions which give rise to the principal-value integrals and the δ functions in (6b).

Now, the δ function $\delta(E_n-E_m+E_r-E_s)$ can be satisfied by the two choices:[2] (1) $n=m$ and simultaneously $r=s$, and (2) $n=s$ and simultaneously $r=m$. With this, W_1 becomes

$$W_1=-2\pi\frac{e^2}{2}\sum\!\!\!\!\!\!\int_{n,s}\ \int d^3x\,\overline{\psi}_n(\mathbf{x})\gamma^{\mu}\psi_n(\mathbf{x})\int d^3y\,\overline{\psi}_s(\mathbf{y})\gamma_{\mu}\psi_s(\mathbf{y})\times\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left[\frac{i\pi}{2k}[\delta(k)+\delta(-k)]+\frac{\mathrm{P}}{2k}\left[-\frac{1}{k}-\frac{1}{k}\right]\right]$$
$$-2\pi\frac{e^2}{2}\sum\!\!\!\!\!\!\int_{n,s}\ \int d^3x\,\overline{\psi}_n(\mathbf{x})\gamma^{\mu}\psi_s(\mathbf{x})\int d^3y\,\overline{\psi}_s(\mathbf{y})\gamma_{\mu}\psi_n(\mathbf{y})$$
$$\times\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left[\frac{i\pi}{2k}[\delta(E_s-E_n+k)+\delta(E_s-E_n-k)]+\frac{\mathrm{P}}{2k}\left[\frac{1}{E_s-E_n-k}-\frac{1}{E_s-E_n+k}\right]\right]\ . \tag{7}$$

Notice that the term proportional to $\delta(k)+\delta(-k)=2\delta(k)$ does not contribute because of the integration over k. From here, one could proceed to the derivation of the equations of motion by minimizing the total action and subsequently solving the coupled Hartree-type equations thus obtained for the energies and wave functions. Instead of following this path, though, we can avoid the nonlinear equations and use the following approach. If we find the equations of motion and insert them back into the action, the action will assume its minimum value, which is $W=0$. In other words, an exact solution to our problem would be to find that set of wave functions $\{\psi_n(\mathbf{x})\}$ which would make $W_0+W_1=0$. Now, in the absence of the nonlinear self-energy part W_1, which is proportional to e^2, W_0 vanishes precisely for the solutions of the Dirac equation of an electron in the external nondynamical field A^e_μ.

If we, therefore, take for $\{\psi_n(\mathbf{x})\}$ the complete set of solutions of Dirac's equation in such a field, $\{\psi^c_n(\mathbf{x})\}$, with their corresponding energies $\{E^c_n\}$, and set $E_n = E^c_n + \Delta E_n$, then as a first iteration of the action, W_0 will contribute a term $2\pi \sum_n \Delta E_n$ and W_1 is evaluated with the functions $\{\psi^c_n(\mathbf{x})\}$. Thus we get from the vanishing of the action in the first iteration,

$$W_1^{(1)} = -2\pi \sum_n \Delta E_n \ , \tag{8}$$

where the superscript on $W_1^{(1)}$ is added to indicate that we are considering a first iteration of the action. In particular, for our problem A^e_μ is a Coulomb field and $\{\psi^c_n(\mathbf{x})\}$ and $\{E^c_n\}$ are therefore the sets of Dirac-Coulomb wave functions and eigenenergies, respectively. From (7) and (8) we immediately identify the shift in the nth energy level as a sum of three terms having the following physical interpretations. (From here on we shall drop the superscript c on ψ_n.)

(1) *Vacuum polarization*

$$\Delta E_n^{\rm VP} = -\frac{e^2}{2} \sum\!\!\!\!\!\!\int_s \int d^3x\, \bar\psi_n(\mathbf{x})\gamma^\mu\psi_n(\mathbf{x}) \mathrm{P}\left[\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^2}\right]\int d^3y\, \bar\psi_s(\mathbf{y})\gamma_\mu\psi_s(\mathbf{y}) \ . \tag{9}$$

(2) *Spontaneous emission and absorption (including bremsstrahlung)*

$$\Delta E_n^{\rm SE} = \frac{e^2}{2} \sum\!\!\!\!\!\!\int_s \int d^3x\, \bar\psi_n(\mathbf{x})\gamma^\mu\psi_s(\mathbf{x}) \int d^3y\, \bar\psi_s(\mathbf{y})\gamma_\mu\psi_n(\mathbf{y}) \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left[\frac{i\pi}{2k}[\delta(E_s-E_n+k)+\delta(E_s-E_n-k)]\right] \ . \tag{10}$$

(3) *The Lamb shift*

$$\Delta E_n^{\rm LS} = \frac{e^2}{2} \sum\!\!\!\!\!\!\int_s \int d^3x\, \bar\psi_n(\mathbf{x})\gamma^\mu\psi_s(\mathbf{x}) \int d^3y\, \bar\psi_s(\mathbf{y})\gamma_\mu\psi_n(\mathbf{y}) \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{2k}\mathrm{P}\left[\frac{1}{E_s-E_n-k}-\frac{1}{E_s-E_n+k}\right] \ . \tag{11}$$

The vacuum-polarization term has been treated elsewhere[3] and so has the Lamb-shift term.[4] We therefore do not discuss them here any further. The spontaneous-emission term is evaluated in detail in Sec. III and numerical examples are presented in Sec. IV.

III. RELATIVISTIC DECAY RATES

The focus of our attention in this work is Eq. (10) of Sec. II. The first thing to notice is that the first δ function, $\delta(E_s-E_n+k)$, implies that $E_n > E_s$, and hence corresponds to the decay of the state n to a set of lower states s. On the other hand, the second δ function, by the same argument, corresponds to the absorption of radiation by the atom in the state n, causing it to be elevated to a higher state s. We choose the second δ function for treating the phenomenon of photoexcitation.[4] The fact that both of these terms come out in a single equation is one of the advantages of using an action approach.

We make two remarks at this point. First, it should be emphasized that the choice of δ function we have just made is in no way as arbitrary as it may seem at first sight. In fact, it is dictated by the remaining k integration over the interval $(0,\infty)$, and choosing one of the two functions automatically precludes the other. If it is an emission process that we study, then $E_n > E_s$ and, since k is positive, only the function $\delta(E_s-E_n+k)$ contributes and not $\delta(E_s-E_n-k)$. Conversely, in the case of absorption, the other δ function will contribute.

The second remark concerns the relation of $\Delta E_n^{\rm SE}$ to the decay rate of the nth level. When the atomic state of some system of energy ε decays in time, the time dependence of its wave function is written as[5]

$$e^{-i(\varepsilon_r - i\Gamma/2)t} = e^{-i\varepsilon_r t} e^{-\Gamma/2t} \ ,$$

where Γ is the decay rate of the state or twice its inverse mean lifetime. In other words,

$$\Gamma = -2\,\mathrm{Im}(\varepsilon) \ . \tag{12}$$

So, taking the correct δ function in (10) and using (12), we get the following general formula for the decay rate of the nth level:

$$\Gamma_n = -e^2 \sum\!\!\!\!\!\!\int_{s<n} \int d^3x\, \bar\psi_n(\mathbf{x})\gamma^\mu\psi_s(\mathbf{x}) \times \int d^3y\, \bar\psi_s(\mathbf{y})\gamma_\mu\psi_n(\mathbf{y}) \times \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\frac{\pi}{2k}\delta(E_s-E_n+k) \ . \tag{13}$$

The total decay rate of a state n is an incoherent sum of rates of decay to all states s whose energy is less than E_n. It follows that only the ground state is stable. All other states $\psi^c_n(\mathbf{x})$ (which are not true eigenstates of the total Hamiltonian) acquire shifts and are unstable.

At this point it is instructive to digress a little and try to recover the decay rate in the dipole approximation which is familiar from old-fashioned perturbation theory. In this approximation,

$$e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\approx 1 ,$$

and hence (13) becomes

$$\Gamma_n = -\frac{\alpha}{4\pi} \sum\!\!\!\!\!\!\int_{s(<n)} \int d^3x\, \bar{\psi}_n(\mathbf{x})\gamma^\mu\psi_s(\mathbf{x}) \times \int d^3y\, \bar{\psi}_s(\mathbf{y})\gamma_\mu\psi_n(\mathbf{y}) \times \int \delta(E_s - E_n + k)k\, dk\, d\Omega_k .$$

Carrying out the integration over k and using $\gamma^\mu\gamma_\mu = \gamma_0^2 - \boldsymbol{\gamma}\cdot\boldsymbol{\gamma}$, we get

$$\Gamma_n \approx -\frac{\alpha}{4\pi} \sum\!\!\!\!\!\!\int_{s(<n)} \omega_{ns} \left[\int d^3x\, \bar{\psi}_n(\mathbf{x})\gamma^0\psi_s(\mathbf{x}) \int d^3y\, \bar{\psi}_s(\mathbf{y})\gamma_0\psi_n(\mathbf{y}) - \left[\int d^3x\, \bar{\psi}_n(\mathbf{x})\boldsymbol{\gamma}\psi_s(\mathbf{x}) \right]\cdot\left[\int d^3y\, \bar{\psi}_s(\mathbf{y})\boldsymbol{\gamma}\psi_n(\mathbf{y}) \right] \right] d\Omega_k ,$$

where $\omega_{ns} = E_n - E_s$. Also, $\bar{\psi}\gamma^0 = \psi^\dagger$ and $\bar{\psi}\boldsymbol{\gamma} = \psi^\dagger\boldsymbol{\alpha}$. These, together with the orthogonality of the wave functions, yields

$$\Gamma_n \approx \frac{\alpha}{4\pi} \sum\!\!\!\!\!\!\int_{s(<n)} \int \omega_{ns} \,|\langle n \,|\, \boldsymbol{\alpha} \,|\, s\rangle|^2 d\Omega_k$$

$$= \frac{\alpha}{4\pi} \sum\!\!\!\!\!\!\int_{s(<n)} \int \omega_{ns} \,|\mathbf{v}_{ns}|^2 d\Omega_k , \quad (\mathbf{v} = c\boldsymbol{\alpha},\ c = 1) .$$

On the other hand, the Heisenberg equations of motion give

$$\mathbf{v}_{ns} = i\langle n \,|\, [H,\mathbf{r}] \,|\, s\rangle = i\omega_{ns}\mathbf{r}_{ns} .$$

Thus,

$$\Gamma_n \approx \frac{\alpha}{4\pi} \sum\!\!\!\!\!\!\int_{s(<n)} \omega_{ns}^3 \int |\mathbf{r}_{ns}|^2 d\Omega_k .$$

If we then introduce the photon polarization via the two polarization vectors $\mathbf{e}_{\mathbf{k}\lambda}$ ($\lambda = 1,2$), orthogonal to the propagation vector $\mathbf{k}$, we get

$$\Gamma_n \approx \frac{\alpha}{4\pi} \sum\!\!\!\!\!\!\int_{s(<n)} \omega_{ns}^3 \sum_\lambda \int |\mathbf{e}_{\mathbf{k}\lambda}\cdot\mathbf{r}_{ns}|^2 d\Omega_k .$$

Finally, after carrying out the angular integration, we arrive at

$$\Gamma_n \approx \tfrac{2}{3}\alpha \sum\!\!\!\!\!\!\int_{s(<n)} \omega_{ns}^3 \,|\mathbf{r}_{ns}|^2 .$$

Still, relativistic wave functions are to be used in the evaluation of the matrix element $|\mathbf{r}_{ns}|$. The squared matrix element $|\mathbf{r}_{ns}|^2$ thus has implicit in it a spin dependence contributing ultimately the factor

$$\sum_{\mu,\mu'} \chi_\mu^\dagger \chi_{\mu'} = \sum_\mu \delta_{\mu\mu'} = 2 .$$

Hence, the famous factor $\frac{4}{3}$ in the electric dipole formula is automatically restored.

Now we go back to our general formula (13) and evaluate it exactly. In the next step, the expression for Γ is simplified by expanding $e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$ in terms of partial waves and subsequently carrying out the integrations over k (see Appendix A). When this is done, Γ takes the following form ($e^2 = 4\pi\alpha$):

$$\Gamma_n = -4\pi\alpha \sum\!\!\!\!\!\!\int_{(s<n)} \sum_{\bar{l},\bar{m}} \omega_{ns}\, {}_nT_s^\mu(\omega)\, {}_sT_{n\mu}(\omega) , \tag{14}$$

where the indices $\bar{l}$ and $\bar{m}$ have been temporarily suppressed in ${}_nT_s^\mu$ and ${}_sT_n^\mu$ which in turn are form factors defined by

$${}_nT_s^\mu = \int Y_{\bar{l}\bar{m}}\, j_{\bar{l}}(\omega r)\bar{\psi}_n(\mathbf{x})\gamma^\mu\psi_s(\mathbf{x})d^3x , \tag{15a}$$

$${}_sT_n^\mu = \int Y_{\bar{l}\bar{m}}^*\, j_{\bar{l}}(\omega r)\bar{\psi}_s(\mathbf{x})\gamma^\mu\psi_n(\mathbf{x})d^3x , \tag{15b}$$

and where $\omega \equiv \omega_{ns} = E_n - E_s$ and $\mathbf{x} = (r,\theta,\phi)$. From (15) it can easily be shown that ${}_sT_{n0} = {}_nT_s^0$ and that ${}_s\mathbf{T}_n = {}_n\mathbf{T}_s^\dagger$, which together simplify (14) into

$$\Gamma_n = -4\pi\alpha \sum\!\!\!\!\!\!\int_{(s<n)} \sum_{\bar{l},\bar{m}} \omega(|{}_nT_s^0|^2 - |{}_n\mathbf{T}_s|^2) . \tag{16}$$

Using relativistic Coulomb wave functions (see Appendix B), ${}_nT_s^0$ and ${}_n\mathbf{T}_s$ can be put into the following forms:

$${}_nT_s^0 = \left[\frac{(2J_n+1)(2J_s+1)}{4\pi}\right]^{1/2} (W_{ns}^{\bar{l}\bar{m}} R_1^{\bar{l}} + W_{n's'}^{\bar{l}\bar{m}} R_2^{\bar{l}}) , \tag{17a}$$

$${}_n\mathbf{T}_s = \left[\frac{(2J_n+1)(2J_s+1)}{4\pi}\right]^{1/2} (\mathbf{K}_{ns'}^{\bar{l}\bar{m}} R_3^{\bar{l}} - \mathbf{K}_{n's}^{\bar{l}\bar{m}} R_4^{\bar{l}}) , \tag{17b}$$

where

$$W_{ns}^{\bar{l}\bar{m}} = \left[\frac{(2J_n+1)(2J_s+1)}{4\pi}\right]^{-1/2} \int Y_{\bar{l}\bar{m}}\, \Omega_n^\dagger \Omega_s\, do ,$$

$$W_{n's'}^{\bar{l}\bar{m}} = \left[\frac{(2J_n+1)(2J_s+1)}{4\pi}\right]^{-1/2} \int Y_{\bar{l}\bar{m}}\, \Omega_{n'}^\dagger \Omega_{s'}\, do ,$$

$$\mathbf{K}_{ns'}^{\bar{l}\bar{m}} = \left[\frac{(2J_n+1)(2J_s+1)}{4\pi}\right]^{-1/2} \int Y_{\bar{l}\bar{m}}\, \Omega_n^\dagger \boldsymbol{\sigma} \Omega_{s'}\, do , \tag{18a}$$

$$\mathbf{K}_{n's}^{\bar{l}\bar{m}} = \left[\frac{(2J_n+1)(2J_s+1)}{4\pi}\right]^{-1/2} \int Y_{\bar{l}\bar{m}}\, \Omega_{n'}^\dagger \boldsymbol{\sigma} \Omega_s\, do$$

($do \equiv \sin\theta\, d\theta\, d\phi$) and

$$
\begin{aligned}
R_1^{\tilde l} &= \int_0^\infty j_{\tilde l}(\omega r) g_n^*(r) g_s(r) r^2 dr , \\
R_2^{\tilde l} &= \int_0^\infty j_{\tilde l}(\omega r) f_n^*(r) f_s(r) r^2 dr , \\
R_3^{\tilde l} &= \int_0^\infty j_{\tilde l}(\omega r) g_n^*(r) f_s(r) r^2 dr , \\
R_4^{\tilde l} &= \int_0^\infty j_{\tilde l}(\omega r) f_n^*(r) g_s(r) r^2 dr .
\end{aligned}
\tag{18b}
$$

With the help of Eqs. (17), Eq. (16) becomes

$$
\Gamma_n = -\alpha \sum_{(s<n)}\!\!\!\!\!\!\!\int \omega (2J_n+1)(2J_s+1) \times \sum_{\tilde l, \tilde m} (| W_{ns}^{\tilde l \tilde m} R_1^{\tilde l} + W_{n's'}^{\tilde l \tilde m} R_2^{\tilde l} |^2 - | \mathbf{K}_{ns'}^{\tilde l \tilde m} R_3^{\tilde l} - \mathbf{K}_{n's}^{\tilde l \tilde m} R_4^{\tilde l} |^2) . \tag{19}
$$

In Eq. (19) there is a sum over M_n and M_s (these are the total magnetic quantum numbers of the initial and final states, respectively) which we have suppressed all along. Moreover, since the electron has a probability $1/g_n$ of being in any one of the magnetic sublevels $|nJ_nM_n\rangle$, where g_n is the degeneracy given by

$$
g_n = 2l_n + 1 , \tag{20}
$$

we have to multiply the total decay rate of level n by this probability. With the above considerations taken into account, the decay rate of the nth atomic level finally becomes

$$
\Gamma_n = -\alpha \sum_{(s<n)}\!\!\!\!\!\!\!\int \sum_{\tilde l, \tilde m} \sum_{M_n M_s} \omega_{ns} \frac{(2J_n+1)(2J_s+1)}{2l_n+1} \times (| W_{ns}^{\tilde l \tilde m} R_1^{\tilde l} + W_{n's'}^{\tilde l \tilde m} R_2^{\tilde l} |^2 - | \mathbf{K}_{ns'}^{\tilde l \tilde m} R_3^{\tilde l} - \mathbf{K}_{n's}^{\tilde l \tilde m} R_4^{\tilde l} |^2) . \tag{21}
$$

In the remainder of this section, we elaborate upon the various terms appearing in Eq. (21). We shall refer to the W's and $\mathbf{K}$'s as *the angular matrix elements* and to the R's as *the radial matrix elements.* In their calculation the angular matrix elements involve a number of $3j$ and $6j$ symbols. This calculation is quite lengthy and most of its details can be found in Appendix C. Only the main results are given here. The first W matrix element is given by

$$
W_{ns}^{\tilde l \tilde m} = (-1)^{M_n - 1/2} \sqrt{(2\tilde l+1)(2l_n+1)(2l_s+1)} \times \begin{pmatrix} l_n & l_s & \tilde l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} J_n & J_s & \tilde l \\ -M_n & M_s & \tilde m \end{pmatrix} \begin{Bmatrix} J_n & J_s & \tilde l \\ l_s & l_n & \frac{1}{2} \end{Bmatrix} . \tag{22}
$$

The range of $\tilde l$ in this expression is restricted to the values given by

$$
| l_n - l_s | < \tilde l < l_n + l_s , \quad l_n + \tilde l + l_s = \text{an even integer} . \tag{23}
$$

$W_{n's'}^{\tilde l \tilde m}$ can readily be found from (22) by merely letting $l_n \to l_n'$ and $l_s \to l_s'$. The same transformation, of course, applied to the condition (23), yields the values of $\tilde l$ that go with $W_{n's'}^{\tilde l \tilde m}$. For the $\mathbf{K}$ matrix elements, we find

$$
\mathbf{K}_{ns'}^{\tilde l \tilde m} = (-1)^{J_n + J_s - M_s + 3/2} \sqrt{(2\tilde l+1)(2l_n+1)(2l_s'+1)} \times \begin{pmatrix} l_n & l_s' & \tilde l \\ 0 & 0 & 0 \end{pmatrix} [(a_1 - a_2)\hat{\mathbf{i}} - i(a_1 + a_2)\hat{\mathbf{j}} + (b_1 + b_2)\hat{\mathbf{k}}] , \tag{24}
$$

where

$$
\begin{aligned}
a_1 &= \begin{pmatrix} l_n & J_n & \frac{1}{2} \\ M_n - \frac{1}{2} & -M_n & \frac{1}{2} \end{pmatrix} \begin{pmatrix} l_s' & J_s & \frac{1}{2} \\ M_s + \frac{1}{2} & -M_s & -\frac{1}{2} \end{pmatrix} \times \begin{pmatrix} l_n & l_s' & \tilde l \\ -M_n + \frac{1}{2} & M_s + \frac{1}{2} & \tilde m \end{pmatrix} , \\
a_2 &= \begin{pmatrix} l_n & J_n & \frac{1}{2} \\ M_n + \frac{1}{2} & -M_n & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} l_s' & J_s & \frac{1}{2} \\ M_s - \frac{1}{2} & -M_s & \frac{1}{2} \end{pmatrix} \times \begin{pmatrix} l_n & l_s' & \tilde l \\ -M_n - \frac{1}{2} & M_s - \frac{1}{2} & \tilde m \end{pmatrix} ,
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
b_1 &= \begin{pmatrix} l_n & J_n & \frac{1}{2} \\ M_n - \frac{1}{2} & -M_n & \frac{1}{2} \end{pmatrix} \begin{pmatrix} l_s' & J_s & \frac{1}{2} \\ M_s - \frac{1}{2} & -M_s & \frac{1}{2} \end{pmatrix} \times \begin{pmatrix} l_n & l_s' & \tilde l \\ -M_n + \frac{1}{2} & M_s - \frac{1}{2} & \tilde m \end{pmatrix} , \\
b_2 &= \begin{pmatrix} l_n & J_n & \frac{1}{2} \\ M_n + \frac{1}{2} & -M_n & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} l_s' & J_s & \frac{1}{2} \\ M_s + \frac{1}{2} & -M_s & -\frac{1}{2} \end{pmatrix} \times \begin{pmatrix} l_n & l_s' & \tilde l \\ -M_n - \frac{1}{2} & M_s + \frac{1}{2} & \tilde m \end{pmatrix} ,
\end{aligned}
$$

with the range of $\tilde l$ defined by

$$
| l_n - l_s' | < \tilde l < l_n + l_s' , \quad l_n + \tilde l + l_s' = \text{an even integer} . \tag{26}
$$

Here, too, the expression for $\mathbf{K}_{n's}^{\tilde l \tilde m}$ as well as the defining equation of $\tilde l$ that goes with it can trivially be written down from (25) and (26), respectively, by letting $l_n \to l_n'$ and $l_s' \to l_s$.

On the other hand, all the radial matrix elements can be calculated exactly with the help of the substitution[6]

$$
j_{\tilde l}(\omega r) = \left[\frac{\pi}{2\omega r} \right]^{1/2} J_{\tilde l + (1/2)}(\omega r) .
$$

The final results are

$$R_1^{\bar{l}} = \left[\left(1+\frac{E_n}{m}\right)\left(1+\frac{E_s}{m}\right)\right]^{1/2} U_n U_s \times \{I_1^{\bar{l}} - I_2^{\bar{l}} - I_3^{\bar{l}} + I_4^{\bar{l}}\} ,$$

$$R_2^{\bar{l}} = \left[\left(1-\frac{E_n}{m}\right)\left(1-\frac{E_s}{m}\right)\right]^{1/2} U_n U_s \times \{I_1^{\bar{l}} + I_2^{\bar{l}} + I_3^{\bar{l}} + I_4^{\bar{l}}\} ,$$

$$R_3^{\bar{l}} = -\left[\left(1+\frac{E_n}{m}\right)\left(1-\frac{E_s}{m}\right)\right]^{1/2} U_n U_s \times \{I_1^{\bar{l}} + I_2^{\bar{l}} - I_3^{\bar{l}} - I_4^{\bar{l}}\} , \tag{27a}$$

$$R_4^{\bar{l}} = -\left[\left(1-\frac{E_n}{m}\right)\left(1+\frac{E_s}{m}\right)\right]^{1/2} U_n U_s \times \{I_1^{\bar{l}} - I_2^{\bar{l}} + I_3^{\bar{l}} - I_4^{\bar{l}}\} ,$$

where

$$I_1^{\bar{l}} = n_r s_r (2\lambda_n)^{\gamma_n - 1}(2\lambda_s)^{\gamma_s - 1} \times \sum_{p=0}^{n_r - 1}\sum_{q=0}^{s_r - 1} \frac{(-n_r+1)_p(-s_r+1)_q}{(2\gamma_n+1)_p(2\gamma_s)_q}\frac{(2\lambda_n)^p(2\lambda_s)^q}{p!q!}H_{pq}^{\bar{l}} ,$$

$$I_2^{\bar{l}} = n_r (N_s - \kappa_s)(2\lambda_n)^{\gamma_n - 1}(2\lambda_s)^{\gamma_s - 1} \times \sum_{p=0}^{n_r - 1}\sum_{q=0}^{s_r} \frac{(-n_r+1)_p(-s_r)_q}{(2\gamma_n+1)_p(2\gamma_s)_q}\frac{(2\lambda_n)^p(2\lambda_s)^q}{p!q!}H_{pq}^{\bar{l}} , \tag{27b}$$

$$I_3^{\bar{l}} = (N_n - \kappa_n) s_r (2\lambda_n)^{\gamma_n - 1}(2\lambda_s)^{\gamma_s - 1} \times \sum_{p=0}^{n_r}\sum_{q=0}^{s_r - 1} \frac{(-n_r)_p(-s_r+1)_q}{(2\gamma_n+1)_p(2\gamma_s)_q}\frac{(2\lambda_n)^p(2\lambda_s)^q}{p!q!}H_{pq}^{\bar{l}} ,$$

$$I_4^{\bar{l}} = (N_n - \kappa_n)(N_s - \kappa_s)(2\lambda_n)^{\gamma_n - 1}(2\lambda_s)^{\gamma_s - 1} \times \sum_{p=0}^{n_r}\sum_{q=0}^{s_r} \frac{(-n_r)_p(-s_r)_q}{(2\gamma_n+1)_p(2\gamma_s)_q}\frac{(2\lambda_n)^p(2\lambda_s)^q}{p!q!}H_{pq}^{\bar{l}} ,$$

$$H_{pq}^{\bar{l}} = \left[\frac{\pi}{4}\right]^{1/2}\left[\frac{\omega}{2}\right]^{\bar{l}}\frac{\Gamma(\gamma_n+\gamma_s+p+q+\bar{l}+1)}{\Gamma(\bar{l}+\frac{3}{2})} \times \frac{(\lambda_n+\lambda_s)^{-(\gamma_n+\gamma_s+p+q+\bar{l}+1)}}{(1+x^2)^{\gamma_n+\gamma_s+p+q}} \times {}_2F_1(a,b,\bar{l}+\tfrac{3}{2};-x^2) , \tag{28}$$

and

$$a = -\frac{\gamma_n+\gamma_s+p+q-\bar{l}-1}{2} ,$$

$$b = -\frac{\gamma_n+\gamma_s+p+q-\bar{l}-2}{2} ,$$

$$x = \frac{\omega}{(\lambda_n+\lambda_s)} .$$

The definitions of the remaining parameters in Eqs. (27) and (28) are collected in Appendix B [Eqs. (B3)].

We now have expressions for all of the matrix elements needed for the calculation of relativistic decay rates using Eq. (21). Owing to the restrictions imposed upon the values of the index $\bar{l}$, Eqs. (23) and (26), the sums over the indices $\bar{l}$ and $\bar{m}$ are no longer infinite. In fact, Eqs. (23) and (26) can be regarded as the selection rules of the theory. The first part of (26), namely, $|l_n - l_s'| < \bar{l} < l_n + l_s'$, is similar to the selection rule familiar from the electric field multipole expansion,[7] because we can effectively interpret $\bar{l}$ as the carrier of the photon angular momentum, although we did not use the concept of a photon as such. In this respect, Eq. (26) is an expression of the law of conservation of angular momentum. In Sec. IV we apply Eq. (21) to the calculation of some decay rates in hydrogen and muonium. Notice that the dependence of the decay rate Γ upon the atomic number Z is solely in the radial matrix elements $R_i^{\bar{l}}$, $i=1,\ldots,4$.

IV. EXAMPLES

In this section, we apply our equation to some of the radiative decay processes of some of the low-lying excited states in hydrogen and muonium. Our aim in presenting these examples is to demonstrate the correctness of the approach as it stands in comparison with the standard well-understood theory.

As has been explained in Sec. III, when it comes to a specific calculation of the decay rate using Eq. (21), the sums over the indices $\bar{l}$ and $\bar{m}$ are finite. In fact, for each allowed value of the index $\bar{l}$, the remaining sums over $\bar{m}$, M_n, and M_s can easily be carried out explicitly without the need to evaluate the $3j$ and $6j$ symbols in most cases as will be shown shortly. The general procedure for calculating a decay rate is outlined as follows.

(a) Identify the quantum numbers n, l_n, and J_n of the initial and final states and calculate the derived ones, namely l_n', κ_n, and n_r [see Appendix A and Table I].

(b) Use Eq. (23) and similar ones to calculate the allowed values of the index $\bar{l}$ for each of the angular matrix elements. The results of doing so for the examples we study are collected in Table II.

TABLE I. Quantum numbers of the states under investigation.

Level	n	l_n	J_n	l'_n	κ_n	n_r
$1S_{1/2}$	1	0	$\frac{1}{2}$	1	-1	0
$2S_{1/2}$	2	0	$\frac{1}{2}$	1	-1	1
$2P_{1/2}$	2	1	$\frac{1}{2}$	0	1	1
$2P_{3/2}$	2	1	$\frac{3}{2}$	2	-2	0

(c) Use the results of (a) and (b) in order to write out Eq. (21) with the sum over $\tilde{l}$ carried out explicitly. Only the sums over $\tilde{m}$, M_n, and M_s remain to be carried out in the remaining steps.

(d) Calculate the numbers

$$\sum_{\tilde{m},M_n,M_s} |W_{ns}^{\tilde{l}\tilde{m}}|^2, \quad \sum_{\tilde{m},M_n,M_s} |\mathbf{K}_{ns'}^{\tilde{l}\tilde{m}}|^2, \ldots, \text{etc} ,$$

utilizing the symmetry properties of the $3j$ and $6j$ symbols and by quoting their tabulated values[8] if necessary. In the case of the K's, the scalar products are obviously carried out first, i.e.,

$$\sum |\mathbf{K}|^2 \propto \sum [2(a_1^2+a_2^2+b_1b_2)+b_1^2+b_2^2] \tag{29}$$

and

$$\sum \mathbf{K}\cdot\mathbf{K}' \propto \sum [2(a_1a_1'+a_2a_2')+b_1b_1'+b_1b_2' +b_2b_1'+b_2b_2'] . \tag{30}$$

Notice that in the process of calculation some angular matrix elements whose $\tilde{l}$ index is allowed by Eqs. (23) and (26) may vanish due to the vanishing of some $3j$ or $6j$ symbol that enters into their definitions. An example of this is the vanishing of $W_{n's'}^{3\tilde{m}}$ in the decay rate $\Gamma(2P_{2/3}\rightarrow 1S_{1/2})$.

(e) Use Eqs. (27) and (28) in order to calculate all the radial matrix elements. This process is also quite tedious. In the example we present here, the radial matrix elements as well as the decay rates as given by Eqs. (32)–(35) below, were calculated to double precision using a simple FORTRAN program. In the program a series representation of the hypergeometric function in Eq. (28) was employed, whereby[9]

$${}_2F_1(a,b,c,z)=1+\frac{ab}{c}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} +\cdots+O(z^5) . \tag{31}$$

In our examples, $z=-x^2<O(\alpha^2)$ and a and b are extremely close to negative integers, which justifies the use of Eq. (31). Of course, Eqs. (32)–(35) are used for calculating the decay rates of both hydrogen and muonium, the only difference being in the reduced mass[10] m. We follow the procedure outlined above in calculating the following decay rates for both hydrogen and muonium. Everywhere in the examples below, $\sum$ stands for sums over the indices $\tilde{m}$, M_n, and M_s, which we do not show for convenience.

TABLE II. Values assumed by the index $\tilde{l}$ for each of the angular matrix elements.

Transition	$W_{ns}^{\tilde{l}\tilde{m}}$	$W_{n's'}^{\tilde{l}\tilde{m}}$	$\mathbf{K}_{ns'}^{\tilde{l}\tilde{m}}$	$\mathbf{K}_{n's}^{\tilde{l}\tilde{m}}$
$2S_{1/2}\rightarrow 1S_{1/2}$	0	0,2	1	1
$2S_{1/2}\rightarrow 2P_{1/2}$	1	1	0	0,2
$2P_{1/2}\rightarrow 1S_{1/2}$	1	1	0,2	0
$2P_{3/2}\rightarrow 1S_{1/2}$	1	1,3	0,2	2

(1) *The $2S_{1/2}\rightarrow 1S_{1/2}$ transition,*

$$\Gamma=-4\alpha\omega\sum(|W_{ns}^{00}|^2|R_1^0|^2+|W_{n's'}^{00}|^2|R_2^0|^2 +2W_{ns}^{00}W_{n's'}^{00}R_1^0R_2^0+|W_{n's'}^{2\tilde{m}}|^2|R_2^2|^2 -|\mathbf{K}_{ns'}^{1\tilde{m}}|^2|R_3^1|^2-|\mathbf{K}_{n's}^{1\tilde{m}}|^2|R_4^1|^2 +2\mathbf{K}_{ns'}^{1\tilde{m}}\cdot\mathbf{K}_{n's}^{1\tilde{m}}R_3^1R_4^1) ,$$

$$\sum|W_{ns}^{00}|^2=\sum|W_{n's'}^{00}|^2=\sum W_{ns}^{00}W_{n's'}^{00}=\tfrac{1}{2} ,$$

and

$$\sum|W_{n's'}^{2\tilde{m}}|^2=0 ,$$

$$\sum|\mathbf{K}_{ns'}^{1\tilde{m}}|^2=3\sum[2(a_1^2+a_2^2+b_1b_2)+b_1^2+b_2^2] =3[2(\tfrac{1}{2}+\tfrac{1}{2}+0)+\tfrac{1}{2}+\tfrac{1}{2}]=\tfrac{3}{2} .$$

Similarly,

$$\sum|\mathbf{K}_{n's}^{1\tilde{m}}|^2=3\sum[2(a_1'^2+a_2'^2+b_1'b_2')+b_1'^2+b_2'^2] =3[2(\tfrac{1}{2}+\tfrac{1}{2}+0)+\tfrac{1}{2}+\tfrac{1}{2}]=\tfrac{3}{2} ,$$

$$\sum\mathbf{K}_{ns'}^{1\tilde{m}}\cdot\mathbf{K}_{n's}^{1\tilde{m}}=3\sum[2(a_1a_1'+a_2a_2')+b_1b_1'+b_2b_2' +b_1b_2'+b_2b_1'] =3[2(-\tfrac{1}{36}-\tfrac{1}{36})+\tfrac{1}{36}+\tfrac{1}{36}-\tfrac{1}{18}-\tfrac{1}{18}]=-\tfrac{1}{2} .$$

Thus,

$$\Gamma(2S_{1/2}\rightarrow 1S_{1/2})=-2\alpha\omega[(R_1^0)^2+(R_2^0)^2+2(R_1^0)(R_2^0) -3(R_3^1)^2-3(R_4^1)^2 -2(R_3^1)(R_4^1)] . \tag{32}$$

(2) *The $2S_{1/2}\rightarrow 2P_{1/2}$ transition,*

$$\Gamma(2S_{1/2}\to 2P_{1/2})=-\tfrac{4}{3}\alpha\omega\sum(|W_{ns}^{1\bar m}|^2|R_1^1|^2+|W_{n's'}^{1\bar m}|^2|R_2^1|^2+2W_{ns}^{1\bar m}W_{n's'}^{1\bar m}R_1^1R_2^1-|\mathbf{K}_{ns'}^{00}|^2|R_3^0|^2$$
$$-|\mathbf{K}_{n's}^{00}|^2|R_4^0|^2+2\mathbf{K}_{ns'}^{00}\cdot\mathbf{K}_{n's}^{00}R_3^0R_4^0-|\mathbf{K}_{n's}^{2\bar m}|^2|R_4^2|^2)\ ,$$

$$\sum|W_{ns}^{1\bar m}|^2=\sum|W_{n's'}^{1\bar m}|^2=\sum W_{ns}^{1\bar m}W_{n's'}^{1\bar m}=\tfrac{1}{2}\ ,$$

$$\sum|\mathbf{K}_{ns'}^{00}|^2=\sum[2(a_1^2+a_2^2+b_1b_2)+b_1^2+b_2^2]=[2(\tfrac14+\tfrac14+0)+\tfrac14+\tfrac14]=\tfrac32\ ,$$

$$\sum|\mathbf{K}_{ns'}^{00}|^2=3\sum[2(a_1^2+a_2^2+b_1b_2)+b_1^2+b_2^2]=3[2(\tfrac14+\tfrac14+0)+\tfrac14+\tfrac14]=\tfrac32\ ,$$

$$\sum|\mathbf{K}_{n's}^{00}|^2=3\sum[2(a_1'^2+a_2'^2+b_1'b_2')+b_1'^2+b_2'^2]=3\sum[2(\tfrac{1}{108}+\tfrac{1}{108}-\tfrac{1}{27})+\tfrac{5}{108}+\tfrac{5}{108}]=\tfrac16\ ,$$

$$\sum\mathbf{K}_{ns'}^{00}\cdot\mathbf{K}_{n's}^{00}=-\sqrt{3}\sum[2(a_1a_1'+a_2a_2')+b_1b_1'+b_2b_2'+b_1b_2'+b_2b_1']$$
$$=-\sqrt{3}\left[2\left[-\frac{1}{12\sqrt3}-\frac{1}{12\sqrt3}\right]-\frac{1}{12\sqrt3}-\frac{1}{12\sqrt3}+\frac{1}{6\sqrt3}+\frac{1}{6\sqrt3}\right]=\frac16\ ,$$

$$\sum|\mathbf{K}_{n's}^{2\bar m}|^2=6\sum[2(a_1'^2+a_2'^2+b_1'b_2')+b_1'^2+b_2'^2]=6[2(\tfrac{1}{27}+\tfrac{1}{27}+\tfrac{1}{54})+\tfrac{1}{54}+\tfrac{1}{54}]=\tfrac43\ ,$$

$$\Gamma(2S_{1/2}\to 2P_{1/2})=-\tfrac{2}{9}\alpha\omega[3(R_1^1)^2+3(R_2^1)^2+6(R_1^1)(R_2^1)-9(R_3^0)^2-(R_4^0)^2-8(R_4^2)^2+2(R_3^0)(R_4^0)]\ . \tag{33}$$

(3) *The $2P_{3/2}\to 1S_{1/2}$ transition,*

$$\Gamma=-\tfrac{8}{3}\alpha\omega\sum(|W_{ns}^{1\bar m}|^2|R_1^1|^2+|W_{n's'}^{1\bar m}|^2|R_2^1|^2+2W_{ns}^{1\bar m}W_{n's'}^{1\bar m}R_1^1R_2^1+|W_{n's'}^{3\bar m}|^2|R_2^3|^2-|\mathbf{K}_{ns'}^{00}|^2|R_3^0|^2$$
$$-|\mathbf{K}_{ns'}^{2\bar m}|^2|R_3^2|^2-|\mathbf{K}_{n's}^{2\bar m}|^2|R_4^2|^2+2K_{ns'}^{2\bar m}\cdot\mathbf{K}_{n's}^{2\bar m}R_3^2R_4^2)\ ,$$

$$\sum|W_{ns}^{1\bar m}|^2=\sum|W_{n's'}^{1\bar m}|^2=\sum W_{ns}^{1\bar m}W_{n's'}^{1\bar m}=\tfrac12\ ,$$

and

$$\sum|W_{n's'}^{3\bar m}|^2=0\ ,$$

$$\sum|\mathbf{K}_{ns'}^{00}|^2=3\sum[2(a_1^2+a_2^2+b_1b_2)+b_1^2+b_2^2]=3[2(\tfrac{1}{27}+\tfrac{1}{27}+\tfrac{1}{54})+\tfrac{1}{54}+\tfrac{1}{54}]=\tfrac23\ ,$$

$$\sum|\mathbf{K}_{ns'}^{2\bar m}|^2=6\sum[2(a_1^2+a_2^2+b_1b_2)+b_1^2+b_2^2]=6[2(\tfrac{5}{216}+\tfrac{5}{216}-\tfrac{1}{108})+\tfrac{7}{216}+\tfrac{7}{216}]=\tfrac56\ ,$$

$$\sum|\mathbf{K}_{n's}^{2\bar m}|^2=5\sum[2(a_1'^2+a_2'^2+b_1'b_2')+b_1'^2+b_2'^2]=5[2(\tfrac{1}{20}+\tfrac{1}{20}+0)+\tfrac{1}{20}+\tfrac{1}{20}]=\tfrac32\ ,$$

$$\sum\mathbf{K}_{ns'}^{2\bar m}\cdot\mathbf{K}_{n's}^{2\bar m}=\sqrt{30}\sum[2(a_1a_1'+a_2a_2')+b_1b_1'+b_2b_2'+b_1b_2'+b_2b_1']$$
$$=\sqrt{30}\left[2\left[-\frac{1}{12\sqrt{30}}-\frac{1}{12\sqrt{30}}\right]+\frac{1}{12\sqrt{30}}-\frac{1}{12\sqrt{30}}-\frac{1}{6\sqrt{30}}-\frac{1}{6\sqrt{30}}\right]=-\frac12\ ,$$

$$\Gamma(2P_{3/2}\to 1S_{1/2})=-\tfrac{4}{9}\alpha\omega[3(R_1^1)^2+3(R_2^1)^2+6(R_1^1)(R_2^1)-4(R_3^0)^2-5(R_3^2)^2-9(R_4^2)^2-6(R_3^2)(R_4^2)]\ . \tag{34}$$

(4) *The $2P_{1/2}\to 1S_{1/2}$ transition,*

$$\Gamma=-\tfrac{4}{3}\alpha\omega\sum(|W_{ns}^{1\bar m}|^2|R_1^1|^2+|W_{n's'}^{1\bar m}|^2|R_2^1|^2+2W_{ns}^{1\bar m}W_{n's'}^{1\bar m}R_1^1R_2^1-|\mathbf{K}_{ns'}^{00}|^2|R_3^0|^2-|\mathbf{K}_{ns'}^{2\bar m}|^2|R_3^2|^2$$
$$-|\mathbf{K}_{n's}^{00}|^2|R_4^0|^2+2\mathbf{K}_{ns'}^{00}\cdot\mathbf{K}_{n's}^{00}R_3^0R_4^0)\ ,$$

$$\sum|W_{ns}^{1\bar m}|^2=\sum|W_{n's'}^{1\bar m}|^2=\sum W_{ns}^{1\bar m}W_{n's'}^{1\bar m}=\tfrac12\ .$$

$$\sum|\mathbf{K}_{ns'}^{00}|^2=3\sum[2(a_1^2+a_2^2+b_1b_2)+b_1^2+b_2^2]=3[2(\tfrac{1}{108}+\tfrac{1}{108}-\tfrac{1}{27})+\tfrac{5}{108}+\tfrac{5}{108}]=\tfrac16\ ,$$

$$\sum|\mathbf{K}_{n's}^{00}|^2=\sum[2(a_1'^2+a_2'^2+b_1'b_2')+b_1'^2+b_2'^2]=[2(\tfrac14+\tfrac14+0)+\tfrac14+\tfrac14]=\tfrac32\ ,$$

$$\sum\mathbf{K}_{ns'}^{00}\cdot\mathbf{K}_{n's}^{00}=-\sqrt{3}\sum[2(a_1a_1'+a_2a_2')+b_1b_1'+b_2b_2'+b_1b_2'+b_2b_1']$$
$$=-\sqrt{3}\left[2\left[\frac{1}{12\sqrt3}+\frac{1}{12\sqrt3}\right]-\frac{1}{12\sqrt3}-\frac{1}{12\sqrt3}+\frac{1}{6\sqrt3}+\frac{1}{6\sqrt3}\right]=-\frac12\ ,$$

$$\sum|\mathbf{K}_{ns'}^{2\bar m}|^2=6\sum[2(a_1^2+a_2^2+b_1b_2)+b_1^2+b_2^2]=6[2(\tfrac{1}{27}+\tfrac{1}{27}+\tfrac{1}{54})+\tfrac{1}{54}+\tfrac{1}{54}]=\tfrac43\ ,$$

$$\Gamma(2P_{1/2}\to 1S_{1/2})=-\tfrac{2}{9}\alpha\omega[3(R_1^1)^2+3(R_2^1)^2+6(R_1^1)(R_2^1)-(R_3^0)^2-9(R_4^0)^2-8(R_4^2)^2-6(R_3^0)(R_4^0)]\ . \tag{35}$$

We collect the results of our calculations in Table III. In Sec. V we discuss these results and compare them with the available data.

V. DISCUSSION AND CONCLUSIONS

In this work, we have derived a general formula for the relativistic decay rates in atoms for transitions from any state n to all lower states s ($s<n$). In applying our general formula to the specific examples presented in Sec. IV we obtained Eqs. (32)–(35) which, in fact, are applicable to a whole host of transitions besides the ones we considered. For example, Eqs. (34) and (35) can be used for calculating $\Gamma(nP \to n'S)$, for any n and n', where $n'<n$. Equations (32) and (33) can be generalized in a similar fashion.

For the $2P \to 1S_{1/2}$ transition our result is in perfect agreement with the most recent and most accurate calculations. We quote here, for the sake of comparison, the result tabulated in Ref. 11 of $\Gamma(2P \to 1S_{1/2}) = 6.265\times10^{8}$ s^{-1}. According to this reference, this figure has an accuracy of better than 1%. Moreover, our result gives the radiative mean lifetime of $\tau_{2P}=1.5962\times10^{-9}$ s. In 1968 Chupp and co-workers[12] obtained experimentally the result $\tau_{2P}=(1.60\pm0.01)\times10^{-9}$ s using the technique of beam-foil excitation.

The calculation involving the $2S_{1/2}$ level, on the other hand, requires some discussion. In the nonrelativistic calculations, based upon the dipole approximation, the transitions from this level are forbidden by the selection rules involving parity for the electric dipole and the total angular momentum for the electric quadrupole transitions, respectively. Also, since this is an S state ($l=0$), the magnetic dipole moment is a purely spin quantity and its matrix element, therefore, vanishes between nonrelativistic wave functions. However, if relativistic wave functions[13] are used instead, one gets the small transition probability of 2.4959×10^{-6} s^{-1}. Of course, there is no reason why two or more photons should not be simultaneously emitted as long as they share the total transition energy in conformity with the conservation of energy principle. With this in mind, and with the interest in this transition in connection with interstellar hydrogen[14] (it contributes to the observed continuum in planetary nebulas), Breit and Teller[15] showed that a double-photon electric dipole transition has a probability that can be bracketed by $6.5\ s^{-1} < \Gamma(2S_{1/2}\to 1S_{1/2}) < 8.7\ s^{-1}$. More accurate calculations followed leading to the most accepted relativistic result[13] of $\Gamma(2S_{1/2}\to1S_{1/2})=(8.2291 \pm 0.0001)\ s^{-1}$. There is also the other calculation involving the transition to the Lamb-shifted level $2P_{1/2}$. We quote here the result of $\Gamma(2S_{1/2}\to 2P_{1/2})=8\times10^{-10}\ s^{-1}$ from Ref. 16, according to which it has been calculated as an electric dipole transition. Shapiro and Breit[17] also gave a rough estimate of the decay rate for this process ($\approx 2\times10^{-10}$ s^{-1}). In our calculation of $\Gamma(2S_{1/2}\to 2P_{1/2})$, we have used the Lamb-shift frequency[18] $\omega=(0.41)m\alpha^5$ in obtaining the statistically weighted rate shown in Table III.

TABLE III. Decay rates (s^{-1}) in hydrogen and muonium as calculated from Eqs. (32)–(35). Notice that $\Gamma(2P\to1S_{1/2}) = \Gamma(2P_{1/2}\to1S_{1/2})+\Gamma(2P_{3/2}\to1S_{1/2})$.

Transition	Hydrogen	Muonium
$2S_{1/2}\to1S_{1/2}$	2.4946×10^{-6}	2.3997×10^{-6}
$2S_{1/2}\to2P_{1/2}$	5.194×10^{-10}	5.172×10^{-10}
$2P_{1/2}\to1S_{1/2}$	2.0883×10^{8}	2.0794×10^{8}
$2P_{3/2}\to1S_{1/2}$	4.1766×10^{8}	4.1587×10^{8}
$2P\to1S_{1/2}$	6.2649×10^{8}	6.2382×10^{8}

TABLE IV. Corrections to the transition rates in hydrogen due to the hyperfine structure.

Transition	$\left\lvert \frac{\partial\Gamma}{\partial\omega}\right\rvert \delta\omega$
$2S_{1/2}\to S_{1/2}$	$4.037\times10^{-13}\delta\omega$
$2P_{1/2}\to1S_{1/2}$	$1.2581\times10^{-12}\delta\omega$
$2P_{3/2}\to1S_{1/2}$	$1.3476\times10^{-8}\delta\omega$

As far as hydrogen is concerned, no experimental measurement of the lifetime of the $2S_{1/2}$ level has come to our knowledge, but observation of the same process of decay in helium and other members of the hydrogen isoelectronic sequence strongly supports the two-photon theory.[19]

In deciding the significant digits in our results, we were guided by a calculation of the corrections to the decay rates due to the hyperfine splitting (of order $m\alpha^4$) and the Lamb shift (of order $m\alpha^5$). These radiative corrections propagate their effect upon the decay rate through the latter's dependence upon the transition frequency ω. We calculate the corrections $\delta\Gamma$ to the decay rates from the equation

$$\delta\Gamma=\frac{\partial\Gamma}{\partial\omega}\delta\omega\ .$$

In Table IV we show the values of $|\partial\Gamma/\partial\omega|$ for all of the transitions except the $2S_{1/2}\to2P_{1/2}$, where the transition frequency has been taken as the Lamb-shift frequency (which is already at least two orders of magnitude smaller than the correction due to hyperfine structure).

We have shown in this work that a simple formulation of the radiative processes that makes no use of the second quantized electromagnetic field and which involves only the first quantized matter field is possible and does produce results for the radiative decay lifetimes of the low-lying excited states in hydrogen that are in excellent agreement with all previous calculations as well as with the results of the experiments performed so far.

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APPENDIX A: THE k INTEGRATION

We have for the k integration,

$$I=\frac{\pi}{2(2\pi)^3}\int\frac{d^3k}{k}\delta(k-\omega)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$$
$$=\frac{\pi}{2(2\pi)^3}\int k\delta(k-\omega)dk\sum_{\bar{l},\bar{m}}\sum_{\bar{l}',\bar{m}'}(4\pi)^2 j_{\bar{l}}(kr)j_{\bar{l}'}(kr')Y_{\bar{l},\bar{m}}(\theta,\phi)Y^*_{\bar{l}',\bar{m}'}(\theta',\phi')\int Y_{\bar{l},\bar{m}}(\theta_k,\phi_k)Y^*_{\bar{l}',\bar{m}'}(\theta_k,\phi_k)d\Omega_k\ .$$

Here we have taken $\mathbf{x}=(r,\theta,\phi)$, $\mathbf{y}=(r',\theta',\phi')$, and $\mathbf{k}=(k,\theta_k,\phi_k)$.

The angular integrations yield $\delta_{\bar{l}\bar{l}'}\delta_{\bar{m}\bar{m}'}$, while the radial one, by virtue of the δ function, gives $\omega j_{\bar{l}}(\omega r)j_{\bar{l}}(\omega r')$. Therefore,

$$I=\sum_{\bar{l}=0}^{\infty}\sum_{\bar{m}=-\bar{l}}^{\bar{l}}\omega j_{\bar{l}}(\omega r)j_{\bar{l}}(\omega r')Y_{\bar{l}\bar{m}}(\theta,\phi)Y^*_{\bar{l}\bar{m}}(\theta',\phi')\ .$$

APPENDIX B: THE DIRAC-COULOMB WAVE FUNCTIONS

We have for the Dirac-Coulomb wave functions,

$$\psi_n(\mathbf{x})=\begin{bmatrix} g_n(r)\Omega_n(\hat{\mathbf{r}}) \\ if_n(r)\Omega_{n'}(\hat{\mathbf{r}})\end{bmatrix}\ . \tag{B1}$$

The subscripts n and n' stand collectively for the principal quantum number n as well as the angular momentum quantum numbers J_n, l_n, and M_n. In other words, $n\equiv(n,J_n,l_n,M_n)$ and $n'\equiv(n,J_n,l_n',M_n)$. Also, $l_n'=2J_n-l_n=l_n\pm1$. The radial parts, the $g_n(r)$ and $f_n(r)$, involve confluent hypergeometric functions with negative-integral first arguments (only true of the wave functions of the discrete spectrum[20] with $|E_n|<m$). This property permits a confluent hypergeometric function to be written as a polynomial:

$$\begin{aligned} g_n(r)&=\left[1+\frac{E_n}{m}\right]^{1/2}U_n(A_n-B_n)\ ,\\ f_n(r)&=-\left[1-\frac{E_n}{m}\right]^{1/2}U_n(A_n+B_n)\ ,\end{aligned} \tag{B2}$$

where

$$\begin{aligned} U_n&=\frac{(2\lambda_n)^{3/2}}{\Gamma(2\gamma_n+1)}\left[\frac{\Gamma(2\gamma_n+n_r+1)}{4N_n(N_n-\kappa_n)n_r!}\right]^{1/2}\ ,\\ A_n(r)&=n_rF(-n_r+1,2\gamma_n+1;2\lambda_n r)e^{-\lambda_n r}(2\lambda_n r)^{\gamma_n-1}\ ,\\ B_n(r)&=(N_n-\kappa_n)F(-n_r,2\gamma_n+1;2\lambda_n r)e^{-\lambda_n r}\\ &\quad\times(2\lambda_n r)^{\gamma_n-1}\ ,\end{aligned} \tag{B3}$$

$$\begin{aligned} F(-n,b;z)&=\sum_{m=0}^{n}\frac{(-n)_m}{(b)_m}\frac{z^m}{m!}\ ,\\ (a)_m&=\frac{\Gamma(a+m)}{\Gamma(a)}\ ,\\ (a)_0&\equiv1\ ,\end{aligned} \tag{B4}$$

and where

$$\begin{aligned} \lambda_n&=\frac{Z\alpha m}{N_n}\ ,\\ N_n&=[n^2-2n_r(|\kappa_n|-\gamma_n)]^{1/2}\ ,\\ E_n^2&=-\lambda_n^2+m^2\ ,\\ \gamma_n&=[\kappa_n^2-(Z\alpha)^2]^{1/2}\ ,\\ n_r&=n-|\kappa_n|\ ,\\ \kappa_n&=\begin{cases}-(l_n+1) & \text{if } J_n=l_n+\frac{1}{2}\ ,\\ l_n & \text{if } J_n=l_n-\frac{1}{2}\ .\end{cases}\end{aligned}$$

The angular parts are defined by[7]

$$\Omega_n=(-1)^{1/2-l_n-M_n}\sqrt{2J_n+1}\times\sum_{m_n,\mu_n}\begin{bmatrix} l_n & \frac{1}{2} & J_n \\ m_n & \mu_n & -M_n\end{bmatrix}|l_nm_n\rangle\chi_{\mu_n}\ , \tag{B5}$$

and $\Omega_{n'}$ is obtained from Ω_n by letting $l_n\to l_n'$ and $m_n\to m_n'$. χ_{μ_n} is the usual two-component Pauli spinor.

APPENDIX C: SPINORIAL ALGEBRA

With the help of the definition of a spherical spinor, Eq. (B5), the first of Eqs. (18) becomes

$$W_{ns}^{\bar{l}\bar{m}}=(4\pi)^{1/2}(-1)^{1-l_n-l_s-M_n-M_s}\times\sum_{m_n,\mu_n,m_s,\mu_s}\begin{bmatrix} l_n & \frac{1}{2} & J_n \\ m_n & \mu_n & -M_n\end{bmatrix}\begin{bmatrix} l_s & \frac{1}{2} & J_s \\ m_s & \mu_s & -M_s\end{bmatrix}\times\langle l_nm_n|\bar{l}\bar{m}|l_sm_s\rangle\chi^\dagger_{\mu_n}\chi_{\mu_s}\ .$$

Now, $\chi^\dagger_{\mu_n}\chi_{\mu_s}=\delta_{\mu_n\mu_s}$ and[21]

$$\langle l_n m_n \,|\, \tilde{l}\tilde{m} \,|\, l_s m_s \rangle = (-1)^{m_n} \left[\frac{(2l_n+1)(2l_s+1)(2\tilde{l}+1)}{4\pi} \right]^{1/2} \begin{pmatrix} l_n & \tilde{l} & l_s \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_n & \tilde{l} & l_s \\ -m_n & \tilde{m} & m_s \end{pmatrix} . \tag{C1}$$

Thus, letting $\mu_n = \mu_s = \mu$ and using (C1), the expression for $W_{ns}^{\tilde{l}\tilde{m}}$ becomes

$$W_{ns}^{\tilde{l}\tilde{m}} = (-1)^{1-l_n-l_s-M_n-M_s} \sqrt{(2l_n+1)(2\tilde{l}+1)(2l_s+1)} \begin{pmatrix} l_n & \tilde{l} & l_s \\ 0 & 0 & 0 \end{pmatrix}$$
$$\times \sum_{m_n, m_s, \mu} (-1)^{m_n} \begin{pmatrix} l_n & \frac{1}{2} & J_n \\ m_n & \mu & -M_n \end{pmatrix} \begin{pmatrix} l_s & \frac{1}{2} & J_s \\ m_s & \mu & -M_s \end{pmatrix} \begin{pmatrix} l_n & \tilde{l} & l_s \\ -m_n & \tilde{m} & m_s \end{pmatrix} . \tag{C2}$$

Next we employ the symmetry properties of the $3j$ symbols under the permutation of their columns and under the change of the signs of the entries in the second row[22] in order to put the sum in Eq. (C2) into the following form:

$$\sum_{m_n, m_s, \mu} = \sum_{m_n, m_s, \mu} (-1)^{m_n} \begin{pmatrix} J_n & l_n & \frac{1}{2} \\ -M_n & m_n & \mu \end{pmatrix} \times \begin{pmatrix} l_s & J_s & \frac{1}{2} \\ -m_s & M_s & -\mu \end{pmatrix} \begin{pmatrix} l_s & l_n & \tilde{l} \\ m_s & -m_n & \tilde{m} \end{pmatrix} .$$

Also, since the spin index μ can take only the values $\pm\frac{1}{2}$, the sum will be invariant under the replacement of μ everywhere by $-\mu$,

$$\sum_{m_n, m_s, \mu} = \sum_{m_n, m_s, \mu} (-1)^{m_n} \begin{pmatrix} J_n & l_n & \frac{1}{2} \\ -M_n & m_n & -\mu \end{pmatrix} \times \begin{pmatrix} l_s & J_s & \frac{1}{2} \\ -m_s & M_s & \mu \end{pmatrix} \begin{pmatrix} l_s & l_n & \tilde{l} \\ m_s & -m_n & \tilde{m} \end{pmatrix} . \tag{C3}$$

From the properties of the $3j$ symbols, we get that $M_n = m_n - \mu$ and $M_s = m_s - \mu$, which together permit the phase factor of $W_{ns}^{\tilde{l}\tilde{m}}$ to be written as

$$(-1)^{1-l_n-l_s-M_n-M_s+m_n} = (-1)^{(1/2)-M_n} (-1)^{\mu+m_n+m_s+l_n+l_s+(1/2)} . \tag{C4}$$

Inserting (C3) and (C4) into (C2), we get

$$W_{ns}^{\tilde{l}\tilde{m}} = (-1)^{(1/2)-M_n} \sqrt{(2l_n+1)(2\tilde{l}+1)(2l_s+1)} \begin{pmatrix} l_n & l_s & \tilde{l} \\ 0 & 0 & 0 \end{pmatrix}$$
$$\times \sum_{m_n, m_s, \mu} (-1)^{\mu+m_n+m_s+l_n+l_s+(1/2)} \begin{pmatrix} J_n & l_n & \frac{1}{2} \\ -M_n & m_n & -\mu \end{pmatrix} \begin{pmatrix} l_s & J_s & \frac{1}{2} \\ -m_s & M_s & \mu \end{pmatrix} \begin{pmatrix} l_s & l_n & \tilde{l} \\ m_s & -m_n & \tilde{m} \end{pmatrix} . \tag{C5}$$

With a little hindsight, the series in Eq. (C5) can be summed using the following formula:[23]

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \sum_{\mu_1, \mu_2, \mu_3} (-1)^{\mu_1+\mu_2+\mu_3+l_1+l_2+l_3} \begin{pmatrix} j_1 & l_2 & l_3 \\ m_1 & \mu_2 & -\mu_3 \end{pmatrix} \begin{pmatrix} l_1 & j_2 & l_3 \\ -\mu_1 & m_2 & \mu_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & j_3 \\ \mu_1 & -\mu_2 & m_3 \end{pmatrix} .$$

We finally get

$$W_{ns}^{\tilde{l}\tilde{m}} = (-1)^{(1/2)-M_n} \sqrt{(2l_n+1)(2\tilde{l}+1)(2l_s+1)} \times \begin{pmatrix} l_n & l_s & \tilde{l} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} J_n & J_s & \tilde{l} \\ -M_n & M_s & \tilde{m} \end{pmatrix} \begin{Bmatrix} J_n & J_s & \tilde{l} \\ l_s & l_n & \frac{1}{2} \end{Bmatrix} . \tag{C6}$$

Notice at this point that $\begin{pmatrix} l_n & l_s & \tilde{l} \\ 0 & 0 & 0 \end{pmatrix} = 0$, unless (a) $l_n + l_s + \tilde{l} =$ an even integer, and (b) $\{l_n, l_s, \tilde{l}\}$ satisfy the following triangular conditions.

(1) $l_n + l_s - \tilde{l} > 0$, which implies $\tilde{l} \le l_n + l_s$.
(2) $l_n - l_s + \tilde{l} > 0$, implying that $\tilde{l} \ge -(l_n - l_s)$.
(3) $-l_n + l_s + \tilde{l} > 0$ or $\tilde{l} \ge l_n - l_s$.

The above-mentioned conditions, taken together, require that $\tilde{l}$ should satisfy the following inequalities:

$$|l_n - l_s| < \tilde{l} < l_n + l_s ,$$
$$l_n + \tilde{l} + l_s = \text{an even integer} . \tag{C7}$$

This completes the derivation of Eqs. (23) and (25). Next we do the vector angular matrix elements exemplified by

$$\mathbf{K}_{ns'}^{\tilde{l}\tilde{m}} = \int Y_{\tilde{l}\tilde{m}} \Omega_n^\dagger \sigma \Omega_s' do$$
$$= (\mathbf{K}_{ns'}^{\tilde{l}\tilde{m}})_x \hat{\mathbf{i}} + (\mathbf{K}_{ns'}^{\tilde{l}\tilde{m}})_y \hat{\mathbf{j}} + (\mathbf{K}_{ns'}^{\tilde{l}\tilde{m}})_z \hat{\mathbf{k}} ,$$

where

$$(\mathbf{K}_{ns'}^{\tilde{l}\tilde{m}})_x=(-1)^{1-l_n-l_s'-M_n-M_s}\sqrt{4\pi}\sum_{m_n,m_s',\mu_n,m_s'}\begin{pmatrix} l_n & \frac{1}{2} & J_n \\ m_n & \mu_n & -M_n \end{pmatrix}\begin{pmatrix} l_s' & \frac{1}{2} & J_s \\ m_s' & \mu_s & -M_s \end{pmatrix}\langle l_n m_n \mid \tilde{l}\tilde{m} \mid l_s' m_s'\rangle \chi_{\mu_n}^{\dagger}\sigma_x\chi_{\mu_s} .$$

Using $\chi_{\mu_n}^{\dagger}\sigma_x\chi_{\mu_s}=\delta_{\mu_n,-\mu_s}$ and (C1), this becomes

$$(\mathbf{K}_{ns'}^{\tilde{l}\tilde{m}})_x=(-1)^{1-l_n-l_s'-M_n-M_s}\sqrt{(2l_n+1)(2\tilde{l}+1)(2l_s'+1)}\begin{pmatrix} l_n & l_s' & \tilde{l} \\ 0 & 0 & 0 \end{pmatrix}$$
$$\times\sum_{m_n,m_s',\mu}(-1)^{m_n}\begin{pmatrix} l_n & \frac{1}{2} & J_n \\ m_n & \mu & -M_n \end{pmatrix}\begin{pmatrix} l_s' & \frac{1}{2} & J_s \\ m_s' & -\mu & -M_s \end{pmatrix}\begin{pmatrix} l_n & l_s' & \tilde{l} \\ -m_n & m_s' & \tilde{m} \end{pmatrix} . \quad \text{(C8)}$$

We finally utilize the property that, for a $3j$ symbol not to vanish, the sum of the entries that make up its second row should be zero in order to eliminate the sums over the indices m_n and m_s'. If we then carry out the remaining sum over the index $\mu=\pm\frac{1}{2}$ explicitly and play around with the indices in the phase factor, we get

$$(\mathbf{K}_{ns'}^{\tilde{l}\tilde{m}})_x=(-1)^{J_n+J_s-M_s+(3/2)}\sqrt{(2l_n+1)(2\tilde{l}+1)(2l_s'+1)}\begin{pmatrix} l_n & l_s' & \tilde{l} \\ 0 & 0 & 0 \end{pmatrix}$$
$$\times\left[\begin{pmatrix} l_n & J_n & \frac{1}{2} \\ M_n-\frac{1}{2} & -M_n & \frac{1}{2} \end{pmatrix}\begin{pmatrix} l_s' & J_s & \frac{1}{2} \\ M_s+\frac{1}{2} & -M_s & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} l_n & l_s' & \tilde{l} \\ -M_n+\frac{1}{2} & M_s+\frac{1}{2} & \tilde{m} \end{pmatrix}\right.$$
$$\left.-\begin{pmatrix} l_n & J_n & \frac{1}{2} \\ M_n+\frac{1}{2} & -M_n & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} l_s' & J_s & \frac{1}{2} \\ M_s-\frac{1}{2} & -M_s & \frac{1}{2} \end{pmatrix}\begin{pmatrix} l_n & l_s' & \tilde{l} \\ -M_n-\frac{1}{2} & M_s-\frac{1}{2} & \tilde{m} \end{pmatrix}\right]$$

or

$$(\mathbf{K}_{ns'}^{\tilde{l}\tilde{m}})_x=(-1)^{J_n+J_s-M_s+(3/2)}\sqrt{(2l_n+1)(2\tilde{l}+1)(2l_s'+1)}\{a_1-a_2\} . \quad \text{(C9)}$$

Following the exact same procedure that led to (C9), we can derive expressions for the remaining two components of $\mathbf{K}_{ns'}^{\tilde{l}\tilde{m}}$, the only difference being in the Pauli spin products, namely,

$$\chi_{\mu_n}^{\dagger}\sigma_y\chi_{\mu_s}=(-1)^{1-\mu_n}\delta_{\mu_n,-\mu_s}$$

and

$$\chi_{\mu_n}^{\dagger}\sigma_z\chi_{\mu_s}=(-1)^{(1/2)-\mu_n}\delta_{\mu_n\mu_s} .$$

Also, by manipulating the $3j$ symbol in a way similar to what has been done in deriving Eq. (C7), we get the restrictive conditions (26). Notice that since the angular matrix elements occur in the final formula for the relativistic decay rate either squared or multiplied by each other, the phase factor in each can be dropped. This is because, for example, $(-1)^{2M_n-1}=1$, owing to the fact that

$$2M_n-1=2\frac{2t+1}{2}-1=2t ,$$

where t is some integer.

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