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ON THE COVARIANCE OF TWO-FERMION EQUATION
FOR QUANTUM ELECTRODYNAMCIS

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ON THE COVARIANCE OF TWO-FERMION EQUATION FOR QUANTUM ELECTRODYNAMICS *

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ABSTRACT

The covariance properties of a two (or more)-body equation which has been recently derived from field theory and applied to the precision tests of QED-bound state problems are established.

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The formulation of a fully relativistic, in fact covariant, two and many-body equations, in quantum electrodynamics, and in general, is still an important and open problem. It is also a necessary task if wish to describe the interactions and self energies of fully relativistic systems such as positronium. For example, we do not have at present, to my knowledge, an adequate formulation of Lamb-shift in positronium using positronium wave functions, that is to all orders in $(Z\alpha)$.

There exists a global covariant approach to relativistic dynamics in the form of infinite component wave equations, as has been used extensively to describe relativistic H-atom and hadrons. These equations are in the simplest case of the form

$$(\Gamma^{\mu}P_{\mu} + K)\psi(P) = 0$$

where P_{μ} is the total momentum of the system, and Γ^{μ} and K are operators in the internal coordinates of the system, e.g. differential operators with respect to the relative coordinates $\vec{r}$, or infinite-dimensional matrices in the case of a discrete basis for the internal degrees of freedom, hence the name "infinite-component wave equations" [1]. However, when such equations are postulated, we would like, in addition, to connect them with the microscopic properties of two or more physical particles that we think are the constituents of the system. It is known that given an equation of this type (9) leading to a certain mass spectrum, the introduction of internal coordinates is not unique [2]. Hence, equations of this type should be complemented by a direct microscopic approach which takes fully into account the spin properties of the constituents. Furthermore, in a precise theory like quantum electrodynamics, where we know the minute details of the interactions involving fine-, hyperfine structures, Lamb shift and vacuum polarization, it is very difficult to guess the form of the operator K which would include all those effects, although a large part of it has been accomplished [3]

Since infinite-component wave equations have nice covariance properties, simple geometric interpretations, and are easy to work with, it is important to obtain them from a field theory of microscopic interactions, and to be able to translate both theories into each other.

There are many approaches to the microscopic relativistic two-body problem as seen from the contributions to this conference. I shall describe an approach which is based on the first quantized quantum electrodynamics, that is on coupled Maxwell-Dirac equations. A covariant two-body equation can be derived from such a field theory by a variational principle which has many nice properties that we generally require from a wave equation, and which, moreover, is quite accurate in its application to QED-bound state problems, such as the positronium. The theory can be generalized to three or more particles.

The wave equation is

$$\left[(\overset{(1)}{\gamma^\mu} p_{1\mu} - m_1) \otimes \overset{(2)}{1} + \overset{(1)}{1} \otimes (\overset{(1)}{\gamma^\mu} p_{2\mu} - m_2) + e_1 e_2 \frac{\overset{(1)}{\gamma^\mu} \otimes \overset{(2)}{\gamma_\mu}}{r_\perp} + V_{magn} + V_{rad} \right] \Phi(x_1, x_2) = 0 \quad . \tag{1}$$

The notation is as follows: $\overset{(1)}{\gamma^\mu}$, $\overset{(2)}{\gamma^\mu}$ are the Dirac spinors for each particle, n^μ a unit vector which we shall discuss below, and $\gamma \cdot n \equiv \gamma^\mu n_\mu$; $r_\perp$ is the invariant distance $r_\perp = [(x\cdot n)^2 - x^2]^{1/2} = \sqrt{x_\perp^2}$, (length of the component of the relative coordinate $x = x_1 - x_2$ perpendicular to n). The first relativistic potential $e_1 e_2 \frac{\overset{(1)}{\gamma^\mu} \otimes \overset{(2)}{\gamma_\mu}}{r_\perp}$ comes from the minimal coupling in the Maxwell-Dirac equations, V_{magn} comes from the Pauli coupling in the Maxwell-Dirac equations, and V_{rad} comes from the self-energy terms also present in the Maxwell-Dirac equations, as we shall indicate. The wave function $\Phi(x_1, x_2)$ is the analogue of the nonrelativistic configuration space wave-function. Eq. (1) is actually a <u>one-time equation</u>, since it turns out automatically that Φ does <u>not</u> depend on the parallel component of the relative coordinate $x_{||} = (x\cdot n)n$. We shall always write in the direct product of spin tensors particle 1 first, particle 2 second; $\overset{(1)}{\gamma^\mu} \otimes \overset{(2)}{\gamma_\mu} = \gamma^\mu \cdot \gamma_\mu$, hence we can omit the superscripts (1) and (2).

Eq. (1) is most directly derived from the action of quantum electrodynamics for two Dirac fields $\psi_1(x)$ and $\psi_2(x)$ [4],[5]. In the derivation below I shall give and emphasize the relativistic covariance. The action W is

$$W = \int dx \left\{ \bar{\psi}_1 (\overset{(1)}{\gamma^\mu} p_{1\mu} - m_1)\psi_1 + \bar{\psi}_2 (\overset{(2)}{\gamma^\mu} p_{2\mu} - m_2)\psi_2 - \sum_{i=1}^{2} e_i \bar{\psi}_i(x) \overset{(i)}{\gamma^\mu} \psi_i(x) A_\mu(x) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\} \quad . \tag{2}$$

We eliminate A_μ and $F_{\mu\nu}$ from this action using the solutions of the Maxwell equations

$$A_\mu(x) = \int dy \, D_{\mu\nu}(x-y) j^\nu(y) \quad . \tag{3}$$

The argument of the δ-function in $D(x-y)$ can be decomposed in a general way as

$$\delta(x^2) = \delta[(x\cdot n)^2 - x_\perp^2] = \frac{\delta(x\cdot n) - r_\perp) \pm \delta(x\cdot n) + r_\perp)}{r_\perp} \quad .$$

This is one instance where a vector n^μ enters, namely when "time" is measured in the direction of n^μ, the distance from the light source is measured by $r_\perp$. There is a second, related, place where a vector n^μ will enter into the theory, namely, in the normalization of the wave functions,

$$\int d^4x \, \delta(x\cdot n - \tau) \bar{\psi} \gamma^\mu n_\mu \psi = \int_\Sigma d\sigma \, n^\mu \, \bar{\psi} \gamma_\mu \psi = 1 \tag{4}$$

Here $d\sigma$ is the volume element on the space-like surface perpendicular to n^μ. We need such a vector n^μ in these two places, (3) and (4), although its choice is arbitrary. Note further that we can choose the Dirac matrices $\overset{(1)}{\gamma^\mu}$, $\overset{(2)}{\gamma_\mu}$ also arbitrarily.

It is important, in the discussion of the covariance of quantum equations, to emphasize that the Hilbert space of states is defined by a foliation of the Minkowski-space into space-like surfaces Σ_τ for each invariant time parameter τ. On the other hand, Lorentz transformations map solutions of the wave function $\psi(t,x)$ into other solutions $\psi'(t',x')$, but not in general a state into another state. Transformations in the space of solutions in the Minkowski space are not the same as the transformations of one state into another on a space-like surface Σ_τ. The occurrence of n^μ in Eq. (1) tells us how we normalize our states, Eq.(4), and how we calculate the field produced by the current of the particle in a definite state. Eq. (1) is covariant in the sense that for a different observer we have also to transform n^μ like a 4-vector, so that we have the same form of the equations as seen from a different frame. Since observed quantities are independent of the choice of n in Eqs. (3) and (4) we expect also the observable quantities following from Eq. (1) to be independent of n, such as the spectrum of bound states and transition probabilities. Let me first indicate how Eq. (1) is derived from the action (2). We insert the solution (3), and the corresponding $F_{\mu\nu}$, into Eq. (2) and multiply each kinetic energy term in (2) with the normalization

condition (4) of the other particle, and obtain for the kinetic energy and mutual interaction terms

$$W = \int dx\, do \left\{ \bar{\psi}_1^{(1)}(\gamma^\mu p_{1\mu} - m_1)\psi_1 \otimes \bar{\psi}_2^{(2)} n^\mu \gamma_\mu \psi_2 + \bar{\psi}_1^{(1)} n^\mu \gamma_\mu \psi_1 \otimes \bar{\psi}_2^{(2)}(\gamma^\mu p_{2\mu} - m_1)\psi_2 \right.$$
$$\left. - \frac{1}{2}\sum_{i,j} e_i e_j \, \bar{\psi}_i^{(i)} \gamma^\mu \psi_j \cdot \frac{1}{r_\perp} \bar{\psi}_j^{(j)} \gamma_\mu \psi_j \right\} . \qquad (5)$$

The terms $i \neq j$ and $i = j$ will be treated separately.

The interaction terms in (5) are such that two states $\psi_i(x_1)$ and $\psi_j(x_2)$ contribute to the action only if x_1 and x_2 are light-like separated: $(x_1 - x_2)^2 = 0$. Similarly, we can arrange the kinetic terms such that x_1 and x_2 are also light-like separated. At each point x_1, the first term in (2) is multiplied with a suitable normalization integral of particle 2 depending on x_1. In this way, and with suitable change of variables, W is a seven-dimensional integral.

We now define the configuration space wave-functions

$$\Phi(x_1,x_2) = \psi_1(x_1)\psi_2(x_2) \qquad (6)$$

It will turn out that the equation for $\Phi(x_1,x_2)$ automatically reflects that the configuration space is actually seven-dimensional and not eight. Our functions $\psi_1(x_1)$, $\psi_2(x_2)$ are all c-numbers, hence we can write the action W in terms of Φ and $\bar{\Phi}$:

$$W = \int dx_1 d\vec{x}_2 \, \bar{\Phi} \left[(\gamma^\mu p_{1\mu} - m_1) \otimes \gamma\cdot n + \gamma\cdot n \otimes (\gamma^\mu p_{2\mu} - m_2) \right.$$
$$\left. + e_1 e_2 \frac{\gamma^\mu \cdot \gamma_\mu}{r_\perp} \right] \Phi(x_1 x_2) \qquad (7)$$

Now the variation of this action with respect to $\bar{\Phi}$ gives the equation (1) that we wrote at the beginning.

The significance of this variational principle should be stated. Whereas the variation of the action W with respect to individual fields $\psi_1(x_1)$, $\psi_2(x_2)$ leads to coupled nonlinear equations of the Hartree-type, the variation of W with respect to Φ leads to a linear equation. After the variational principle the solution of the configuration space equation (1) for Φ is no longer factorizable into a product $(\psi_1(x)\cdot\psi_2(x))$ although to derive it we have started from the product form (6). This is because the latter variational principle is weaker than the first, and reflects the nonlocality of the quantum two-body configuration space equation. This is the same situation as in nonrelativistic quantum mechanics.

The difference between Hartree-equations and configuration space equations is experimentally observable. Hence it seems that we must take the variational principle for Φ as one of the additional basic postulates of Quantum Theory when going from a single particle to two, or more, particles.

For identical particles we shall proceed again as in non-relativistic quantum mechanics, namely choose solutions which are antisymmetric under the exchange of two particles.

The treatment of the Pauli-coupling term in the action is slightly more involved, but is essentially similar[5],[6] and we do not need to go into it here. But the treatment of the self-energy terms is more subtle, interesting, and for many reasons, essential for the whole theory. Consider one of the terms

$$I = \frac{1}{2} e_1^2 \int dxdy \bar{\psi}_1(x)\gamma^\mu \psi_1(x)\, D(x-y)\bar{\psi}_1(y)\gamma_\mu\psi_1(y) .$$

We multiply this expression twice with the normalization (4) of the second field and change variables such that it becomes

$$I = \frac{1}{2} e_1^2 \int dxdydzdu\ \bar{\psi}_1(x)\gamma^\mu\psi_1(x)\bar{\psi}_2(y)\gamma\cdot n\psi_2(y)\, D(x-z)\bar{\psi}_1(z)\gamma_\mu\psi_1(z)$$
$$\times\ \bar{\psi}_2(u)\gamma\cdot n\psi_2(u) .$$

Or, in terms of the composite field Φ, Eq. (6),

$$I = \frac{1}{2} e_1^2 \int dxdy\ \bar{\Phi}(x,y)\gamma^\mu \otimes \gamma\cdot n\Phi(x,y) \Big] dzdu\, D(x-z)\bar{\Phi}(z,u)\gamma_\mu \otimes \gamma\cdot n\Phi(z,u) .$$

Thus, the variation of the action W with respect to $\bar{\Phi}$ gives finally the nonlocal or nonlinear term in Eq. (1),

$$V_{rad} = + \frac{1}{2} e_1^2\, \gamma^\mu \otimes \gamma\cdot n \int dzdu\, D(x-z)\bar{\Phi}(z,u)\gamma_\mu \otimes \gamma\cdot n\Phi(zu)$$
$$+ \frac{1}{2} e_2^2\ \ \gamma\cdot n \otimes \gamma^\mu \int dzdu\, D(y-u)\bar{\Phi}(z,u)\, \gamma\cdot n \otimes \gamma_\mu \Phi(z,u) . \qquad (8)$$

Self energy terms therefore introduce self-consistent nonlocal potentials into our equation as we expect. These terms have been treated for Lamb-shift and spontaneous emission in the Coulomb field [7],[8],[9], and in the above covariant form will be applied to the Lamb-shift problem for positronium.

We now introduce the relative and center of mass coordinates and momenta by

$$P_\mu = p_{1\mu} + p_{2\mu}, \quad x_\mu = x_{1\mu} - x_{2\mu}, \quad p = (1-a)p_1 - a\, p_2, \quad R = ax_1 + (1-a)x_2$$

$$p_1 = aP_\mu + p_\mu, \quad p_2 = (1-a)P_\mu - p_\mu \qquad x_1 = (1-a)r + R$$
$$(a = m_1/(m_1+m_2)) \qquad x_2 = -ar + R$$

So that Eq. (1) becomes

$$\Big\{(a\,\gamma^{\mu}\cdot\gamma_n + (1-a)\,\gamma_n\cdot\gamma^{\mu})P_{\mu} - (\gamma^{\mu}\cdot\gamma_n - \gamma_n\cdot\gamma^{\mu})p_{\mu}$$
$$-m_1 I\cdot\gamma_n - m_2\,\gamma_n\cdot I - V(r_\perp)\Big\}\Phi(R,r) = 0 \qquad (\gamma_n = \gamma\cdot n = \gamma^{\mu}p_{\mu}) \tag{9}$$

We separate the components of the relative momentum with respect to n^{μ} and perpendicular to it

$$p = p_{\parallel} + p_{\perp} = (p\cdot n)n + p_{\perp} \quad .$$

Then in the second term (9), the contribution of $p_{\parallel}$ vanishes

$$(\gamma^{\mu}\cdot\gamma_n - \gamma_n\cdot\gamma^{\mu})p_{\mu} = (\gamma^{\mu}\cdot\gamma_n - \gamma_n\cdot\gamma^{\mu})p_{\perp\mu} \quad .$$

Hence Φ depends only on R_{μ} and $r_\perp$, $\Phi(R, r_\perp)$, as it should be according to our general discussion above, because the potentials V also depend only on $r_\perp$.

The coordinates (P_{μ}, P_{ν}) and (R, r) are conjugate pairs satisfying canonical commutation relations. The perpendicular components of coordinates and momenta satisfy

$$[p_\perp^{\mu}, r_\perp^{\nu}] = i(g^{\mu\nu} - n^{\mu}n^{\nu})$$

which means that the internal motion takes place indeed on a three-dimensional hypersurface perpendicular to n^{μ}.

We can also decompose γ^{μ} into its parallel and transverse components: $\gamma_{\mu} = (\gamma\cdot n)n_{\mu} + \gamma_{\mu\perp}$, $n\gamma_\perp^{\mu} = 0$. The set (γ_n and $\gamma_\perp^{\mu}$) play the role of γ^0 and γ^i for an arbitrary n^{μ} with

$$\{\gamma_n, \gamma_\perp^{\mu}\} = 0, \quad \{\gamma_\perp^{\mu}, \gamma_\perp^{\nu}\} = 2(g^{\mu\nu} - n^{\mu}n^{\nu}) \quad .$$

Finally we can decompose the total momentum P_{μ} into its parallel and perpendicular components in the same way so that our basic Eq. (9) becomes

$$\Big\{(\gamma_n\otimes\gamma_n)(P_n) + (a\gamma_\perp\otimes\gamma_n + (1-a)\ \gamma_n\otimes\gamma_\perp)P_\perp$$
$$- (\gamma_\perp\otimes\gamma_n - \gamma_n\otimes\gamma_\perp)P_\perp - m_1 I\cdot\gamma_n - m_2\gamma_n\cdot I - V(r_\perp)\Big\}\Phi = 0$$

or, multiplying all terms with $\gamma_n\otimes\gamma_n$:

$$\Big\{P_{\parallel} + (a\alpha_\perp\cdot 1 + (1-a)1\cdot\alpha_\perp)P_\perp$$
$$- (\alpha_\perp\cdot I - I\cdot\alpha_\perp)p_\perp - m_1\beta_{1\perp} - m_2\beta_{2\perp} - \tilde{V}(r_\perp)\Big\}\Phi = 0 \quad , \tag{10}$$

where we have put

$$\alpha^{\mu} = (\gamma_n)\,\gamma_\perp^{\mu} \tag{11}$$

in analogy to the Dirac's $\alpha^i = \gamma^0\gamma^i$, and

$$\tilde{V} = \gamma_n\otimes V\otimes\gamma_n \tag{12}$$

in the dyadic notation.

Now the vector n^{μ} has disappeared in Eq. (10) with proper labelling of coordinates and momenta. Here $P_{\parallel}$ is the total energy (or Hamiltonian) of the system and the second term in (10) represents the total kinetic energy of the center of mass, while the remaining terms refer to the relative or internal motion. On the hypersurface with normal n^{μ} we can now choose, if we like, three-dimensional coordinates and Dirac matrices such that Eq. (10) becomes

$$\Big\{P_{\parallel} - (a\vec{\alpha}_1 + (1-a)\vec{\alpha}_2)\cdot\vec{p} - (\vec{\alpha}_1 - \vec{\alpha}_2)\cdot\vec{p}$$
$$- m_1\beta_1 - m_2\beta_2 - V(\vec{r})\Big\}\Phi = 0 \tag{13}$$

or

$$H\Phi = \{(a\vec{\alpha}_1 + (1-a)\vec{\alpha}_2)\vec{p} + H_{rel}\}\Phi \tag{13'}$$

In the rest frame, $\vec{P} = 0$, and we are left with H_{rel} and we see that center of mass and relative coordinates are completely separable. The internal Hamiltonian is

$$H_{rel} = (\vec{\alpha}_1 - \vec{\alpha}_2)\cdot\vec{p} + \beta_1 m_1 + \beta_2 m_2 + \tilde{V}(\vec{r}) \tag{14}$$

For the minimal coupling, the potential $\tilde{V}$ is

$$\tilde{V} = e_1 e_2\ \frac{1 + \alpha_\perp^{\mu}\otimes\alpha_{\perp\mu}}{r_\perp} \quad \Rightarrow \quad e_1 e_2\ \frac{1 - \vec{\alpha}_1\cdot\vec{\alpha}_2}{r}$$

The rest frame Eq. (10) has been extensively studied and analyzed [10],[14]. It works very well when applied to realistic systems like positronium, muonium and hydrogen, including the magnetic potential coming from a Pauli coupling. The agreement with perturbative QED is up to order α^5, because at order α^5 there are also self energy terms given in Eq. (8) which have not yet been fully analyzed. We should emphasize that these results are now obtained from a bona-fida two-body wave equation, and not by perturbative QED graphs.

To conclude I summarize the salient features of the wave equation (1):

i) it is fully relativistic and covariant;

ii) It takes full account of the spins and recoil of both constituents;
it is a 16 x 16 spinor equation; it can be generalized to a 64 x 64 three-body equation, etc;

iii) It can be fully justified and derived from first quantized QED (coupled Maxwell-Dirac fields) by a variational principle. The existence of the Green function $D(x-y)$, hence the exchange of massless photons is essential in the derivation;

iv) It is a one-time equation, the relative time automatically drops out. There are no retardation problems;

v) Center of mass and relative coordinates are exactly separable [5],[10]. Center of mass itself is described by a soluble 16 or 64 component spinor equation (14);

vi) Angular and radial parts of the relative motion are exactly separable [10];

vii) Radiative corrections can be included by additional anomalous magnetic moment and self-energy potentials;

viii) For QED, the relativistic potentials up to order α^4 turn out to be exactly soluble [11]. The exactly soluble genuinly two-body problem provides relativistic quantum numbers for a system like positronium (where L and S are not good quantum numbers) and can be used as a basis for a Furry-picture radiative corrections, and for a nonperturbative scattering theory. The remaining terms in the potentials have been treated perturbatively.

ix) Applications to (e^+e^-), $(\mu^-\mu^+)$ and H give results in agreement with perturbative QED [12],[13].

REFERENCES

[1] For a review of the axiomatic properties of infinite component wave equations see A.O. Barut, Chapter 6 in Groups, Systems and Many-Body Physics, Eds. P. Kramer et al. (Vieweg Verlag, 1980) pp. 285-317; A.O. Barut and R. Raczka, Theory of Group Representations and Applications, (Second Edition, PWN Warsaw, 1980).

[2] A.O. Barut and I.H. Duru, Phys. Rev. D10, 3448 (1974); Proc. Roy. Soc. A333, 217 (1973).

[3] A. Baiquni, in Structure of Matter, Ed. B. Wybourne (University of Canterbury Press, 1972).

[4] A.O. Barut, in Lecture Notes in Physics, Vol. 180 (Springer, 1983) pp. 332.

[5] A.O. Barut and S. Komy, Fortschritte der Physik 33, 309 (1985).

[6] A.O. Barut and B.W. Xu, Physica Scripta 26, 129 (1982).

[7] A.O. Barut and J. Kraus, Found. of Phys. 13, 189 (1983).

[8] A.O. Barut and J.F. van Huele, Phys. Rev. A32, 3887 (1985).

[9] A.O. Barut and J. Kraus, "Nonperturbative QED without infinities" (submitted for publication).

[10] A.O. Barut and N. Ünal, Fortschritte der Physik 33, 319 (1985).

[11] A.O. Barut and N. Ünal, J. Math. Phys. (in press).

[12] A.O. Barut and N. Ünal, submitted to Physica.

[13] A.O. Barut and N. Ünal, ICTP, Trieste, preprint IC/86/119 (1986), submitted to Physica.

[14] A.O. Barut and G. Strobel, Few Body Systems (1986).