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Vacuum Polarization in Self-Field Quantum Electrodynamics

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We have evaluated analytically the vacuum polarization in a Coulomb field using the relativistic Dirac–Coulomb wave functions by a new method. The result is made finite by an appropriate choice of contour integrations and gives the standard result in the lowest order of iteration. We used the formalism of self-field quantum electrodynamics in the evaluation of the vacuum polarization which needs neither field quantization nor renormalization. There are no infrared or ultraviolet divergences.

1. INTRODUCTION

One of the most important effects in quantum electrodynamics is the vacuum polarization (VP). It is important both practically and conceptually, because it represents the most divergent term in perturbation theory and enters significantly in the idea of a "running coupling constant" and "renormalization group." In the standard treatment of VP we evaluate the photon propagator in perturbation theory by renormalizing the charge, getting an effective potential by taking the Fourier transform of it and then taking the matrix elements of the potential between Coulomb wave functions.

Schwinger gave the field theoretical formulation of VP.(1) Wichmann and Kroll performed calculations of the VP with relativistic Coulomb wave functions first,(2) and others studied it further, also numerically.(3–11) The work started by Wichmann and Kroll has not been finished.

Barut developed a new formulation of QED for the bound states of the electron, based on the self-energy.(12) This new approach has been

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applied to VP problem recently[13] as well as to Lamb shift,[14] spontaneous emission,[15] and (g-2)calculations.[16] In Ref. 13, a closed expression and a finite result has been obtained for VP.

Since this study contains the sum over infinitely many bound and continuum Coulomb wave functions, which is very complex and intricate, we present a new method. Furthermore, the finite part of the lowest-order contribution of VP has been rederived by using this new method with a regularization and without any renormalization. This result is presented in a short paper elsewhere.[17]

In Ref. 17 we presented the unambigous definition of all the integrals by their Mellin–Barnes transform.[18] This is the new regularization method and we obtain a finite result by using this method without any renormalization procedure. The aim of this paper is to present the details of the sum over infinitely many wave functions in Ref. 17, derive the regularized form of the integrals, and show that the physical contribution of the divergent terms are zero, because of the sum over all energy values. The derivation of the finite part of VP without any renormalization procedure is very important for the proof of the fundamental question of the finiteness of the theory.

2. THE NEW ANALYTIC CALCULATION OF VACUUM POLARIZATION

In the present picture all the QED effects come from a single expression without going into the separate Feynman diagrams, such as a loop diagram. To first order of iteration one obtains a general expression for the energy shift of a level n directly from the actions:[13, 19]

$$\begin{aligned}\Delta E_n = {} & \frac{e^2}{2}\int d\mathbf{x}\, \bar{\Psi}_n(\mathbf{x})\, \gamma_\mu \Psi_n(\mathbf{x})\, P\int \frac{d\mathbf{k}}{(2\pi)^3}\int d\mathbf{y}\,\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^2}\sum_s \bar{\Psi}_s(\mathbf{y})\, \gamma^\mu \Psi_s(\mathbf{y}) \\ & -\frac{e^2}{2}\sum_S \int d\mathbf{x}\, d\mathbf{y}\, \bar{\Psi}_n(\mathbf{x})\, \gamma_\mu \Psi_s(\mathbf{x})\int \frac{d\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\bar{\Psi}_s(\mathbf{y})\, \gamma^\mu \Psi_n(\mathbf{y}) \\ & \times\left[\frac{1}{E_s-E_n-k}-\frac{1}{E_s-E_n+k}\right] \\ & -\frac{e^2}{2}\sum_{s<n}\int d\mathbf{x}\, d\mathbf{y}\, \bar{\Psi}_n(\mathbf{x})\, \gamma_\mu \Psi_s(\mathbf{x})\int \frac{d\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\bar{\Psi}_s(\mathbf{y})\, \gamma^\mu \Psi_n(\mathbf{y}) \\ & \times \frac{i\pi}{2k}\,\delta(E_s-E_n-k)\end{aligned} \tag{2.1}$$

where Ψ_n is a fixed level, and we sum over all levels Ψ_s of the external field, i.e., Coulomb field. The first term is the contribution of vacuum polarization, the second that of self-energy (or proper Lamb shift including the contribution of (g-2)), and the third term gives the spontaneous emission rate which has been evaluated exactly and analytically.

We are going to evaluate the first term in Eq. (2.1), vacuum polarization. This term can be interpreted as the interaction energy of two current distributions. If we use the relativistic Coulomb wave functions and perform the spin algebra because of the spherical symmetry we obtain[16]

$$\Delta E^{\mathrm{VP}} = 4\alpha(2j_n + 1) \sum_{l,m} \int (2j_s + 1) \int dr\, dr'\, V_\ell(r, r') \\ \times (|f(r)|^2 + |g(r)|^2)(|f(r')|^2 + |g(r')|^2) \tag{2.2}$$

where we have introduced a potential $V_\ell(r', r)$ by

$$V_\ell(r', r) = \frac{2}{\pi} r^2 r'^2 \int_0^\infty j_\ell(kr')\, j_\ell(kr)\, dk = \frac{r^2 r'^2 r'^{\ell}_{<}}{(2\ell + 1)\, r'^{\ell+1}_{>}} \tag{2.3}$$

and $f_n(r)$ and $g_n(r)$ the radial Dirac wave functions. The most difficult part of the calculations is the sum over all discrete and continuous states. Here we use the method of Green's functions initiated by Wichmann and Kroll.[2]

Since the completeness of Green's function we have

$$e \sum_s (|f_s|^2 + |g_s|^2) = \frac{e}{2} \sum_{E_s > 0} (|f_s|^2 + |g_s|^2) - \frac{e}{2} \sum_{E_s < 0} (|f_s|^2 + |g_s|^2) \tag{2.4}$$

The Green's function involves both negative and positive energy solutions and the negative-energy solutions correspond to positive-energy solutions with the sign of charge reversed. Then the above sum can be represented as a well-known contour integral around the positive and negative energy spectrum in the z (energy)-plane (Fig. 1)

$$e \sum_s (|f_s|^2 + |g_s|^2) = \frac{e}{4\pi i} \left[\int_{C+} + \int_{C-} \right] dz\, \mathrm{Tr}\, K(r, r'; z) \tag{2.5}$$

where $K(r, r'; z)$ is the energy-dependent Green's function of the radial Coulomb problem which is known.

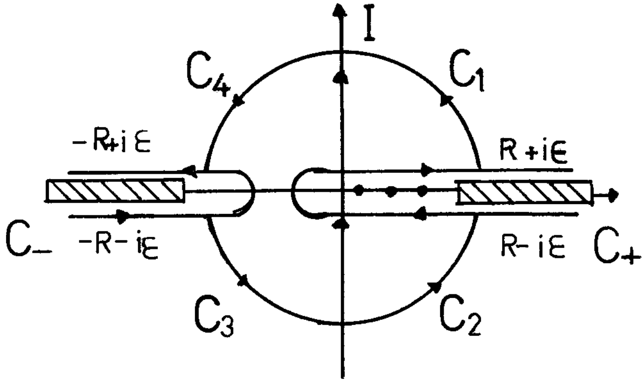

Fig. 1. $C^{\pm}$ contours and the deformed contours for the z-integration.

In the terms of the regular solutions of the Coulomb problem at the origin $W(r_< ; r)$ and at infinity $W(r_> ; z)$ the Green's function is given as follows:

$$K(r, r'; z) = \frac{1}{k(z)} \begin{bmatrix} W_1^{(2)} & (r_> ; z) \\ W_2^{(2)} & (r_> ; z) \end{bmatrix} [W_1^{(1)}(r_< ; z), W_2^{(1)}(r_< ; z)] \tag{2.6}$$

where

$$\begin{aligned} W_{\substack{1\\2}}^{(1)}(r) = {} & (2r\sqrt{z^2-1})^{\gamma} \frac{\exp[ir\sqrt{z^2-1}]}{r} \left[\left(\kappa - \frac{iZ\alpha}{\sqrt{z^2-1}} \right) \right. \\ & \times \begin{pmatrix} i\sqrt{z+1} \\ \sqrt{z-1} \end{pmatrix} \Phi(\gamma - i\nu, 2\gamma + 1; -2ir\sqrt{z^2-1}) \\ & \left. + \begin{pmatrix} i\sqrt{z+1} \\ -\sqrt{z-1} \end{pmatrix} (\gamma - i\nu)\, \Phi(\gamma + i\nu - 1, 2\gamma + 1; -2ir\sqrt{z^2-1}) \right] \end{aligned}$$

$$\begin{aligned} W_{\substack{1\\2}}^{(2)}(r) = {} & (2r\sqrt{z^2-1})^{\gamma} \frac{\exp[ir\sqrt{z^2-1}]}{r} \left[\left(\kappa - i\frac{Z\alpha}{\sqrt{z^2-1}} \right) \right. \\ & \times \begin{pmatrix} i\sqrt{z+1} \\ \sqrt{z-1} \end{pmatrix} \chi(\gamma - i\nu, 2\gamma + 1; -2ir\sqrt{z^2-1}) \\ & \left. + \begin{pmatrix} i\sqrt{z+1} \\ -\sqrt{z-1} \end{pmatrix} (\gamma - i\nu)\, \chi(\gamma - i\nu + 1, 2\gamma + 1; -2ir\sqrt{z^2-1}) \right] \end{aligned}$$

and

$$k(z) = -2\sqrt{z^2-1}\left(\kappa - i\frac{Z\alpha}{\sqrt{z^2-1}}\right)\frac{\Gamma(-\gamma-i\nu)\,\Gamma(2\gamma+1)}{\Gamma(-2\gamma)\,\Gamma(\gamma-i\nu)}\exp\left[\frac{i\pi}{2}(2\gamma+1)\right]$$

with

$$\gamma = [\kappa^2 - (Z\alpha)^2]^{1/2} \qquad \text{and} \qquad \nu = \frac{Z\alpha}{\sqrt{z^2-1}}\,z$$

The functions $\Phi(a, c; z)$ and $\chi(a, c; z)$ are the regular solutions of the confluent hypergeometric equation at the origin and infinity respectively. They are given by

$$\Phi = {}_1F_1(a, c; z)$$

and

$$\chi(a, c; z) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\,{}_1F_1(a, c; z) + \frac{\Gamma(c-1)}{\Gamma(a)}\,{}_1F_1(a-c+1, 2-c; z)$$

The discrete radial wave functions are

$$\begin{bmatrix} f_n(r) \\ g_n(r) \end{bmatrix} = \left\{\frac{\Gamma(2\gamma_n + n_r + 1)}{4N_n(N_n - K_n)\,n_r!}\right\}^{1/2} \frac{(2P_N)^{3/2}\,(2P_N r)^{\gamma_n - 1}\,e^{-P_N r}}{\Gamma(2\gamma_n + 1)} \times \begin{bmatrix} (1+E_n)\,F_n(r) \\ (1-E_n)\,G_n(r) \end{bmatrix} \tag{2.7}$$

where

$$F_n(r) = n_r\,\Phi(1-n_r, 2\gamma_n + 1; 2P_N r)$$

and

$$G_n(r) = (N_n - \kappa_n)\,\Phi(-n_r, 2\gamma_n + 1; 2P_N r)$$

with

$$\gamma_n = [\kappa_n^2 - (Z\alpha)^2]^{1/2}$$

and

$$\kappa_n = \pm(j_n + 1/2)$$

After substituting the required expressions and definitions into Eq. (2.2) we obtain

$$
\begin{aligned}
\Delta E_n^{\mathrm{VP}} = {} & \frac{\alpha}{2} \sum_{n_1, n_2} A_{n_1, n_2} \left[\int_{C+} + \int_{C-} \right] \frac{dz}{2\pi i} \frac{4i}{\sqrt{z^2 - 1}} \\
& \times \sum_{\kappa = 1}^{\infty} 2 |\kappa| \, T_{\alpha\alpha'} \int_0^1 dt \, t^{\alpha - 1} (1 - t)^{2\gamma - \alpha} \\
& \times \int_0^{\infty} dt' \, t'^{\alpha' - 1} (1 + t')^{2\gamma - \alpha'} \int_0^{\infty} r^2 \, dr \int_0^{\infty} r'^2 \, dr' \\
& \times \frac{(2P_N)^3 \, e^{-2P_N r} (-2 i r' \sqrt{z^2 - 1})^{2\gamma}}{r'^2} \\
& \times e^{2i(1 - t + t') r' \sqrt{z^2 - 1}} \, \frac{r_<^{\ell}}{r_>^{\ell + 1}} \, \frac{(2P_N r)^{2\gamma_n + n_1 + n_2 - 2}}{2\ell + 1}
\end{aligned} \tag{2.8}
$$

where we have set

$$
\begin{aligned}
T_{\alpha\alpha'} = {} & \frac{iZ\alpha}{\sqrt{z^2 - 1}} \left[\frac{\delta_{\alpha, \gamma - i\nu} \delta_{\alpha', \gamma - i\nu} + \delta_{\alpha, \gamma + 1 - i\nu} \delta_{\alpha', \gamma + 1 - i\nu}}{\Gamma(\alpha) \, \Gamma(2\gamma + 1 - \alpha')} \right] \\
& - z \left[\frac{\delta_{\alpha, \gamma - i\nu} \delta_{\alpha', \gamma + 1 - i\nu}}{\Gamma(\alpha) \, \Gamma(2\gamma + 1 - \alpha')} - \frac{\delta_{\alpha, \gamma + 1 - i\nu} \delta_{\alpha', \gamma - i\nu}}{\Gamma(\alpha') \, \Gamma(2\gamma + 1 - \alpha)} \right] \Bigg\}
\end{aligned} \tag{2.9}
$$

and

$$
A_{n_1, n_2} = \left(\frac{\Gamma(2\gamma_n + n_r + 1) \, \Gamma(n_1 - n_r) \, \Gamma(n_2 - n_r) \times [(N_n - \kappa_n)^2 + (n_1 - n_r)(n_2 - n_r) + 2E_n (N_n - \kappa_n)(n_1 - n_r)]}{2N_n (N_n - \kappa_n) \, \Gamma^2(-n_r) \, \Gamma(2\gamma_n + n_1 + 1) \, \Gamma(2\gamma_n + n_2 + 1) \, n_1! \, n_2! \, n_r!} \right)
$$

and γ_n, n_1, n_2 and n_r are quantum numbers.

The radials integrals

$$
\begin{aligned}
R = {} & \int_0^{\infty} dr \, e^{-2P_N r} r^{-\ell + 1} (2P_N r)^{2\gamma_n + n_1 + n_2 - 2} \int_0^r dr' \, r'^{2\gamma + \ell} e^{2ir'(1 - t + t') \sqrt{z^2 - 1}} \\
& + \int_0^{\infty} dr \, e^{-2P_N r} r^{\ell} (2P_N r)^{2\gamma_n + n_1 + n_2 - 2} \int_r^{\infty} dr' \, r'^{2\gamma - \ell - 1} e^{2ir'(1 - t + t') \sqrt{z^2 - 1}}
\end{aligned} \tag{2.10}
$$

can be exactly evaluated by converting them into the Mellin transforms of hypergeometric functions or using the integral representations of the hypergeometric functions in the complex plane and carefully analyzing the poles in the Mellin plane. Then we have

$$R = (2P_N)^{-2\gamma-3}\Bigg[\int_{-i\infty}^{i\infty}\frac{d\nu}{2\pi i}\,\frac{\Gamma(2\gamma+2\gamma_n+n_1+n_2+1+\nu)}{2\gamma+\ell+1+\nu}\left(\frac{a}{2P_N}\right)^{\nu}$$
$$+\left(\frac{2P_N}{a}\right)^{2\gamma+2\gamma+n_1+n_2+1}$$
$$\times\int_{-i\infty}^{i\infty}\frac{d\nu}{2\pi i}\,\frac{\Gamma(2\gamma_n+n_1+n_2+2\gamma+1+\nu)}{2\gamma_n+\ell+1+n_1+n_2+\nu}\left(\frac{a}{2P_N}\right)^{-\nu}\Bigg] \tag{2.11}$$

where $a = 2i\sqrt{z^2-1}\,(1-t+t')$.

We do the contour integrals in Eq. (2.11). Because of the asymptotic reasons we close the contour in the first integral from the right- and in the second integral from the left-hand side of the complex ν-plane. Then the contributions of the terms are equal. In the first term the poles of

$$\Gamma(2\gamma+2\gamma_n+n_1+n_2+1+\nu)$$

are at

$$\nu_r = -(2\gamma+2\gamma_n+n_1+n_2+1+r)$$

where

$$r = 0, 1, 2, \ldots$$

If we take $\gamma_n \cong 1$, $n_1 = n_2 = 0$ (for the s-waves), then we obtain $\nu_r = -2\gamma-3-r$.

These poles give the power series expansion of ΔE_n in terms of $(Z\alpha)$. If we evaluate the Eq. (2.11) at the first pole, we obtain

$$R = -\frac{\Gamma(2\gamma+3)}{2a^{2\gamma+3}} \tag{2.12}$$

Then we are left with

$$\Delta E_n = \alpha A_{00}\sum_{\kappa=1}^{\infty}|\kappa|\left[\int_{C+}+\int_{C-}\right]\frac{dz}{2\pi i}\,\frac{(2P_N)^3}{(z^2-1)^2}\,T_{\alpha\alpha'}\int_0^1 dt\, t^{\alpha-1}(1-t)^{2\kappa-\alpha}$$
$$\times\int_0^{\infty} dt'\, t'^{\alpha'-1}(1+t')^{2\kappa-\alpha'}\,\frac{\Gamma(2\kappa+3)}{(-2)(1-t+t')^{2\kappa+3}} \tag{2.13}$$

where we approximated $\gamma \approx |\kappa|$ for $Z\alpha \ll 1$. For s-waves ($n_1 = n_2 = 0$, $n_r = n - 1$, $\kappa_n = -1$, $l = 0$) we find $A_{00} = 1/2$ independent of n.

In the previous paper[16] the t- and t'-integrals were done first and κ-summation afterwards. The main point here is to do the reverse. The κ-summation can be reduced to hypergeometric functions. If these hypergeometric functions are expanded into power series we are left with the following expressions containing t- and t'-integrations:

$$(K, L, M)_{a,a'} = \int_0^1 dt \int_0^\infty dt' \frac{t^a(1-t)^{1-a} t'^{a'}(1+t')^{1-a'}}{(1-t+t')^5} \left(\frac{1-t+t'}{1-t+t'-2tt'} \right)^{5,7,9} \tag{2.14a}$$

$$(N, P, R)_{a,a'} = \int_0^1 dt \int_0^\infty dt' \frac{t^{1+a}(1-t)^{2-a} t'^{1+a'}(1+t')^{2-a'}}{(1-t+t')^7} \left(\frac{1-t+t'}{1-t+t'-2tt'} \right)^{7,9,11} \tag{2.14b}$$

where a and a' are the new indices which are defined in the following way. If in Eq. (2.13) α or α' is $\gamma - i\nu$, then a or a' is $(-i\nu)$ respectively. If α or α' is $\gamma + 1 - i\nu$ then a or a' is $(1 - i\nu)$.

In Eqs. (2.14a) and (2.14b) the t-integration can be performed by using the hypergeometric functions. The remaining t'-integration formally diverges at the lower limit $t' = 0$. In order to regularize the integrals we change the lower limit to ε and represent the result of t'-integration as Mellin–Barnes type complex integrals. Then, in these complex integrals we choose the integration contour such that they contain only the nonnegative powers of ε. The details of calculation are given in the Appendix. The results are

$$K_{a=a'=-i\nu} = \tfrac{13}{24} + \tfrac{2}{3} \ln \varepsilon$$

$$L_{a=a'=-i\nu} = \tfrac{5}{24} + \tfrac{1}{5} \ln \varepsilon$$

$$M_{a=a'=-i\nu} = \tfrac{37}{280} + \tfrac{4}{35} \ln \varepsilon$$

$$K_{a=a'=1-i\nu} = \tfrac{7}{24} - \tfrac{2}{3} \ln \varepsilon$$

$$L_{a=a'=1-i\nu} = -\tfrac{7}{120} - \tfrac{1}{5} \ln \varepsilon$$

$$M_{a=a'=1-i\nu} = -\tfrac{49}{1960} - \tfrac{4}{35} \ln \varepsilon$$

$$K_{a=a'-1=i\nu} = (-1)^{i\nu} [-\tfrac{5}{12} - \tfrac{17}{72} i\nu - \tfrac{1}{3} \ln \varepsilon]$$

$$L_{a=a'-1=i\nu} = (-1)^{i\nu} [\tfrac{491}{1260} - \tfrac{11}{72} i\nu + \tfrac{1}{3} \ln \varepsilon]$$

$$M_{a=a'-1=-i\nu} = (-1)^{i\nu}\left[-\tfrac{73}{420} - \tfrac{263}{2520}i\nu - \tfrac{1}{35}\ln\varepsilon\right]$$

$$N_{a=a'-1=-i\nu} = (-1)^{i\nu}\left[\tfrac{7}{90} + \tfrac{5}{216}i\nu + \tfrac{1}{15}\ln\varepsilon\right]$$

$$P_{a=a'-1=-i\nu} = (-1)^{i\nu}\left[\tfrac{4}{315} + \tfrac{37}{3780}i\nu + \tfrac{1}{105}\ln\varepsilon\right]$$

$$R_{a=a'-1=-i\nu} = (-1)^{i\nu}\left[\tfrac{1}{210} + \tfrac{2021}{7560}i\nu + \tfrac{113}{315}\ln\varepsilon\right]$$

$$K_{a=a'+1=1-i\nu} = (-1)^{i\nu}\left[\tfrac{1}{12} + \tfrac{1}{24}i\nu + \tfrac{1}{3}\ln\varepsilon\right]$$

$$L_{a=a'+1=1-i\nu} = (-1)^{i\nu}\left[\tfrac{1}{60} + \tfrac{7}{360}i\nu + \tfrac{1}{15}\ln\varepsilon\right]$$

$$M_{a=a'+1=1-i\nu} = (-1)^{i\nu}\left[\tfrac{17}{20} + \tfrac{29}{2520}i\nu - \tfrac{11}{15}\ln\varepsilon\right]$$

$$N_{a=a'+1=1-i\nu} = (-1)^{i\nu}\left[-\tfrac{1}{60} - \tfrac{1}{80}i\nu - \tfrac{1}{15}\ln\varepsilon\right]$$

$$P_{a=a'+1=1-i\nu} = (-1)^{i\nu}\left(\tfrac{17}{840} - \tfrac{1}{504}i\nu - \tfrac{1}{105}\ln\varepsilon\right)$$

$$R_{a=a'+1=1-i\nu} = (-1)^{i\nu}\left(\tfrac{173}{1890} - \tfrac{1}{1050}i\nu + \tfrac{89}{630}\ln\varepsilon\right) \tag{2.15}$$

where we take $\nu = 0$ in the terms $(K, L, M)_{a=a'}$ and expand all the gamma functions in the other terms in power series of ν and take the first-order terms in ν only ($\nu = Z\alpha z/(z^2 - 1)^{1/2}$).

Then we substitute these results into Eq. (2.13). For the s-waves we obtain

$$\Delta E_n^{\mathrm{VP}} = -2\alpha\left(\frac{Z\alpha}{N_n}\right)^3 \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i}\frac{z}{(z^2-1)^2}\left[-\frac{(Z\alpha)^2}{(z^2-1)}z\left(\frac{11}{3} + 4\ln\varepsilon\right) + \frac{2(Z\alpha)}{z^2-1}z^3\left(22 + \frac{4}{3}\ln\varepsilon\right) + i\frac{Z\alpha}{\sqrt{z^2-1}}z^2\left(\frac{2}{5}\right)\right]\left(1 - \pi\frac{Z\alpha}{\sqrt{z^2-1}}z\right) \tag{2.16}$$

For the lowest order of $(Z\alpha)$, i.e., $(Z\alpha)^4$, we find that the following term will contribute:

$$\Delta E_n^{\mathrm{VP}} = -2\alpha\left(\frac{Z\alpha}{N_n}\right)^3 \frac{2}{5}\int_{-i\infty}^{i\infty} \frac{dz}{2\pi i}\frac{z^2}{(z^2-1)^2}\, i\,\frac{Z\alpha z^2}{(z^2-1)^{1/2}} + O((Z\alpha)^5) \tag{2.17}$$

Finally using

$$\int_{-i\infty}^{i\infty} \frac{dz}{2\pi i}\frac{z^2}{(z^2-1)^{5/2}} = -i/3\pi \tag{2.18}$$

we are left with a finite result for the vacuum polarization energy shift

$$\Delta E_n^{\mathrm{VP}} = -\frac{4\alpha}{3\pi}\left(\frac{1}{5}\right) Z\alpha\left(\frac{Z\alpha}{N_n}\right)^3 + O((Za)^5)$$

We conclude that the divergences in QED are due to the use of plane waves in the intermediate states in the loop diagrams. With the use of Coulomb waves as intermediate states in the loop diagrams, a prescription has been found for the choice of contour integrations such that divergent integrals cancel, leaving a finite residual piece which is correct to lowest nonvanishing order. Further work remains to be done to establish whether or not the technique can be extended to higher-order terms.

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APPENDIX: $t - t'$ INTEGRALS

We have the following t and t' integrals:

$$(K, L, M)_{a,a'} = \int_0^1 dt \int_0^\infty dt' \frac{t^a(1-t)^{1-a} t'^{a'}(1+t')^{1-a'}}{(1-t+t')^5}\left(\frac{(1-t+t')}{\pi(t,t')}\right)^{5,7,9} \quad \text{(A.1)}$$

and

$$(N, P, R)_{a,a'} = \int_0^1 dt \int_0^\infty dt' \frac{t^{1+a}(1-t)^{2-a} t'^{1+a'}(1+t')^{2-a'}}{(1-t+t')^7}\left(\frac{(1-t+t')}{\pi(t,t')}\right)^{7,9,11} \quad \text{(A.2)}$$

where

$$\pi(t, t') = 1 - t + t' - 2tt' \quad \text{(A.3)}$$

In order to regularize these integrals we take the lower limit of t'-integration as ε. Then we perform t-integration. We can represent them in the following form:

$$(K, L, M)_{a,a'} = (-1)^{-a}\,\Gamma(2-a) \times \Bigg\{ -A_{aa'} \frac{\Gamma(1+a)}{\Gamma(3)} {}_2F_1(-2/-4/-6, 1+a; 3; 2) + 2B_{aa'} \frac{\Gamma(2+a)}{\Gamma(4)} {}_2F_1(-1/-3/-5, 2+a; 4; 2) - C_{aa'} \times \frac{\Gamma(3+a)}{\Gamma(5)} {}_2F_1(0/-2/-4, 3+a; 5; 2) \Bigg\} \tag{A.4}$$

and

$$(N, P, R)_{a,a'} = (-1)^{-a}\,\Gamma(3-a) \Bigg\{ A_{aa'} \frac{\Gamma(2+a)}{\Gamma(5)} {}_2F_1(-2/-4/-6, 2+a; 5; 2) - 2B_{aa'} \times \frac{\Gamma(3+a)}{\Gamma(6)} {}_2F_1(-1/-3/-5, 3+a; 6; 2) + C_{aa'} \frac{\Gamma(1+a)}{\Gamma(7)} {}_2F_1(0/-2, -4, 4+a; 7; 2) \Bigg\} \tag{A.5}$$

where $(A, B, C)_{aa'}$ are

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \int_{\varepsilon}^{\infty} dt' \frac{t'^{a'-a}}{(1+t')^{a'-a+3}} \begin{pmatrix} t'^{-1} \\ t'^{-2} \\ t'^{-3} \end{pmatrix} \tag{A.6}$$

We do the t'-integration in Eq. (A.5) by using the hypergeometric functions. The result is

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix}_{aa'} = \begin{pmatrix} \frac{1}{3\varepsilon^3} {}_2F_1\left(a'-a+3, 3; 4; -\frac{1}{\varepsilon}\right) \\ \frac{1}{4\varepsilon^4} {}_2F_1\left(a'-a+3, 4; 5; -\frac{1}{\varepsilon}\right) \\ \frac{1}{5\varepsilon^5} {}_2F_1\left(a'-a+3, 5; 6; -\frac{1}{\varepsilon}\right) \end{pmatrix} \tag{A.7}$$

The Mellin–Barnes type integral representation of (A.6) is

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix}_{aa'} = \int_{c-i\infty}^{c+\infty} \frac{dv}{2\pi i}\,\Gamma(-v)\,\frac{\Gamma(a'-a+3+v)}{\Gamma(a'-a+3)} \begin{pmatrix} \dfrac{1}{(3+v)\,\varepsilon^{3+v}} \\ \dfrac{1}{(4+v)\,\varepsilon^{4+v}} \\ \dfrac{1}{(5+v)\,\varepsilon^{5+v}} \end{pmatrix} \tag{A.8}$$

where c defines the contour of the v integration. The result depends on $a-a'$. In our calculation there are three values of $a-a'$: 0 and ± 1. Then we rewrite, for example, $A_{aa'}$ in Eq. (A.8) as

$$\begin{aligned} A_{aa'} = {} & \delta_{aa'} \int_{c-i\infty}^{c+i\infty} \frac{dv}{2\pi i}\,\Gamma(-v)\,\frac{\Gamma(3+v)}{\Gamma(3)(3+v)\,\varepsilon^{3+v}} + \delta_{a,a'-1} \\ & \times \int_{c-i\infty}^{c+i\infty} \frac{dv}{2\pi i}\,\Gamma(-v)\,\frac{\Gamma(2+v)}{\Gamma(2)(3+v)\,\varepsilon^{3+v}} \\ & + \delta_{a,a'+1} \int_{c-i\infty}^{c+i\infty} \frac{dv}{2\pi i}\,\frac{\Gamma(-v)\,\Gamma(4+v)}{\Gamma(4)(4+v)\,\varepsilon^{3+v}} \end{aligned} \tag{A.9}$$

Figure 2 shows the poles of $A_{aa'}$, where the pole at $v=-3$ is a double pole. The others are single poles. The poles at $v=0, 1, 2, \ldots$ give the asymptotic expansion of $A_{aa'}$ in terms of ε. Contributions of the single poles

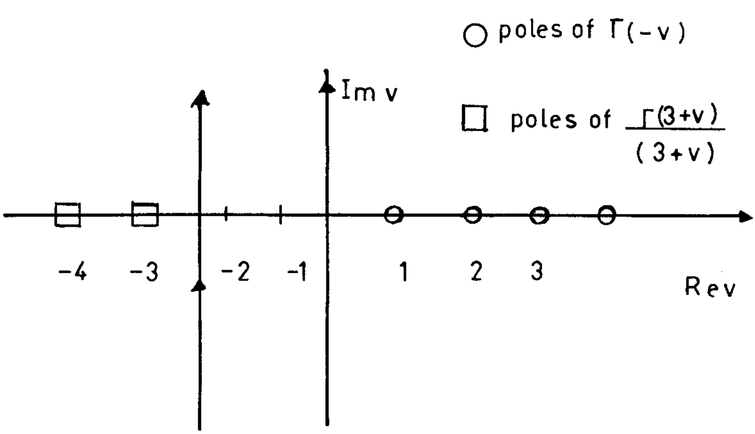

Fig. 2. The poles of $A_{a,a'}$.

at $v = -4, -5, \ldots$ are proportional to the positive powers of ε and when ε goes to zero, their contributions also go to zero. But $v = -3$ is a double pole and gives a constant term and a logaritmic term in ε. We choose the contour as shown in the Fig. 2. Then

$$A_{a,a} = \frac{1}{\Gamma(3)} \operatorname{Re} s \left\{ \frac{\Gamma(-v)\,\Gamma(4+v)}{(3+v)^2\,\varepsilon^{3+v}},\ v = -3 \right\} = -\frac{3}{2} - \ln \varepsilon \qquad \text{(A.10)}$$

Figure 3 shows the poles of $A_{a,a-1}$. There is a double pole at $v = -3$. All the others are single poles. All the poles at the right-hand side of $v = -3$ are contributing to the asymptotic expansion of $A_{a,a-1}$. For this reason, we choose the contour such that the series expansion of $A_{a,a-1}$ does not include the negative powers of ε. Then, for the small values of ε we get a contribution only from the double pole at $v = -3$, when we close the contour from the left-hand side of the υ-plane. It is

$$A_{a,a-1} = \frac{1}{\Gamma(2)} \operatorname{Re} s \left\{ \frac{\Gamma(-v)\,\Gamma(4+v)}{(2+v)(3+v)^2\,\varepsilon^{3+v}},\ v = -3 \right\} = 1 + 2 \ln \varepsilon \qquad \text{(A.11)}$$

In the same way we calculate $A_{\alpha,\alpha+1}$. It is

$$A_{a,a+1} = \frac{1}{\Gamma(4)} \operatorname{Re} s \left\{ \frac{\Gamma(-v)\,\Gamma(4+v)}{(3+v)\,\varepsilon^{3+v}},\ v = -3 \right\} = \frac{1}{3} \qquad \text{(A.12)}$$

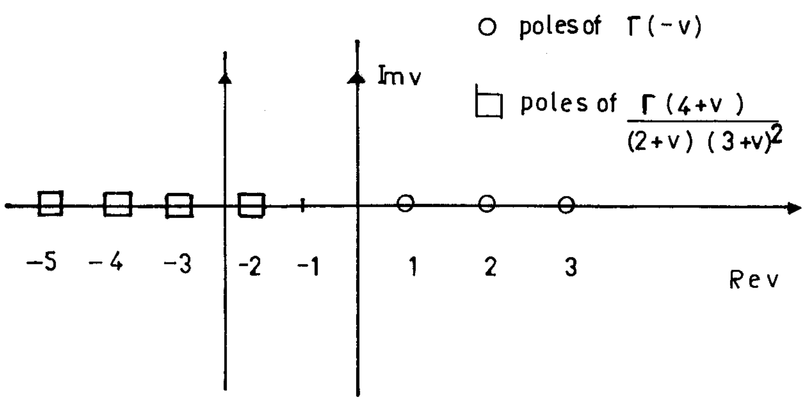

Fig. 3. The poles of $A_{a,a-1}$.

The results of the other integrals are obtained by the same procedure. They are

$$\begin{aligned} B_{a,a} &= \tfrac{5}{2} + 3\ln\varepsilon, & C_{a,a} &= -\tfrac{7}{2} - 6\ln\varepsilon \\ B_{a,a+1} &= \tfrac{11}{6} - \ln\varepsilon, & C_{a,a+1} &= -\tfrac{13}{3} + 4\ln\varepsilon \\ B_{a,a-1} &= -1 - 3\ln\varepsilon, & C_{a,a-1} &= 1 + 4\ln\varepsilon \end{aligned} \tag{A.13}$$

We substitute the results of Eqs. (A.10)–(A.13) into (A.4) and (A.5). Then we find the values of $(K, L, M)_{aa'}$ and $(N, P, R)_{aa'}$ as in Eq. (2.15).

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