An exactly soluble relativistic quantum two-fermion problem

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The two-fermion problem of quantum electrodynamics in which both particles are treated relativistically and full spin degrees of freedom are taken into account is shown to be exactly soluble when potentials up to order α^4 are kept. It is therefore a good starting point for radiative corrections of higher order for the precision tests of QED in bound-state problems. Recoil corrections are included to all orders.

I. INTRODUCTION

There are, to our knowledge, no examples of exactly soluble realistic spin-$\frac{1}{2}$ two-body problems in which both particles are treated relativistically. We present here a case that has been extracted from a fully covariant two-body equation in quantum electrodynamics. It is realistic in the sense that it gives a spectrum for the H or positronium atoms correct up to order α^4 and contains moreover the recoil corrections to all orders. It can therefore be used as a good starting point for radiative corrections in the precision tests of quantum electrodynamics for the remaining terms of the order of α^5 and higher.

We shall also compare this system with the covariant infinite-component wave equation with exactly the same spectrum. In the latter case the composite structure of the system is characterized algebraically by a dynamical group rather than in terms of the parameters of the constituents as a dynamical bound state of two particles.

II. COVARIANT TWO-BODY EQUATION

The starting point is the covariant two-body equation

$$\Big[(\overset{(1)}{\gamma^{\mu}}\cdot p_{1\mu}-m_1)\otimes\overset{(2)}{\gamma}\cdot n + \overset{(1)}{\gamma}\cdot n\otimes(\overset{(2)}{\gamma}\cdot p_2-m_2)+V(d)\Big]\Phi=0\,, \tag{1}$$

derived directly from the coupled Maxwell–Dirac equations in a nonperturbative way by a variational principle.[1–3]

Here $\overset{(1)}{\gamma_{\mu}}$ and $\overset{(2)}{\gamma_{\mu}}$ are the Dirac algebras for both particles so that Eq. (1) is a (16×16)-spinor equation. Further n_μ is a four-vector normal to the spacelike surface associated with the relative coordinate and $\gamma\cdot n\equiv\gamma^\mu n_\mu$. The relativistic potential $V(d)$ is a function of the covariant relative distance of the two particles $d=\sqrt{(x\cdot n)^2-x^2}$. For the explicit solutions in this paper we shall from now on choose $n^\mu=(1,0,0,0)$, whence $\gamma\cdot n=\gamma^0$ and $d=r$, the magnitude of the relative three-vector $\mathbf{r}$. The spin matrices we write always as the direct products $A\otimes B$, where A refers to particle 1 and B to particle 2.

Equation (1) has many remarkable properties, among them the exact separability of the center of mass and relative coordinates. One then sees that it is actually a *one-time equation*. The dependence on the relative time drops out automatically. The equation for the center of mass is[2]

$$((m_1/M)\boldsymbol{\alpha}_1+(m_2/M)\boldsymbol{\alpha}_2)\cdot\mathbf{P}\phi(R)=(E_0-E)\phi(R)\,, \tag{2}$$

whereas the relative motion is given by

$$[(\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2)\cdot\mathbf{p}+\beta_1m_1+\beta_2m_2+\beta_1\otimes V\otimes\beta_2]\psi(r)=E\psi(r)\,, \tag{3}$$

where E is the energy in the center of mass frame (total mass of the system) and E_0 the total energy of the moving system so that the difference (E_0-E) in (2) is the relative kinetic energy of the center of mass: $M=m_1+m_2$.

For the coupling of the spinor fields to a vector field A_μ of the form $e\bar\psi\gamma^\mu\psi A_\mu$ and for an effective anomalous magnetic moment coupling of the form $a\bar\psi\sigma_{\mu\nu}\psi F^{\mu\nu}$ the form of the relativistic potential has been derived. The first coupling gives

$$V(r)=(e_1e_2/r)\overset{(1)}{\gamma^{\mu}}\otimes\overset{(2)}{\gamma_{\mu}}\,. \tag{4}$$

The second potential coming from the Pauli coupling is rather lengthy and since it has been given elsewhere,[2,3] we do not write it here but shall give its radial form later.

In the derivation of Eq. (3) from field theory there are also self-energy terms corresponding to Lamb shift and spontaneous emission. These are of order of $\alpha(Z\alpha)^4$ and higher and will be taken into account separately.

III. RADIAL EQUATIONS

For Eq. (3) we can also separate completely the radial and angular parts.[3] This results in two sets of eight first-order radial wave equations. In each set four of the eight equations are algebraic and the other four are first-order differential equations. Eliminating some of the components of the wave functions we arrive *for the first set* at the following *two coupled second-order equations* (including Pauli terms):

$$\begin{aligned}&\bigg\{\frac{V_1V_6}{4V_3}+\bigg[\frac{V_6}{V_3}\partial_+\frac{V_3}{V_6}+\frac{2\lambda Mj(j+1)}{Er^4V_3}\Big(\frac{1}{E}+\frac{3}{V_5}\Big)\bigg]\partial_-\\&\quad-\frac{2\lambda Mj(j+1)V_6}{EV_3}\partial_+\\&\quad\times\frac{1}{r^4V_6}\Big(\frac{1}{E}+\frac{3}{V_5}\Big)-\frac{M^2j(j+1)}{E^2r^2V_3}V_2\bigg\}(ru_2)\\&\quad+M\sqrt{j(j+1)}\bigg\{\frac{V_6}{V_3}\partial_+\frac{V_3}{V_6rV_5}\end{aligned}$$

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$$+\frac{2\lambda M j(j+1)}{E r^5 V_3 V_5}\left(\frac{1}{E}+\frac{3}{V_5}\right)$$
$$-\left[\frac{V_2}{rEV_3}+\frac{2\lambda V_6}{MV_3}\partial_+\frac{1}{V_6 r^3}\left(\frac{1}{E}+\frac{3}{V_5}\right)\right]\tilde{\partial}_+\Bigg\}(rv_0)$$
$$=0, \tag{5}$$

$$\Bigg\{\frac{V_4V_6}{4V_2}+\left[\frac{V_6}{V_2}\tilde{\partial}_-\frac{V_2}{V_6}+\frac{2\lambda M j(j+1)}{r^4 V_2 V_5}\left(\frac{1}{E}+\frac{3}{V_5}\right)\right]\tilde{\partial}_+$$
$$-\frac{2\lambda M j(j+1)V_6}{V_2}\tilde{\partial}_-\frac{1}{V_6V_5r^4}\left(\frac{1}{E}+\frac{3}{V_5}\right)$$
$$-\frac{M^2 j(j+1)}{r^2V_2V_5^2}V_3\Bigg\}(rv_0)-M\sqrt{j(j+1)}$$
$$\times\Bigg\{-\frac{V_6}{EV_2}\tilde{\partial}_-\frac{V_2}{rV_6}-\frac{2\lambda M j(j+1)}{Er^5V_2V_5}\left(\frac{1}{E}+\frac{3}{V_5}\right)$$
$$+\left[\frac{V_3}{rV_5V_2}+\frac{2\lambda}{M}\frac{V_6}{V_2}\tilde{\partial}_-\frac{1}{r^3V_6}\left(\frac{1}{E}+\frac{3}{V_5}\right)\right]\partial_-\Bigg\}(ru_2)$$
$$=0, \tag{6}$$

where the following abbreviations have been used:

$$V_1(r)\equiv E+\frac{2\alpha}{r}-\frac{M^2}{E}-\frac{4a_1a_2}{r^3}-\frac{\tau^2}{Er^4},$$
$$V_2(r)\equiv E+\frac{2\alpha}{r}-\frac{\Delta m^2}{E}-\frac{4j(j+1)}{r^2V_5}+\frac{4a_1a_2}{r^3}-\frac{\lambda^2}{Er^4},$$
$$V_3(r)\equiv E+\frac{4\alpha}{r}-\frac{\Delta m^2}{E-2\alpha/r}-\frac{4j(j+1)}{Er^2}-\frac{9\lambda^2}{r^4V_5},$$
$$V_4(r)\equiv E-\frac{M^2}{V_5}-\frac{\tau^2}{r^4(E-2\alpha/r)}, \tag{7}$$
$$V_5(r)\equiv E+\frac{2\alpha}{r}+\frac{8a_1a_2}{r^3},$$
$$V_6(r)\equiv V_2V_3-\frac{4\lambda^2 j(j+1)}{r^6}\left(\frac{1}{E}+\frac{3}{V_5}\right),$$

and

$$\partial_\pm\equiv\partial_r\pm\frac{\lambda M+\tau\Delta m}{2Er^2},$$
$$\tilde{\partial}_\pm\equiv\partial_r\pm\frac{1}{r}\mp\left(\frac{3\lambda M}{2r^2V_5}-\frac{\tau\Delta m}{2r^2(E-2\alpha/r)}\right). \tag{8}$$

Further

$$\lambda=e_1a_2+e_2a_1,\quad \tau=e_1a_2-e_2a_1. \tag{9}$$

We are interested in the solutions of Eq. (5). They are rather complicated. However, if we consider some of the small terms (which are, in the electromagnetic problem, of order α^5 and smaller) as perturbations, we have found that these coupled equations are exactly soluble.

In order to motivate the method of solution and to interpret the angular momentum quantum numbers, we begin with a much simpler case, namely the radial equations of two relativistic free particles in the center of mass frame. Even this case is not trivial in this form[3] and provides us actually the tools to solve the case with interactions.

IV. SOLUTIONS OF THE RADIAL EQUATIONS FOR TWO RELATIVISTIC FREE PARTICLES

First we set all the coupling constants equal to zero:

$$\alpha=0,\quad \lambda=0,\quad \tau=0,\quad a_1a_2=0. \tag{10}$$

Then Eqs. (5) and (6) become

$$\Bigg\{\frac{1}{4}\left(E-\frac{M^2}{E}\right)\left(E-\frac{\Delta m^2}{E}\right)-\frac{j(j+1)}{r^2}+\partial_r^2$$
$$-\frac{j(j+1)/r^3}{(E^2-\Delta m^2)/4-j(j+1)/r^2}\partial_r\Bigg\}(ru_2)$$
$$-\frac{2M}{Er^2}\sqrt{j(j+1)}\Bigg\{\frac{E^2-\Delta m^2}{4}$$
$$\times\frac{1}{(E^2-\Delta m^2)/4-j(j+1)/r^2}\Bigg\}(rv_0)=0, \tag{11}$$

$$\Bigg\{\frac{1}{4}\left(E-\frac{M^2}{E}\right)\left(\frac{E-\Delta m^2}{E}\right)-\frac{j(j+1)+2}{r^2}+\partial_r^2$$
$$-\frac{j(j+1)/r^3}{(E^2-\Delta m^2)/4-j(j+1)/r^2}\partial_r\Bigg\}(rv_0)$$
$$-\frac{2M}{Er^2}\sqrt{j(j+1)}\Bigg\{\frac{E^2-\Delta m^2}{4}$$
$$\times\frac{1}{(E^2-\Delta m^2)/4-j(j+1)/r^2}\Bigg\}(ru_2)=0. \tag{12}$$

The only difference between Eqs. (11) and (12) is the term $(-(2/r^2)(rv_0))$ in the first part of the second equation. The other terms are completely symmetrical. In the dimensionless units these equations can be written as

$$\Bigg\{\partial_\rho^2+1-\frac{j(j+1)}{\rho^2}-\frac{2j(j+1)/\rho^3}{\epsilon^2-j(j+1)/\rho^2}\partial_\rho\Bigg\}f(\rho)$$
$$-\frac{2a/\rho^3}{\epsilon^2-j(j+1)/\rho^2}g(\rho)=0, \tag{13}$$
$$\Bigg\{\partial_\rho^2+1-\frac{j(j+1)+2}{\rho^2}-\frac{2j(j+1)/\rho^3}{\epsilon^3-j(j+1)/\rho^2}\partial_\rho\Bigg\}g(\rho)$$
$$-\frac{2a/\rho^3}{\epsilon^2-j(j+1)/\rho^2}f(\rho)=0, \tag{14}$$

with

$$\rho\equiv kr,$$
$$4k^2\equiv(E^2-M^2)(E^2-\Delta m^2)/E^2,$$

where k has the meaning of momentum in the center of mass frame when E is the center of mass energy

$$\epsilon^2\equiv\frac{E^2-\Delta m^2}{4k^2}=\frac{E^2}{E^2-M^2},\quad a\equiv\epsilon^2M/E,$$
$$f(\rho)\equiv\rho u_2(\rho),\quad g(\rho)\equiv\rho v_0(\rho).$$

We note the following.

(i) Except the coupling terms and the

$$\frac{-2j(j+1)/\rho^3}{\epsilon^2-j(j+1)/\rho^2}\partial_\rho$$

terms these are the equations for the spherical Bessel functions $(\rho j_l(\rho))$.

(ii) Although the Bessel differential equations have singularities at $\rho=0$ and $\rho=\infty$, Eqs. (13) and (14) have thus more singularities at $\rho=\pm\epsilon/\sqrt{j(j+1)}$. These additional singularities are artificial, since the original first-order equations have only the two singularities at $\rho=0$ and $\rho=\infty$. The additional singularities have been introduced in the process of going from the first-order differential equations to second-order differential equations. For this reason we search a regular solution of Eqs. (13) and (14) at $\rho = \pm\epsilon\sqrt{j(j+1)}$. Now we try a solution of Eqs. (13) and (14) in the form of a series of spherical Bessel functions (times ρ). These are

$$f(\rho) = \rho\sum_{n=0}^{\infty} A_n j_{n+s}(\rho)\,,$$

$$g(\rho) = \sum_{n=0}^{\infty} B_n j_{n+s}(\rho)\,.$$

The first and second derivations of $f(\rho)$ and $g(\rho)$ are

$$\partial_\rho f(\rho) = \sum_{n=0}^{\infty} A_n\Big[j_{n+s}(\rho) + \frac{\rho}{2(n+s)+1}((s+n)j_{n+s-1} - (s+n+1)j_{n+s+1})\Big]\,, \tag{15}$$

$$\partial_\rho^2 f(\rho) = \rho\sum_{n=0}^{\infty} A_n\Big(-1 + \frac{(n+s)(n+s+1)}{\rho^2}\Big)j_{n+s}\,, \tag{16}$$

and similar expressions for $g(\rho)$. Inserting these relations into Eqs. (13) and (14) we obtain

$$\sum_{n=0}^{\infty}\Big\{\Big[-\frac{j(j+1)}{\rho^3}\Big(\frac{(n+s+1)(n+s+2)-j(j+1)}{2(n+s)+1}j_{n+s-1} + \frac{(n+s)(n+s-1)-j(j+1)}{2(n+s)+1}j_{n+s+1}\Big)A_n\Big] + \frac{1}{\rho^2}\Big[\epsilon^2((n+s)(n+s+1)-j(j+1))A_n - 2aB_n\Big]j_{n+s}\Big\} = 0\,, \tag{17}$$

$$\sum_{n=0}^{\infty}\Big\{\Big[-\frac{j(j+1)}{\rho^3}\Big(\frac{(n+s+1)(n+s+2)-j(j+1)}{2(n+s)+1}j_{n+s-1} + \frac{(n+s)(n+s-1)-j(j+1)}{2(n+s)+1}j_{n+s+1}\Big)B_n\Big] + \frac{1}{\rho^2}\Big[\epsilon^2((n+s)(n+s+1)-j(j+1)-2)B_n - 2aA_n\Big]j_{n+s}\Big\} = 0\,. \tag{18}$$

For $n = 0$, these equations give the following condition:

$$-\frac{j(j+1)}{\rho^3}\,\frac{(s+1)(s+2)-j(j+1)}{2s+1}j_{s-1}\begin{Bmatrix}A_0\\ B_0\end{Bmatrix} = 0\,. \tag{19}$$

The solutiuon of these individual equations gives

$$s = j-1 \quad\text{or}\quad s = j\,, \tag{20}$$

with $A_0 \neq 0$; $B_0 \neq 0$. We choose the positive one: $s = j-1$. In order to get the recursion relations for the coefficients we eliminate the $(1/\rho)$-term in the first part of Eqs. (17) and (18). This can be done by using the following functional relations of spherical Bessel functions

$$\frac{1}{\rho}j_{l-1}(\rho) = \frac{1}{2l-1}[j_{l-2}(\rho) + j_l(\rho)]\,. \tag{21}$$

By inserting this into Eqs. (17) and (18) we find the following recursion relations:

$$-j(j+1)\Big[\frac{(n+4)(n+5+2j)}{(2(n+4+j)-1)(2(n+j+4)-3)}A_{n+4} + \frac{1}{2(n+j+2)-1}\Big(\frac{(n+2)(n+3+2j)}{2(n+j+2)-3} + \frac{n(n+2j+1)}{2(n+j+2)+1}\Big)A_{n+2} + \frac{(n-2)(n+2j-1)}{2(n+j+1)(2(n+j)-1)}A_n\Big] + \Big[\epsilon^2((n+j+1)(n+j+2)-j(j+1))A_{n+2} - 2aB_{n+2}\Big] = 0\,, \tag{22}$$

$$-j(j+1)\Big[\frac{(n+4)(n+2j+5)}{(2(n+j)+7)(2(n+j)+5)}B_{n+4}\Big] + \frac{1}{2(n+j)+3}\Big(\frac{(n+2)(n+2j+3)}{2(n+j)+1} + \frac{n(n+2j+1)}{2(n+j)+5}\Big)B_{n+2} + \frac{(n-2)(n+2j-1)}{(2(n+j)+1)(2(n+j)-1)}B_n\Big] + [-2aA_{n+2} + \epsilon^2((n+j+1)(n+j+2)-j(j+1)-2)B_{n+2}] = 0\,. \tag{23}$$

Starting from $n = -2$, with $A_{-2} = B_{-2} = 0$, we then obtain the following relations between A_2, B_2, A_0, and B_0:

$$\frac{j+1}{2j+1}A_2 + \Big(\epsilon^2 - \frac{j+1}{2j+1}\Big)A_0 + \frac{aB_0}{j} = 0\,, \tag{24a}$$

$$\Big(\epsilon^2 - \frac{j}{2j+1}\Big)A_2 - \frac{a}{j+1}B_2 + \frac{j}{2j+1}A_0 = 0\,, \tag{24b}$$

$$-\frac{a}{j}A_2 + \Big(\epsilon^2 - \frac{j+1}{2j+1}\Big)B_2 + \frac{j+1}{2j+1}B_0 = 0\,, \tag{24c}$$

$$\frac{j}{2j+1}B_2 + \frac{a}{j+1}A_0 + \Big(\epsilon^2 - \frac{j}{2j+1}\Big)B_0 = 0\,. \tag{24d}$$

The determinant of the coefficients of Eq. (24) is zero. So it has a nontrivial solution given by

$$A_2 = \left[\left(1-\epsilon^2\frac{2j+1}{j+1}\right)A_0 - \frac{2j+1}{j(j+1)}aB_0\right],$$
$$B_2 = \left[-\frac{2j+1}{j(j+1)}aA_0 + \left(1-\epsilon^2\frac{2j+1}{j}\right)B_0\right]. \qquad (25)$$

Going back to the recursion relations (22) and (24) with $n=0$ and the solution given by (25) we get

$$A_4 = B_4 = 0.$$

Next for $n=2$ we obtain

$$A_6 = B_6 = 0.$$

This means that

$$A_{2n+4} = B_{2n+4}, \quad \text{for all } n>0.$$

Hence the solutions of the coupled differential Eqs. (13) and (14) are

$$f(\rho) = A_0 j_{j-1}(\rho) + A_2 j_{j+1}(\rho), \qquad (26)$$
$$g(\rho) = B_0 j_{j-1}(\rho) + B_2 j_{j+1}(\rho), \qquad (27)$$

where the relation between A_2, B_2 and A_0, B_0 is given in Eq. (25).

Physically we see that the components (ρu_2) and (ρv_0) of our wave functions in Eqs. (11) and (12) represent states that are superpositions of two angular momenta $l=j+1$ and $l=j-1$.

The spectrum is given by

$$E^2 = 2k^4 + m_1^2 + m_2^2 \pm 2\sqrt{k^4 + k^2(m_1^2+m_2^2) + m_1^2 m_2^2}.$$

V. INTERACTING PARTICLES

In this section we discuss a second limit of Eqs. (5) and (6). This limit is obtained by expanding the potentials as a power series of α/r and taking the terms up to the fifth power of α. In the power counting l/r is counted as α. This process gives the following set of coupled second-order differential equations:

$$\left[\frac{1}{4}\left(E-\frac{M^2}{E}\right)\left(E-\frac{\Delta m^2}{E}\right) + \frac{\alpha}{r}\left(E-\frac{M^2+\Delta m^2}{2E}\right) - \frac{j(j+1)-\alpha^2}{r^2} + \partial_r^2\right](ru_2) - \frac{2\sqrt{j(j+1)}}{r^2} rv_0(r) = 0, \qquad (28)$$

$$\left[\frac{1}{4}\left(E-\frac{M^2}{E}\right)\left(E-\frac{\Delta m^2}{E}\right) + \frac{\alpha}{r}\left(E-\frac{M^2+\Delta m^2}{2E}\right) - \frac{j(j+1)+2-\alpha^2}{r^2} + \partial_r^2\right](rv_0) - \frac{2\sqrt{j(j+1)}}{r^2} ru_2(r) = 0. \qquad (29)$$

Here again, the only difference between Eqs. (28) and (29) is the term $-(2/r^2)(rv_0)$ in Eq. (29). The remaining terms are symmetrical in both equations. In the dimensionless units these equations are

$$\left[-\frac{1}{4} + \frac{2Z}{\rho} - \frac{l(l+1)}{\rho^2} + \partial_\rho^2\right] f(\rho) - \frac{2\sqrt{j(j+1)}}{\rho^2} g(\rho) = 0, \qquad (30)$$

$$\left[-\frac{1}{4} + \frac{2Z}{\rho} - \frac{l(l+1)+2}{\rho^2} + \partial_\rho^2\right] g(\rho) - \frac{2\sqrt{j(j+1)}}{\rho^2} f(\rho) = 0, \qquad (31)$$

where

$$\rho = 2\lambda r, \qquad (32a)$$
$$4\lambda^2 = (M^2-E^2)(E^2-\Delta m^2)/E^2, \qquad (32b)$$
$$2Z = \frac{\alpha}{2\lambda}\left(E - \frac{M^2+\Delta m^2}{2E}\right), \qquad (32c)$$
$$l(l+1) = j(j+1) - \alpha^2. \qquad (32d)$$

These equations have two singular points, $\rho=0$ and $\rho=\infty$. The point $r=0$ is a regular singularity, while $\rho=\infty$ is an irregular singularity. At $\rho=\infty$ the equations simplify

$$\left(-\frac{1}{4} + \frac{d^2}{d\rho^2}\right)\begin{pmatrix} f(\rho) \\ g(\rho)\end{pmatrix} = 0, \qquad (33)$$

so that the regular solution at infinity is

$$\begin{pmatrix} f(\rho) \\ g(\rho)\end{pmatrix} \to e^{-(1/2)\rho}. \qquad (34)$$

At $\rho \simeq 0$ the equations are

$$\left(\frac{d^2}{d\rho^2} - \frac{l(l+1)}{\rho^2}\right) f(\rho) - \frac{2\sqrt{j(j+1)}}{\rho^2} g(\rho) = 0, \qquad (35)$$

$$\left(\frac{d^2}{d\rho^2} - \frac{l(l+1)+2}{\rho^2}\right) g(\rho) - \frac{2\sqrt{j(j+1)}}{\rho^2} f(\rho) = 0. \qquad (36)$$

We assume a powerlike behavior of the solution at the origin

$$\begin{pmatrix} f(\rho) \\ g(\rho)\end{pmatrix} \to \begin{pmatrix} A_0 \\ B_0\end{pmatrix}\rho^s. \qquad (37)$$

Insertion of this ansatz into Eqs. (35) and (36) gives the following relation:

$$(s(s-1) - l(l+1))A_0 - 2\sqrt{j(j+1)}B_0 = 0,$$
$$-2\sqrt{j(j+1)}A_0 + (s(s-1) - l(l+1) - 2)B_0 = 0. \qquad (38)$$

Hence the condition for the existence of a nontrivial solution is

$$s(s-1) = j(j-1) - \alpha^2 \qquad (39)$$

or

$$s = \frac{1}{2} + \sqrt{\frac{1}{4} + j(j-1) - \alpha^2} = j + \left(j-\frac{1}{2}\right)\left[\sqrt{1 - \frac{\alpha^2}{(j-\frac{1}{2})^2}} - 1\right] \simeq j - \frac{\alpha^2}{2(j-\frac{1}{2})} + O(\alpha^4). \qquad (40)$$

This s-value is in agreement with the s-value in Eq. (17) for $\alpha=0$ case. In order to find a regular solution for all ρ's we write $f(\rho)$ and $g(\rho)$ in the form

$$f(\rho) = e^{-\rho/2}\rho^s y(\rho), \qquad (41)$$
$$g(\rho) = e^{-\rho/2}\rho^s z(\rho). \qquad (42)$$

Now instead of searching a power series solution for $y(\rho)$ and $g(\rho)$ we assume a solution that is a series of confluent hypergeometric functions. In the free-particle case we already obtained in the previous section such a two-term series with orbital angular momentum $l=j-1$ and $l=j+1$. Except for the term $-2/r^2 g(\rho)$ and the coupling terms Eqs. (30) and (31) are the same as the Schrödinger equation for the hydrogen atom. The Coulomb problem has the following solutions:

$$R_{n,l}(\rho)=e^{-\rho/2}\rho^{l+1}\,{}_1F_1(-n+l,2l+2;\rho)\,, \tag{43}$$

where F_1 is the confluent hypergeometric function.

For Eqs. (30) and (31) we try again a two-term solution with $l=s$ and $l=s'$ of the form

$$f(\rho)=e^{-\rho/2}[A_0\rho^s\,{}_1F_1(-n+s,2s;\rho)$$
$$+A_2\rho^{s'+2}\,{}_1F_1(-n+s'+2,2s'+4;\rho)]\,, \tag{44}$$

$$g(\rho)=e^{-\rho/2}[B_0\rho^s\,{}_1F_1(-n+s,2s;\rho)$$
$$+B_2\rho^{s'+2}\,{}_1F_1(-n+s'+2,2s'+4;\rho)]\,. \tag{45}$$

We shall make use of the following property of the functions R_{nl} that can be proved by using the functional relations of confluent hypergeometric functions:

$$\frac{d^2}{d\rho^2}R_{nl}(\rho)$$
$$=\frac{d^2}{d\rho^2}\left[e^{-\rho/2}\rho^{l+1}\,{}_1F_1(-n+l+1,2l+2;\rho)\right]$$
$$=\left(\frac{1}{4}+\frac{n}{\rho}+\frac{l(l+1)}{\rho^2}\right)R_{nl}\,. \tag{46}$$

We insert (44) and (45) into (30) and (31), and, by using (46), we obtain the following relations

$$\left[\frac{2Z-n}{\rho}+\frac{s(s-1)-l(l+1)}{\rho^2}\right]A_0R_{ns}$$
$$+\left[\frac{2Z-n}{\rho}+\frac{(s'+2)(s'+1)-l(l+1)}{\rho^2}\right]A_2R_{n,s'+2}$$
$$-\frac{2\sqrt{j(j+1)}}{\rho^2}[B_0R_{n,s}+B_2R_{n,s'+2}]=0\,, \tag{47}$$

$$\left[\frac{2Z-n}{\rho}+\frac{s(s-1)-l(l+1)-2}{\rho^2}\right]B_0R_{n,s}$$
$$+\left[\frac{2Z-n}{\rho}+\frac{(s'+1)(s'+2)-l(l+1)-2}{\rho^2}\right]$$
$$\times B_2R_{n,s'+2}$$
$$-\frac{2\sqrt{j(j+1)}}{\rho^2}[A_0R_{n,s}+A_2R_{n,s'+2}]=0\,. \tag{48}$$

If we choose

$$2Z=n\,, \tag{49}$$

then the $1/\rho$-terms drop out, and we get relations between the coefficient of $R_{n,s}$ and $R_{n,s'}$. The relation between A_0 and B_0 are the same as Eq. (38). The relation between A_2 and B_2 are

$$((s'+1)(s'+2)-l(l+1))A_2-2\sqrt{j(j+1)}B_2=0\,,$$
$$-2\sqrt{j(j+1)}A_2$$
$$+((s'+1)(s'+2)-l(l+1)-2)B_2=0\,. \tag{50}$$

For the existence of a nontrivial solution of Eq. (50), s' must satisfy the following condition

$$(s'+1)(s'+2)=(j+1)(j+2)-\alpha^2\,. \tag{51}$$

Hence the relation between A_2 and B_2, and A_0 and B_0 are

$$A_2=\sqrt{j/j+1}B_2\,. \tag{52}$$

Equation (38) gives also the following solution:

$$A_0=-\sqrt{j+1/j}B_0\,. \tag{53}$$

Thus the final solutions of our problem are

$$f(\rho)=\rho u_2(\rho)$$
$$=e^{-\rho/2}[A_0\sqrt{j+1}\,\rho^{s_-}\,{}_1F_1(-n+s_-;2s_-;\rho)$$
$$+A_2\sqrt{j}\rho^{s_+}\,{}_1F_1(-n+s_+;2s_+;\rho)]\,, \tag{54}$$

$$g(\rho)=\rho\,v_0(\rho)$$
$$=e^{-\rho/2}[A_0\sqrt{j}\rho^{s_-}\,{}_1F_1(-n+s_-;2s_-;\rho)$$
$$+A_2\sqrt{(j+1)}\,\rho^{s_+}\,{}_1F_1(-n+s_+;2s_+;\rho)]\,, \tag{55}$$

where s_- and s_+ are the obtained from Eqs. (39) and (51), respectively. Finally the quantization condition (49) gives using Eq. (32) the following energy or mass spectrum:

$$E^2=\frac{M^2+\Delta m^2}{2}\pm\frac{M^2-\Delta m^2}{2}\left[1+\frac{\alpha^2}{n^2}\right]^{-1/2}$$
$$=m_1^2+m_2^2\pm 2m_1m_2\left(1+\frac{\alpha^2}{n^2}\right)^{-1/2}. \tag{56}$$

Here the principal quantum number n is related to the radial quantum number n_r by

$$n=n_r+l_0\,, \tag{57}$$

where l_0 (the nonrelativistic label of the angular momentum) is equal to $l_0=j-1$ or $l_0=j+1$, for the two states we have discussed.

The bound states in E^2 are slightly below the continuum $E^2>(m_1+m_2)^2$ for the $(+)$ sign in the spectrum (57) and for the $(-)$ sign, slightly above the negative continuum $E^2<(m_1-m_2)^2$. If we expand Eq. (56) in powers of α and pass to from E^2 to E we obtain

$$E(n,l)=m_1+m_2-\frac{m_1\alpha^2}{2n^2(1+m_1/m_2)}$$
$$-\frac{m_1\alpha^4}{4n^3(1+m_1/m_2)(l+\frac{1}{2})}$$
$$+\frac{3}{8}\frac{m_1\alpha^4}{n^4(1+m_1/m_2)}-\frac{1}{8}\frac{(m_1^2/m_2)\alpha^4}{n^4(1+(m_1/m_2))^2}$$
$$+O(\alpha^6)\,,$$
$$l=-\tfrac{1}{2}+\sqrt{(j+\tfrac{1}{2})^2-\alpha^2}$$
$$\cong j-\alpha^2/(2j+1)+O(\alpha^4)\,, \tag{58}$$

which shows that the mass spectrum agrees with the usual QED up to order α^4. But the exact expression (56) should be used for recoil correction to all orders in α. Usually the nonrelativistic quantum number $n\equiv n_r+j$ is used. But it is better to keep n_r and j separately for really relativistic systems, e.g., positronium, in which l is not quite an integer. In fact one of the interesting problems of relativistic two-body dynamics is to find exact quantum numbers, besides energy and total angular momentum J, to label the states. The usual nonrelativistic labeling of positronium states, for example, as ${}^1S_0$, ${}^3S_1$,... means only that these states have these corresponding values in the nonrelativistic limit.

VI. THE SECOND SET. TOTAL SPIN $S=0$ AND $S=1$ EQUATIONS

In a similar manner we treat the second set of eight radial linear equations arising from Eq. (3). We eliminate half of the components using the algebraic equations and obtain two coupled second-order equations, namely the counter parts of Eqs. (5) and (6). The exactly soluble part of these equations, up to order α^4, are the following two uncoupled equations:

$$\left[\frac{1}{4}\left(E-\frac{M^2}{E}\right)\left(E-\frac{\Delta m^2}{E}\right)+\frac{\alpha}{r}\left(E-\frac{M^2+\Delta m^2}{2E}\right) - \frac{j(j+1)-\alpha^2}{r^2}+\partial_r^2\right](ru_1)=0, \tag{59a}$$

$$\left[\frac{1}{4}\left(E-\frac{M^2}{E}\right)\left(E-\frac{\Delta m^2}{E}\right)+\frac{\alpha}{2r}\left(2E-\frac{M^2+\Delta m^2}{E}\right) - \frac{j(j+1)-\alpha^2\delta_E}{r^2}+\partial_r^2\right](rv_{00})=0. \tag{59b}$$

The free-particle solution of (59a) and (59b) are simpler than in the first set, Eqs. (26) and (27), namely

$$\rho u_2(\rho)=A\rho j_j(\rho),$$
$$\rho v_{00}(\rho)=B\rho j_j(\rho). \tag{60}$$

In Eq. (59b) the factor δ_E is given by

$$\delta_E=\frac{\Delta m^2-M^2}{E^2}+\frac{M^2\Delta m^2}{E^4}. \tag{61}$$

The solutions of (59a) and (59b), because they are uncoupled, can be written down immediately in terms of hydrogenic wave functions

$$\rho u_1(\rho)=Ae^{-\rho/2}\rho^{l+1}\,{}_1F_1(-n+l+1;2l+2;\rho), \tag{62a}$$

$$\rho v_{00}(\rho)=Be^{-\rho/2}\rho^{l_0+1}\,{}_1F_1(-n+l_0+1;2l_0+2;\rho), \tag{62b}$$

and the spectrum has the same general form as in Eq. (56),

$$E^2=\frac{M^2+\Delta m}{2}\pm\frac{M^2-\Delta m^2}{2}\left(1+\frac{\alpha^2}{(n_r+l)^2}\right)^{-1/2}. \tag{63}$$

But the range of the angular momentum l is now given by

$$l(l+1)=j(j+1)-\alpha^2, \quad \text{for Eq. (59a)} \tag{64}$$

and

$$l(l+1)=j(j+1)-\alpha^2\delta_E, \quad \text{for Eq. (59b)}. \tag{65}$$

VII. COMPARISON WITH THE EXACTLY SOLUBLE INFINITE-COMPONENT WAVE EQUATIONS

Infinite-component wave equations appropriate for two-body Coulomb systems are generalizations of the original infinite component Majorana equation.[4] They make use of the dynamical group SO(4,2) rather than the Lorentz group of the Majorana equation and account for the correct degeneracy of states. They have been used to describe the relativistic H-atom and hadrons, and to describe many properties of these composite systems in a relativistic way, such as form factors and transition amplitudes in external fields.[5] It is interesting that our exactly soluble models give precisely the same spectrum as the infinite-composite wave equation for the relativistic Coulomb problem. We thus have in the one hand the group structure of our model, and on the other hand, the infinite-component wave equation acquires an explicit dynamical realization in terms of constituents.

The wave equation is a generalized Dirac equation

$$(J^\mu P_\mu+K)\psi(P)=0, \tag{66}$$

where P_μ is the total momentum of the composite system, and the current and mass operators are given by

$$J_\mu=\alpha_1\Gamma_\mu+\alpha_2P_\mu+\alpha_3P_\mu\Gamma_4,$$
$$K=\beta\Gamma_4+\gamma. \tag{67}$$

Here Γ_μ and Γ_4 are the generators of the dynamical group SO(4,2); P_μ the total momentum of the atom. The choice of the constants[6]

$$\alpha_1=1, \quad \alpha_2=\alpha/2m_2, \quad \alpha_3=1/2m_2,$$
$$\beta=(m_2^2-m_1^2)/2m_2, \quad \gamma=-\alpha(m_1^2+m_2^2)/2m \tag{68}$$

gives the spectrum

$$M_n^{\pm 2}=m_1^2+m_2^2\pm 2m_1m_2(1+\alpha^2/n^2)^{-1/2} \tag{69}$$

which coincides with (56) or (63).

In fact the form of infinite-component wave equation (66) can be inferred directly from our basic equation (1), but the operators J_μ and K have a more complicated form for Eq. (1); the simpler forms given in (67) and (68) correspond to the exactly soluble part of our equation. The infinite-component equation is very useful in treating further the external interactions of our composite atom because it treats the whole atom now as a single relativistic "particle."

The discussion of the perturbations of order α^5 is given elsewhere.[7]

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Comment on the paper "An exactly soluble relativistic quantum two-fermion problem" [J. Math. Phys. 27, 3055 (1986)]

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A comment is made on the separability of the center of mass and relative coordinates in the exact solution of a covariant two-body equation for two spin-$\frac{1}{2}$ particles.

I. INTRODUCTION

In a recent paper[1] a covariant two-body equation for two spin-$\frac{1}{2}$ particles was considered and an exact solution was presented in the center of mass system. Although this solution is correct, it was incorrectly stated at the beginning of the paper that the center of mass and relative coordinates are exactly separable. We wish to correct this statement and indicate the proper treatment of the equation in an arbitrary frame.

II. THEORY

With a choice of the spacelike surface perpendicular to $n_\mu = (1000)$, the covariant equation can be written in the Hamiltonian form as

$$\{(1/M)(m_1\boldsymbol{\alpha}_1 + m_2\boldsymbol{\alpha}_2)\cdot\mathbf{P} + [(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2)\cdot\mathbf{P} - \beta_1 m_1 - \beta_2 m_2 - V(r)]\}\Phi = E\Phi, \tag{1}$$

where $\mathbf{P}$ is the total momentum and $\mathbf{p}$ and $\mathbf{r}$ are the relative coordinates. Thus the Hamiltonian separates into a sum of two terms, one depending on the center of mass momentum, the other on the relative coordinates. However, the coefficients in these two terms depending on the spin matrices do not commute, hence the solution cannot be written as a product of two functions, one depending on the center of mass coordinates and one on the relative coordinates. Since $\mathbf{R}$ does not appear, we have always a factor $e^{i\mathbf{P}\cdot\mathbf{R}}$ so that we can treat $\mathbf{P}$ as a number in the momentum representation.

Equation (1) is a specific case of an infinite component wave equation generally written as

$$(J^\mu P_\mu - K)\Phi = 0,$$

or, with J_0 diagonal and equal to 1 as in our case,

$$(\mathbf{J}\cdot\mathbf{P} - K)\Phi = E\Phi, \tag{2}$$

where $\mathbf{J}$ and $\mathbf{K}$ (which is a function of relative coordinates or internal degrees of freedom) do not commute. There is a general procedure to solve Eq. (2) in an arbitrary frame[2] and it was an oversight not to connect Eq. (1) with Eq. (2). The method consists in finding the appropriate boost operators $\mathbf{M}$, solving the equation in the rest frame ($\mathbf{P} = 0$), and then boosting the result to an arbitrary frame: $\Phi(\mathbf{P}) = e^{i\boldsymbol{\xi}\cdot M}\Phi(0)$.

Another procedure in the present case is to separate the radial and angular part of Eq. (1) for a general $\mathbf{P}$. Since in the momentum representation the coefficient of $\mathbf{P}$ is a finite matrix, the method of separation used in Ref. 3 can easily be extended.

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