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Nonperturbative Quantum Electrodynamics: The Lamb Shift

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The nonlinear integro-differential equation, obtained from the coupled Maxwell–Dirac equations by eliminating the potential A_μ, is solved by iteration rather than perturbation. The energy shift is complex, the imaginary part giving the spontaneous emission. Both self-energy and vacuum polarization terms are obtained. All results, including renormalization terms, are finite.

Dirac has insisted for a long time that the perturbation theory of quantum electrodynamics (QED) with quantized electromagnetic field is a "bad theory"[(1)]: "... it was shown by Lamb that the infinities (of perturbation theory) could be removed by a process of renormalization. ... The general idea of renormalization is quite sensible physically, but the way it is applied here is not sensible, because the factor connecting the original parameters with the new (renormalized) ones is infinitely great." Further problems with renormalization procedures, their limitations and generalizations have been reviewed recently.[(2,3)]

As a method of avoiding the perturbation theory with quantized electromagnetic field we have proposed some time ago[(4)] that electrodynamics could be based on the spinor field ψ alone by eliminating A_μ from the coupled Maxwell–Dirac equations and solving the resultant equation for the localized wave function ψ, never using plane-wave expansions.

We carry now this program for the calculation of the Lamb shift and also show how to obtain the spontaneous emission and the anomalous magnetic moment. All integrals, including the renormalization integral, are

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finite, and closed expressions can be obtained for the above quantum phenomena to all orders of α.

It may come as a surprise that the three major QED effects, namely the Lamb shift, the spontaneous emmission, and the $(g-2)$ can be calculated without explicit use of a quantized A_μ-field, and all within a single equation. The present formulation of QED is conceptually along the same lines as classical electrodynamics,[3,5] where also the self-energy effects can be treated in a closed nonperturbative way, and brings a definite simplicity, compactness, and completeness to quantum radiation theory.

Nonperturbative quantum electrodynamics is based on the following equation, obtained by eliminating A_μ from the coupled Maxwell–Dirac equations[4]:

$$(p - e\not{A}^{\text{ext}} - m)\psi = e\not{A}^{\text{self}}\psi \tag{1}$$

where

$$A_\mu^{\text{self}}(x) = e\int dx' D(x-x')\,\bar{\psi}(x')\,\gamma_\mu\psi(x')$$

Here $\not{p} \equiv \gamma^\mu p_\mu$, etc., and $D(x-y)$ is the Green's function of $\Box A_\mu$, as usual. Eq. (1) is a nonlinear integro-differential equation. In this paper we shall take A_μ^{ext} to be the Coulomb field, but our formalism can be generalized to two (or more) relativistic particles (e.g., positronium) by taking two coupled fields ψ_1 and ψ_2, in which case A_μ^{ext} is the field produced by the second particle.[5]

We look for a solution of Eq. (1) by expanding $\psi(x)$ in a complete set of solutions of the full problem

$$\psi(x) = \sum\!\!\!\!\!\!\int_n \psi_n(\mathbf{r})e^{-iE_n t} \tag{2}$$

If this expansion is inserted into (1), using

$$D(x-x') = -\frac{1}{(2\pi)^4}\int d^4q\,\frac{e^{iq(x-x')}}{q^2}$$

we have

$$\sum\!\!\!\!\!\!\int (E_n\gamma^0 - \mathbf{p}\cdot\boldsymbol{\gamma} - e\not{A}^{\text{ext}} - m)\,\psi_n(\mathbf{r})\,e^{-iE_n t}$$

$$= \sum\!\!\!\!\!\!\int_{r,s,m} \frac{-e^2}{(2\pi)^4}\int d^4q\,\frac{e^{-i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r})}}{q^2}\,\bar{\psi}_r(\mathbf{r}')\,\gamma_\mu\,\psi_s(\mathbf{r}')\,e^{i(E_r-E_s-q_0)t'}\gamma^\mu$$
$$\times e^{-i(E_m-q_0)t}\,\psi_m(\mathbf{r})\,d^3r\cdot dt'$$

or, after the t' and q^0-integration,

$$\sum\!\!\!\!\!\!\int_n (E_n\gamma^0 - \mathbf{p}\cdot\boldsymbol{\gamma} - e\not{A}^{\text{ext}} - m)\,\psi_n(\mathbf{r})\,e^{-iE_nt}$$
$$= -\frac{e^2}{(2\pi)^3}\sum\!\!\!\!\!\!\int_{r,s,m}\int \frac{d^3q e^{-i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}}{(E_r-E_s)^2-\mathbf{q}^2}\,\gamma^\mu\psi_m(\mathbf{r})\,\bar{\psi}_r(\mathbf{r}')\,\gamma_\mu\psi_s(\mathbf{r}')$$
$$\times e^{-i(E_m+E_s-E_r)t}\,d^3r' \tag{3}$$

Comparing the coefficients of both sides, we see that for each n, the sum on the right-hand side goes over those values of r, s, m such that

$$E_m + E_s - E_r = E_n \tag{4}$$

Thus,

$$(E_n\gamma^0 - \mathbf{p}\cdot\boldsymbol{\gamma} - e\not{A}^{\text{ext}} - m)\,\psi_n(\mathbf{r})$$
$$= -\frac{\alpha}{2\pi^2}\sum\!\!\!\!\!\!\int_{\substack{r,s,m\\ E_m+E_s-E_r=E_n}}\int \frac{d^3q\, e^{=i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}}{(E_r-E_s)^2-\mathbf{q}^2}\,\gamma^\mu\psi_m(\mathbf{r})\,\bar{\psi}_r(\mathbf{r}')\,\gamma_\mu\psi_s(\mathbf{r}')\,d^2r'$$
$$(\alpha = e^2/4\pi) \tag{5}$$

This is an infinite set of coupled equations. Now if we write for the energy shift in the state n:

$$E_n = E_n^{\text{Coulomb}} + \Delta E_n \tag{6}$$

multiply from the left with $\bar{\psi}_n^{\text{Coulomb}}$, and integrate, we get, using

$$\bar{\psi}_n^c(E_n\gamma^0 - \mathbf{p}\cdot\boldsymbol{\gamma} - e\not{A}^{\text{ext}} - m) = 0$$

$$\Delta E_n \int \psi_n^{+c}(\mathbf{r})\,\psi_n(\mathbf{r})\,d^2r$$
$$= -\frac{\alpha}{2\pi^2}\sum\!\!\!\!\!\!\int_{\substack{r,s,m\\ E_m+E_s-E_r=E_n^c+\Delta E_n}}\int \frac{d^3q\, e^{-i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}}{(E_r-E_s)^2-\mathbf{q}^2}\,\bar{\psi}_n^c(\mathbf{r})\,\gamma^\mu\psi_m(\mathbf{r})\,\bar{\psi}_r(\mathbf{r}')$$
$$\times \gamma_\mu\psi_s(\mathbf{r}')\,d^3r\,d^3r' \tag{7}$$

Two immediate solutions of Eq. (4) are:

$$E_s = E_r, \quad \text{hence} \quad E_n = E_m \tag{8}$$
$$E_m = E_r, \quad \text{hence} \quad E_n = E_s \tag{9}$$

There are other solutions to Eq. (4), with three or all four E's being different. We shall see that the case (8) corresponds to a sum of vacuum polarization terms and (9) to self energy terms. For the case (8), we obtain

$$\Delta E_n^{(a)} \int \psi_n^{c+} \psi_n \, d^3r = \frac{\alpha}{2\pi^2} \sum_s \int \frac{d^3q \, e^{-i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}}{\mathbf{q}^2} \bar{\psi}_n^c(\mathbf{r}) \, \gamma^\mu \psi_n(\mathbf{r}) \, \bar{\psi}_s(\mathbf{r}') \; \gamma_\mu \psi_s(\mathbf{r}') \; d^3r \, d^3r' \tag{10}$$

and, in the second case (9),

$$\Delta E_n^{(b)} \int \psi_n^{+c} \psi_n \, d^3r = -\frac{\alpha}{2\pi^2} \sum_m \int \frac{d^3q \, e^{-i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}}{(E_m - E_n)^2 + \mathbf{q}^2} \bar{\psi}_n^c(\mathbf{r}) \, \gamma^\mu \psi_m(\mathbf{r}) \, \bar{\psi}_m(r') \, \gamma_\mu \psi_n(r') \, d^3r \, d^3r' \tag{11}$$

The remaining values of energies fulfilling condition (4) seem to lie all in the continuum.

The form of the contributions of (10) and (11) are still exact.

If we replace now all ψ_n by ψ_n^c, Eqs. (10) and (11) become

$$\Delta E_n^{(a)} = \frac{\alpha}{2\pi^2} \sum_s \int \frac{d^3q}{\mathbf{q}^2} \, T_{nn}^\mu(\mathbf{q}) \, T_{ss}(-\mathbf{q})_\mu \tag{12}$$

$$\Delta E_n^{(b)} = -\frac{\alpha}{2\pi^2} \sum_m \int \frac{d^3q}{(E_m - E_n)^2 - \mathbf{q}^2} \, T_{nm}^\mu(\mathbf{q}) \, T_{mn}(-\mathbf{q})_\mu \tag{13}$$

where

$$T_{nm}^\mu(\mathbf{q}) \equiv \int \bar{\psi}_n(\mathbf{r}) \, \gamma^\mu \psi_m(\mathbf{r}) \, e^{-i\mathbf{q}\cdot\mathbf{r}} \, d^3r \tag{14}$$

If in (13) we factorize the denominator and take the contribution of one of the poles, we obtain

$$\Delta E_n^{(b)} = \frac{\alpha}{4\pi^2} \int \frac{d^3q}{q} \sum_m \frac{T_{nm}^\mu(\mathbf{q}) \, T_{mn}(-\mathbf{q})_\mu}{E_n - E_m - q} \tag{13a}$$

The expressions (12) and (13a) agree with the standard formulas[6,7] except that here the ψ's are relativistic Coulomb wave functions (and not plane-waves) and all integrals converge. In fact, the integrals can be carried out exactly.[8]

But we have still to perform a finite renormalization. Since the observable effect of the self energy term on the right-hand side of Eq. (1) is to change the energy, the only measurable contributions in (12) and (13) will come from $s \neq n$ and $m \neq n$, respectively. The terms $s = n$ and $m = n$ are already counted in the mass and charge of the electron, and in the definition of E_n. Furthermore, we extract in the integral (13a) only the term proportional to $(E_n - E_m)$ according to the expansion $q/(E_n - E_m - q) = -1 + (E_n - E_m)/(E_n - E_m - q)$. There is *one* renormalization term for both (12) and (13), namely

$$(\Delta E_n)^{\text{renor}} = \frac{\alpha}{2\pi^2} \int \frac{d^3q}{\mathbf{q}^2} T^{\mu}_{nn}(\mathbf{q})\, T_{nn}(-\mathbf{q})_{\mu} \tag{15}$$

Note also that in the vacuum polarization term, the wave functions of the "virtual pairs" are the relativistic Coulomb wave functions and not plane-waves, which contain negative energy solution when expanded into plane-waves.

It is easy to see the approach of our Eqs. (12) and (13) to their usual form in standard practical calculations. In particular, (13a) goes over into Bethe's formulas in the dipole and nonrelativistic approximation of T^{μ}_{nm} in (14).

Coming back to our general formulas (12) and (13), we find that the energy shifts are complex. The imaginary part of the shift must be identified with the spontaneous emission of the excited states due to self-energy. It is thus gratifying that, in this formulation, Lamb energy and spontaneous emission occur together as they are both due to a self-energy effect. We can furthermore calculate the anomalous magnetic moment-form factor of the electron in the same external field. For this purpose, we go to Eq. (5) and evaluate the self-potential term $e\gamma_{\mu}\psi_n A^{\text{self}}_{\mu}$ (i.e., $m = n$, $r = s = n$). The radial effective potential goes like $f(r)\, 1/r^2$. The coefficient of $1/r^2$-potential is the anomalous magnetic moment.[4,9]

As we have noted, the problem of quantization of the electromagnetic field A_{μ} never arises, since it has been eliminated. The theory sofar behaves like a c-number theory with respect to ψ, because we never had to commute the ψ-fields. However, second quantization of ψ-field will come in if we use identical particles and in processes involving real pair production. Having shown that the standard energy-shift terms are contained in the solution of the fundamental equation (1), one may attempt to solve directly this integro-differential equation, with finite renormalization term (15) subtracted, without making the expansion (2). This would provide indeed a one-shot final solution to the complete Coulomb problem with all relativistic and radiative effects already included.

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