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Quantum-Electrodynamics Based on Self-Energy

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Abstract

We review the basic principles and results of a formulation of quantum-electrodynamics based on the self-energy of the electron, rather than quantized fields. The applications include relativistic spontaneous emission rates, the effect of cavities on Lamb-shift, spontaneous emission and Casimir–Polder forces. We also review the relativistic two-body equations including radiative processes with applications to hydrogen, muonium and positronium spectra.

1. New approach towards testing radiative processes

The traditional way of testing QED-effects (Lamb-shift, anomalous magnetic moment, etc.) consists in calculating higher and higher order Feynman graphs for an isolated quantum system (H-atom, electron) and compare the results, after renormalization, with experiments. We seem to have reached the limits of such a procedure, because the effects of the apparatus, the change of the properties of the system in the external field of the measuring apparatus, and the errors in the calculations in the nth order, say, are all bigger than the whole $(n + 1)$th order to be calculated. There seems to be no reason therefore to go on with perturbation theory. On the other hand, discrepancies still exist between the theory and the experiment in almost all the tests of QED. It is therefore appropriate to try to take into account these effects and others in a non-perturbative way.

In the $(g - 2)$-experiment for the electron, for example, the effect of the cavity surrounding the single electron may have already been seen [1]. Furthermore, the anomalous magnetic moment $(g - 2)$ for the electron is not a constant, intrinsic property of the electron, but a quantity which depends self-consistently on the state of the electron, hence on the external field itself. Moreover, the quantum systems, like an atom, do not have, strictly speaking, a discrete spectrum, due to centre of mass motion and due to radiative processes. The incoming and outgoing fields used to make measurments are not necessarily plane waves. All these effects can be taken into account fully in a natural way in the quantum-electrodynamics based on self-energy. This formulation of quantum-electrodynamics parallels the classical radiation theory, and makes fewer assumptions than QED.

2. The basic ideas

We start from the coupled Maxwell and Dirac (or Schrödinger) equations and treat $\psi(x)$ as a classical field describing the electronic matter. We next eliminate the electromagnetic field A_μ, replacing it by the matter current $j_\mu(x) = e\bar{\psi}\gamma_\mu\psi$, that produces it, using the equation

$$A_\mu(x) = \int \mathrm{d}y D(x - y) j_\mu(y) + A_\mu^{\mathrm{ext}}(x). \tag{1}$$

Here $D(x - y)$ is the Green's function of d'Alembartian $\Box$ in the environment of a cavity, free space, or a temperature bath, etc. satisfying appropriate boundary conditions (see below). If there are external fields whose sources are far away or whose sources are not dynamical variables, they may be introduced as given non-dynamical external fields A_μ^{ext} in eq. (1) in the form of an homogeneous term.

The next step is to insert eq. (1) into the Dirac (or Schrödinger) equation and study the resultant non-linear-integro-differential equation

$$(\gamma^\mu \mathrm{i}\partial_\mu - m)\psi(x) = \mathrm{e}\gamma^\mu A_\mu^{\mathrm{ext}}(x)\psi(x) + \mathrm{e}^2\gamma^\mu\psi(x)\int \mathrm{d}y D(x - y)\bar{\psi}(y)\gamma_\mu\psi(y), \tag{2}$$

where A_μ^{ext} is a given function as we noted.

We introduced $\psi(x)$ as a "classical" field, but of course it is a complex field and the Dirac equation contains $\hbar$ already, so that "classical" here means, *not* second quantized, not an operator-valued field. In fact, we shall see that it is not necessary to quantize the ψ-field, nor, because we have eliminated it, the A_μ-field, at least for one or few-body problems. For genuinly infinitely many particles, it is convenient to use the formalism of second quantization.

The important thing about the "classical" electronic field $\psi(x)$ is not so much its value at a point $\boldsymbol{x}$, but its frequency content, that is its Fourier transform. We already know from the theory of the H-atom that the ψ-field, due to the natural boundary conditions in the atom, contains discrete and continuous frequencies, although it is a classical continuum field, like a classical membrane or shell. Furthermore, atomic transitions with external fields are resonance phenomena of these natural frequencies of ψ, with the frequencies contained in the external fields. We therefore look now at these natural frequencies in the presence of the self-energy by performing a Fourier analysis of ψ:

$$\psi(\boldsymbol{x}, t) = \sum_n \psi_n(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}E_n t}. \tag{3}$$

Here the sum may also contain an integral over continuous frequencies.

We insert eq. (3) into (2) and carry time-integrations. We further multiply the resultant from the left by $\psi_n(\boldsymbol{x})$ and integrate. The result is

$$\int \mathrm{d}^3x\,\bar{\psi}_n(\boldsymbol{x})\,(\gamma^0 E_n - \boldsymbol{\gamma}\cdot\boldsymbol{p} - eA_0^{\mathrm{ext}} - m)\psi_n(\boldsymbol{x}) = \sum_m \int \mathrm{d}^3x\bar{\psi}_n(\boldsymbol{x})\gamma^\mu\psi_m(\boldsymbol{x})A_\mu^{\mathrm{ext}}(\boldsymbol{x}, E_n - E_m) - \frac{e^2}{2}\sum_{ns}\int \mathrm{d}^3x\bar{\psi}_n(\boldsymbol{x})\gamma^\mu\psi_n(\boldsymbol{x})\int \mathrm{d}^3y\bar{\psi}_s(\boldsymbol{y})\gamma_\mu\psi_s(\boldsymbol{y})$$

$$\times \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot(x-y)} \left\{ \frac{i\pi}{2k} [\delta(k) + \delta(-k)] + \frac{P}{2k} \right.$$

$$\left. \times \left[-\frac{1}{k} - \frac{1}{k} \right] \right\} - \frac{e^2}{2} \sum_{nS}^{\wedge} \int d^3x \bar{\psi}_n(x) \gamma^\mu \psi_s(x)$$

$$\times \int d^3y \bar{\psi}_s(y) \gamma_\mu \psi_n(y) \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot(x-y)}$$

$$\times \left\{ \frac{i\pi}{2k} [\delta(E_s - E_n + k) + \delta(E_s - E_n - k)] \right.$$

$$\left. + \frac{P}{2k} \left[\frac{1}{E_s - E_n - k} - \frac{1}{E_s - E_n + k} \right] \right\}_s \qquad (4)$$

Here $A_\mu^{ext}(x, E_n - E_m)$ is the Fourier transform of a time-dependent external field with frequencies $\omega_{nm} = E_n - E_{nm}$. We have further used the causal Green function

$$D_c(x - y) = -\frac{1}{(2\pi)^4} \int \frac{dk}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2} \qquad (5)$$

to take into account the proper propagation of antiparticles and particles in a relativistic theory.

In the absence of the external field and self-energy, the right-hand side of eq. (4) would be zero and then the Fourier coefficients $\psi_n(x)$ in eq. (3) would be the eigensolutions of $\psi(x)$ in a fixed time independent external potential A_0^{ext}. However, $\psi_n(x)$ are not the exact stationary solutions of the system. The first term on the right in eq. (4) tells us which states ψ_n's are coupled together by the Fourier components of the external field. Then we have the self-energy terms. The three self-energy terms on the right of eq. (4) proportional to e^2, can be interpreted as the energy shifts due to vacuum polarization, spontaneous emission and the proper real Lamb shifts, respectively. We expect that their observable contributions will be small, if e and m already describe the physical values of the charge and the mass of the electron. So the sums and integrals here must be properly defined or regularized as we shall see. We also expect that only those terms in the self-energy sum can contribute to observable effects which couple *different* states $\psi_n \neq \psi_m$ or ψ_s, as in the case of the external field. The terms with $\psi_n = \psi_s$ or $\psi_n = \psi_m$, respectively, are the diagonal terms in the $\{\psi_n\}$-basis, and can be absorbed in the definition of m.

The system (4) can be solved by iteration for the self-energy shifts. We now set $A_\mu^{ext}(x, E_n - E_m) = 0$. In the first iteration, we take $\psi_n = \psi_n^{(0)}$ to be the solutions of the external field A_0^{ext}-problem, and $E_n = E_n^{(0)} + \Delta E_n$. Then the left-hand side of eq. (4) is just ΔE_n, the shift of energy from $E_n^{(0)}$ which is then given by the three last terms on the right-hand side of eq. (4) with ψ_n's replaced everywhere by $\psi_n^{(9)}$. If there is need the iteration procedure can be repeated.

This completes the calculation of all radiative processes in the case of a single electron in a time-independent external field, e.g., the H-atom. All processes, spontaneous emission, vacuum polarization, Lamb shift, anomalous magnetic moment, are calculated within a single equation together. Non-trivial properties of the cavity and the external varying fields are all implicitly contained in the basic equation (4) as we shall see.

3. Results

3.1. *Spontaneous emission*

The rate Γ_n of the spontaneous emission of a level $|n\rangle$ is related to the imaginary part of the energy shift ΔE_n by $\Gamma = -2\mathrm{Im}\,(\Delta E_n)$. From the third term in eq. (4) we obtain immediately

$$\Gamma_n = -e^2 \sum_{s<n}^{\wedge} \int d^3x \bar{\psi}_n(x) \gamma^\mu \psi_s(x) \int d^3y \bar{\psi}_s(y) \gamma_\mu \psi_n(y)$$

$$\times \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot(x-y)} \frac{\pi}{2k} \delta(E_s - E_n + k). \qquad (6)$$

The sum here goes over all states which lie lower than the state $|n\rangle$. It follows that only ground state of the system is stable; all others must decay spontaneously.

We have evaluated eq. (6) exactly for relativistic Coulomb problem [3]. The spontaneous emission in H-like system is not yet among the precision tests of QED, which may be due to experimental difficulties or due to the lack of precise theoretical numbers. Up to now only the non-relativistic dipole approximation seems to be used. We have now precise relativistic numbers which, although do not differ much in the case of H from non-relativistic values, could be very different for other relativistic systems, like high Z ions, or $(e^-\mu^+)$, or mesonic atoms. Table I shows some decay rates for hydrogen and muonium $(e^-\mu^+)$.

3.2. *Vacuum polarization and self-energy contributions to the Lamb-shift*

For the Lamb shifts the second and the fourth terms on the right-hand side of eq. (4) have been studied to all orders in $(Z\alpha)$ using relativistic Coulomb wave functions and without the dipole approximation [4, 5]. The new feature here is a sum over the infinitely many continuous Coulomb states ψ_m. This is the source of the ultraviolet divergence of quantum-electrodynamics. In the limit to free particles (when ψ_n are plane wave free particle Dirac wave functions) the sum corresponds to the loop integral. The infrared divergence does not occur in our case, since we are using localized Coulomb wave functions and there are no virtual photons present in our calculation. If we analyze carefully the sum over the continuum by a method using a Mellin transform in the energy plane we can naturally separate and regularize the poles which give spurious infinites to our sum. When this is done, the energy shift can be expressed as a sum of residues of physical poles in the Mellin transform plane. The contribution of the first pole is proportional to $(Z\alpha)^4$ multiplied with a finite integral which has to be evaluated numerically.

3.3 *Cavity quantum-electrodynamics*

The effect of cavities on the radiative processes can be considerable and is now accessible to exeriments. In fact, many

Table I. *Spontaneous decay rates* Γ

	Hydrogen	Muonium $(e^-\mu^+)$
$\Gamma(2s_{1/2} \to 1s_{1/2})$	$2.4946 \times 10^{-6}\,s^{-1}$	$2.3997 \times 10^{-6}\,s^{-1}$
$\Gamma(2p_{1/2} \to 1s_{1/2})$	$2.0883 \times 10^{8}\,s^{-1}$	$2.0794 \times 10^{8}\,s^{-1}$
$\Gamma(2p_{3/2} \to 1s_{1/2})$	$4.1766 \times 10^{8}\,s^{-1}$	$4.1587 \times 10^{8}\,s^{-1}$
$\Gamma(2p \to 1s_{1/2})$	$6.2649 \times 10^{8}\,s^{-1}$	$6.2382 \times 10^{8}\,s^{-1}$

interesting recent experiments have demonstrated these effects. As mentioned in the introduction, our formalism contains these effects implicitly from the beginning by simply inserting for the Green function $D(x - y)$ in eq. (2) the Green function appropriate for the cavity in the experiment. The cavity Green functions in simple cases (parallel plates, sphere) can be constructed by the method of image charges, for example.

The rate of spontaneous emission in cavities can be enhanced or inhibited depending on geometry. We have given the general formulas for various geometries elsewhere [6]. For example, between two parallel plates of spacing L, the rate of spontaneous emission is zero for L between $L = 0$ and $L = \lambda_0/2$ = half the transition wavelength; at $L = \lambda_0/2$, the rate A jumps to $3/2A_0$, where A_0 is the free space rae, and then decreases to the value A_0 as $L \to \infty$. This behavior is in agreement with experiments [7].

Using the same procedure we have also calculated the effect of cavities on the Lamb-shift and on long-range Casimir-Polder forces [8] in agreement with other calculations.

The new features of the present approach are that the final formulas, eqs. (4) essentially, are the same for relativistic or non-relativistic calculations (only the appropriate form factors for the atoms are different) and the calculations are performed on the basis of self-energy alone, without vacuum fluctuations or field quantizations.

4. Relativistic two-body quantum-electrodynamics

The accuracy of experimental tests of QED is such that relativistic and recoil corrections are important even in such a system as H-atom with $\xi = m_1/m_2 = 5.446 \times 10^{-4}$. These corrections become more important for muonium $(e^-\mu^+)$, whereas a system like positronium must be treated fully relativistically. At present recoil corrections in hydrogen and muonium are taken into account perturbatively in the ratio m_1/m_2.

While the application of QED perturbation theory to a problem like $(g - 2)$ is purely mechanical, the test of QED for bound state problems requires an initial wave equation with some potential, and the choice of such a starting equation has been more or less an "art" [9] up to now. We will now show that the method of elimination of the electromagnetic potential A_μ provides a closed covariant two-body equation. The method can be further generalized to more than two particles.

We start with two fields ψ_1 and ψ_2 coupled together by the Maxwell–Dirac action,

$$W = \int \mathrm{d}x \left\{ \bar{\psi}_1(\gamma^\mu \mathrm{i}\partial_\mu - m_1)\psi_1 + \bar{\psi}_2(\gamma^\mu \mathrm{i}\partial_\mu - m_2)\psi_2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - (e_1\bar{\psi}_1\gamma^\mu\psi_1 + e_2\bar{\psi}_2\gamma^\mu\psi_2)A_\mu \right\} \tag{7}$$

The potentials A_μ can again be eliminated by eq. (1), except that the current j_μ is now the sum of the currents of the two fields

$$j_\mu(x) = e_1\bar{\psi}_1(x)\gamma_\mu\psi_1(x) + e_2\bar{\psi}_2(x)\gamma_\mu\psi_2(x) \tag{8}$$

Furthermore, by a partial integration, we can also express the third term in eq. (7) as

$$-\tfrac{1}{4}\int \mathrm{d}x F_{\mu\nu}F^{\mu\nu} = \tfrac{1}{2}\int \mathrm{d}x A_\mu(x) j^\mu(x) \tag{9}$$

Whence the action (7) can be written in an "action-at-a-distance" form

$$W = \int \mathrm{d}x \{\bar{\psi}_1(\gamma^\mu \mathrm{i}\partial_\mu - m_1)\psi_1 + \bar{\psi}_2(\gamma^\mu \mathrm{i}\partial_\mu - m_2)\psi_2 - \sum_{i,j=1}^{2} e_i e_j \int \mathrm{d}y\, \bar{\psi}_i\gamma^\mu\psi_i D(x - y)\bar{\psi}_j\gamma_\mu\psi_j\} \tag{10}$$

We must now specify a variational principle. We could vary the action W with respect to individual fields ψ_1 and ψ_2 separately. This results in non-linear coupled Hartree-type equations for these fields. Instead, we propose a relativistic configuration space formalism [10] to take into account the long-range quantum correlations. We introduce the 16-component composite field

$$\Phi(x_1, x_2) \equiv \psi_1(x_1)\psi_2(x_2). \tag{11}$$

The action can be written in terms of the composite field Φ. In order to do this we multiply the kinetic energy terms with the normalization integrals $\int \psi_2^\dagger\psi_2\,\mathrm{d}y$. We have to do this twice also on the self-energy terms (terms with $e_i = e_j$). Having expressed W as a functional of Φ, we vary it with respect to $\bar{\Phi}$. The result is a wave equation for Φ of the form

$$\{(\gamma^\mu p_{1\mu} - m_1) \otimes \gamma^\nu n_\nu + \gamma^\nu n_\nu \otimes (\gamma^\mu p_{2\mu} - m_2) + e_1e_2\frac{\gamma^\mu \otimes \gamma_\mu}{d} + \frac{e_1^2}{2}\gamma^\mu \otimes \gamma^\nu n_\nu A_\mu^{\mathrm{self}}(1) + \frac{e_2^2}{2}\gamma^\nu n_\nu \otimes \gamma^\mu A_\mu^{\mathrm{self}}(2)\}\,\Phi(x, y) = 0 \tag{12}$$

The notation is as follows. In the spinor direct products, like $\gamma_\mu \otimes \gamma^\mu$, the first factor always refers to particle 1, the second to particle 2. The vector n is a unit normal vector perpendicular to a space-like surface and d is the relativistic distance $d = ((n \cdot (x - y))^2 - (x - y)^2)^{1/2}$. The physical results are independent of the choice of n. For $n = (1000)$ we get $d = r$ and $\gamma \cdot n = \gamma^0$. The radiative self-energy potentials are given by

$$A_\mu^{\mathrm{self}}(1) = \int \mathrm{d}z\,\mathrm{d}u\, D(x - z)\bar{\Phi}(z, u)\gamma_\mu \otimes \gamma^\nu n_\nu\Phi(z, u)$$

$$A_\mu^{\mathrm{self}}(2) = \int \mathrm{d}z\,\mathrm{d}u\, D(y - z)\Phi(z, u)\gamma^\nu n_\nu \otimes \gamma_\mu\Phi(z, u) \tag{13}$$

They represent the spontaneous emission, Lamb-shift, etc. corrections to the relativistic two-body problem. Equation (12) is a one-time equation.

The Hamiltonian form of eq. (12), after we introduce the centre of mass and relative coordinates, is

$$P_0 \equiv H = \mathbf{\Gamma} \cdot \boldsymbol{P} + H_{\mathrm{rel}} - \tfrac{1}{2}e_1^2(A_0^{\mathrm{self}}(1) - \boldsymbol{\alpha}_1 \cdot \boldsymbol{A}^{\mathrm{self}}(1)) - \tfrac{1}{2}e_2^2(A_0^{\mathrm{self}}(2) - \boldsymbol{\alpha}_2 \cdot \boldsymbol{A}^{\mathrm{self}}(2)) \tag{14}$$

where $\mathbf{\Gamma} \cdot \boldsymbol{P}$ corresponds to the kinetic energy of the centre of mass with

$$\mathbf{\Gamma} = a\boldsymbol{\alpha}_1 + (1 - a)\boldsymbol{\alpha}_2, \quad a = m_1/(m_1 + m_2) \tag{15}$$

and

$$H_{\mathrm{rel}} = (\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2) \cdot \boldsymbol{p} - e_1e_2\frac{1 - \boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2}{r} + \beta_1 m_1 + \beta_2 m_2 \quad (\boldsymbol{p} = \boldsymbol{p}_1 - \boldsymbol{p}_2) \tag{16}$$

is the relative Hamiltonian with mutual interactions of particles 1 and 2. The relative time automatically drops out in eqs. (12) or (14). The total Hamiltonian also contains the self-energy terms which we have written separately. These

latters involve however non-local and non-linear potentials given in eq. (13).

We have thus succeeded to have a closed covariant two-body equation that even includes the radiative effects in a non-perturbative way. The recoil effects thus are included to all orders, as well as all the binding effects to all orders of $(Z\alpha)$. The covariance properties of eq. (12) have been discussed elsewhere [11]. In the Hamiltonian form (14), the centre of the mass kinetic energy and the relative energy are simply additive. This is a property of the spinor Dirac particles; for scalar particles such a neat separation is not possible. Equation (12) has two important limiting cases. When one of the particles is heavy, $m_2 \to \infty$, we may recover eqs. (2) and (4) of a single particle m_1 in the Coulomb field of the other. When we treat, in the other extreme, the mutual potential perturbatively, i.e., when we replace everywhere ψ_1 and ψ_2 by plane waves, we recover the Feynman diagrams with self-energy in the lowest approximation.

5. Results

The relative Hamiltonian (16) has been extensively studied [12]. We can separate relative and angular parts exactly. "The resultant 16 radial equations separate into two sets of 8 equations. Keeping the potentials up to order α^4, these equations are exactly soluble [13], with proper interpretation of negative energy states [14]. An important two-body mass formula (containing recoil correction to all order) for potentials up to order α^4 is

$$E^2 = m_1^2 + m_2^2 + 2m_1m_2\left(1 + \frac{\alpha^2}{(n_r + l)^2}\right)^{-1/2}.$$

We have treated potentials of order α^5 as perturbation. The full results are given elsewhere [15]. We quote here a few numerical results excluding self-energy corrections:

Hydrogen Hfs : $\Delta E_{\text{Hfs}} = 1420.348\,\text{MHz}$ (exp. 1420.4057)

Muonium Hfs : $\Delta E_{\text{Hfs}} = 4463.0601\,\text{MHz}$ (exp. 4463.302)

Positronium $n = 2$, $n = 1$ transition : $\Delta E_{21} = \frac{3}{8}\text{Ry} - 0.468093\,\alpha^2\text{Ry}$.

The self-energy effects we propose to evaluate by iteration: the first order wave function Φ will be put in eq. (13) and the resultant potentials A_μ^{self} evaluated; we then use these potentials as perturbation to the energy levels.

Up to the order calculated so far our results agree with perturbation theory. But everything is obtained from a one-time relativistic wave equation which in closed form contains all radiative effects as well.

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