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Momentum flow as an alternative perspective in elementary mechanics

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Through a series of sample problems and solutions, the case is made that the notion of force as momentum flow can serve as the basis for vastly expanding the importance of Newton's third law, "action and reaction," as experienced by students in elementary mechanics courses. The reasons this potential has been neglected seem more "cultural" than pedagogical. The final section raises some particular issues concerning the advisability of actually teaching elementary mechanics from this alternative point of view.

I. INTRODUCTION

There is a rather striking asymmetry in Newton's Laws. The first two laws have an incredibly rich and varied network of techniques, equations, important special cases, heuristic advice, etc., relating to them. The spine of the network consists in (i) selecting the physical system(s) of interest, (ii) identifying all forces acting on it (them), (iii) using $F = ma$ to solve. Branching away, one could organize a great deal of what is taught in elementary mechanics by how it contributes to this general form of analysis. Free-body diagrams contribute prominently on the left-hand side of the equation. The art of selecting a system is a subproblem of free-body analysis. Kinematics provides standardized special cases like uniform acceleration and uniform circular motion as grist for the $F = ma$ machine on the right-hand side. One should not forget such trivialities as vector decomposition and advice about how to carry out such decomposition effectively. (Can any freshman physicist look at an object moving on a flat surface without decomposing an applied force into normal and tangential components?) The list can easily be extended by thinking about special devices used in the varied domains, statics, rope and pulley problems, trajectories, etc., in which one conventionally learns to implant $F = ma$.

On the other hand, contrast the relative sparsity surrounding Newton's Third Law, action and equal and opposite reaction. Beyond the image of a man jumping off a boat, how much more is there? Even in the form of conservation of momentum (a form which surprisingly few students recognize as identical to action–reaction) most students will think of the principle as a constraint equation which will help in collision situations, but little more.

This broad asymmetry is not so much in the laws themselves but in the way that they are conventionally taught and thought about. In particular this paper aims at showing how the action–reaction notion *can* be enriched to come much closer to parity with $F = ma$ as an insightful way of looking at many phenomena of mechanics.

Actually, I will use a reformulation of action–reaction which emphasizes the fundamental conservation law embedded in it: Force is simply the flow of the conserved "stuff," momentum, from one place to another. Technically speaking, force is the rate with which momentum flows.

A useful context for this paper can be set by suggesting in a little more detail what does and does not account for the fact that momentum flow is almost entirely outside the bounds of "normal physics teaching" represented in current texts. As a frame for analysis for elementary mechanics, momentum flow is clearly Newtonian at its formal core. It is in that sense by no means "new physics." The issue is more one of style than content. Moreover, the uniform dominance of one style over another is not usually so much a question of efficiency, but more one of culture. Thus we are led to frame the issue in much the same way in which Thomas Kuhn explained certain discontinuities in the history of science[1]: what is learned by succeeding generations of scientists is to a great extent cultural, carried as much in the selection of problems posed and patterns implicit in paradigm solutions as it is carried in the explicit theory. In that light this paper begins to ask whether or not the culture of physics as experienced by novices is pedagogically justified.

II. EXAMPLES

A. Question of image

What is central here is the overall image of physical interaction: what is happening, where and why. So let us begin with a simple situation and contrast $F = ma$ to momentum flow. Imagine my holding an apple in my hand. $F = ma$ suggests observing that since acceleration is zero, the force on the apple must be zero; that must be *net* force since gravity is acting on the apple. I conclude that my hand is providing a force equal and opposite to that of gravity.

A momentum flow analysis might observe that since gravity is a force it must be "pouring" momentum into that apple. But clearly the apple is not collecting the momentum; you can always see momentum collect in an acquired velocity. So the apple must be passing all that momentum on to my hand. One could stop there, but a natural question arises: where is the momentum going from there? As my hand is stationary, it isn't collecting momentum either. The momentum must continue through my arm and body into the floor. There is no resolution until the momentum spreads out in the earth to all the places from which, after all, it came in the first place. All along the way there are stresses (forces) and strains which are the physical manifestations of momentum flow through solid matter. Note in particular that $F = dp/dt$ is a special case of momentum flow. It cannot generally suffice because, as in this instance, there can be momentum flow without any $p = mv$ visible at all.

The contrast of images is striking. Two opposing forces

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on an isolated "particle," or the apple as a link in a global loop of flow. On the one hand ($F = ma$) one has topological simplicity. On the other hand one has an enriched sense of the mechanism, a sense that each part of the contact of apple with hand is contributing a part of the total flow, a sense for how flows at distant parts of the system are correlated. Break the loop, say remove the hand from the apple, and momentum collects in the apple; it falls. Just as much, in those circumstances what is lost by the earth is not replenished.

We see two themes emerging: Momentum flow provides a better appreciation for the distributed mechanism of forces involving real (nonpoint) bodies. As well, momentum flow analysis provides a different perspective on the relation of localized phenomena (forces, stresses) to the overall, global picture of interactions.

Now, of course, each of these two analyses could parallel the other. For example, $F = ma$ could mirror the "propagation" exhibited in the momentum flow analysis if one choses a sequence of abutting systems down my arm and into the earth. But the emphasis is still very different. What one sees propagating is not the *incremental* stress due to the fact that I am holding an apple, but a total force which is being increased by the weight of successive subsystems. Further analysis is necessary to "see" the incremental stress.

The potential importance of this different point of view can be illustrated with a common mistake. What is the shearing force S in a diving board at D loaded with a man of weight W at L (see Fig. 1)? The canonical wrong answer comes from students who see a lever and conclude S equals the obvious proportion of W. Of course this is a misapplication, but a misapplication which stems from the lack of a sense for the actual mechanism at work. A student who makes a momentum flow analysis sees the whole momentum provided by load W can neither "evaporate" nor augment in its flow along the board back to the earth. S must be independent of D.

B. Elephants

One of the well-known conflicts of naive physical intuition with physics comes from comparing the ability of a pair of elephants to break a rope with that of one elephant and a wall. A momentum flow analysis views the "breaking power" of an elephant as how much momentum it can pump per second, F. Given that, it makes no difference whether the sink for the momentum pumped into the rope is another elephant or a wall. The way to use two pumps effectively is to put them in parallel—not series; hitch two elephants to the same end of the rope (with any suitable sink at the other end).

It's worth doing this analysis a bit more carefully. The muscles of an elephant's legs are normally used to move forward-pointing momentum from the ground to his body,

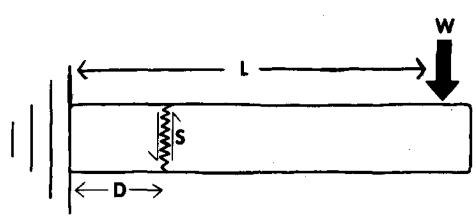

Fig. 1. Diving board loaded with W at L: What is the shear S at D?

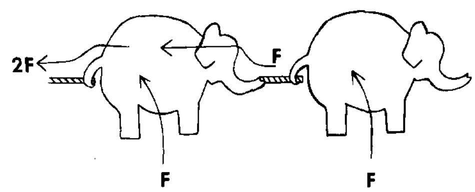

Fig. 2. Two elephants working "in parallel."

i.e., to accelerate him forward. If a rope is attached which keeps the elephant from moving, the momentum pumped by the elephant's muscles (no more, nor less) has to be leaving via the rope.

If we look at a "parallel pump" configuration (see Fig. 2) we see the second elephant's body has two equal sources of forward momentum, one from its legs and one from the rope connecting it to its presumably equally effective forward neighbor. The rope holding both elephants must be a sink for $2F$.

Even after overcoming the initial obvious choice, that two elephants must be better than one plus a wall, students will often want to label the tension in the rope between a pair of elephants as $2F$. Unless tension is explicitly defined (it usually isn't) that is a fair enough definition, but it loses all naturalness in a momentum flow view; you obviously measure a flow by how much crosses any point, not by the sum of how much goes in plus how much goes out.

Part of the trick which makes momentum flow view perspicuous here is that it asymmetrizes the forces on opposite ends of the rope. With different roles, one as source, one as sink, adding their values automatically becomes a more dubious proposition.

Now the rope-pulled-on-two-ends situation really *is* in some sense symmetrical. To see the symmetry one must merely view a source of F as a sink of $-F$ (and vice versa); moving Δp from A to B is exactly equivalent to moving $-\Delta p$ from B to A (see Fig. 3). That one has this choice of point of view for momentum flow is probably the most counter-intuitive aspect of it. But even this has important parallels. From an external point of view, electrical flow of positive particles in one direction is indistinguishable from an opposite flow of negative particles. Unfortunately for Ben Franklin, + and − charges are carried by distinct particles with very different properties, and we can say with conviction that he was wrong in his choice of flow direction. No such thing is true of momentum, so there is no way to choose one direction of flow over the opposite.

C. Stress in a race horse

Suppose a horse is running so fast his hooves can only manage to touch the ground 10% of the time. What can one say about the stress in his legs?

Momentum flow sees gravity inexorably pumping milligrams worth of downward momentum into the horse per second. The horse must, if he's not to collect that momentum (i.e., fall), get rid of it. Clearly during the 10% of the time that he has contact with the ground he must dump momentum on average at 10 times the milligram rate. And if the horse isn't very cleverly designed so that his hooves push at nearly a constant force while they are on the ground, but instead build up gradually to a peak flow and subside, then the peak rate of flow could easily be 2 to 3 times more

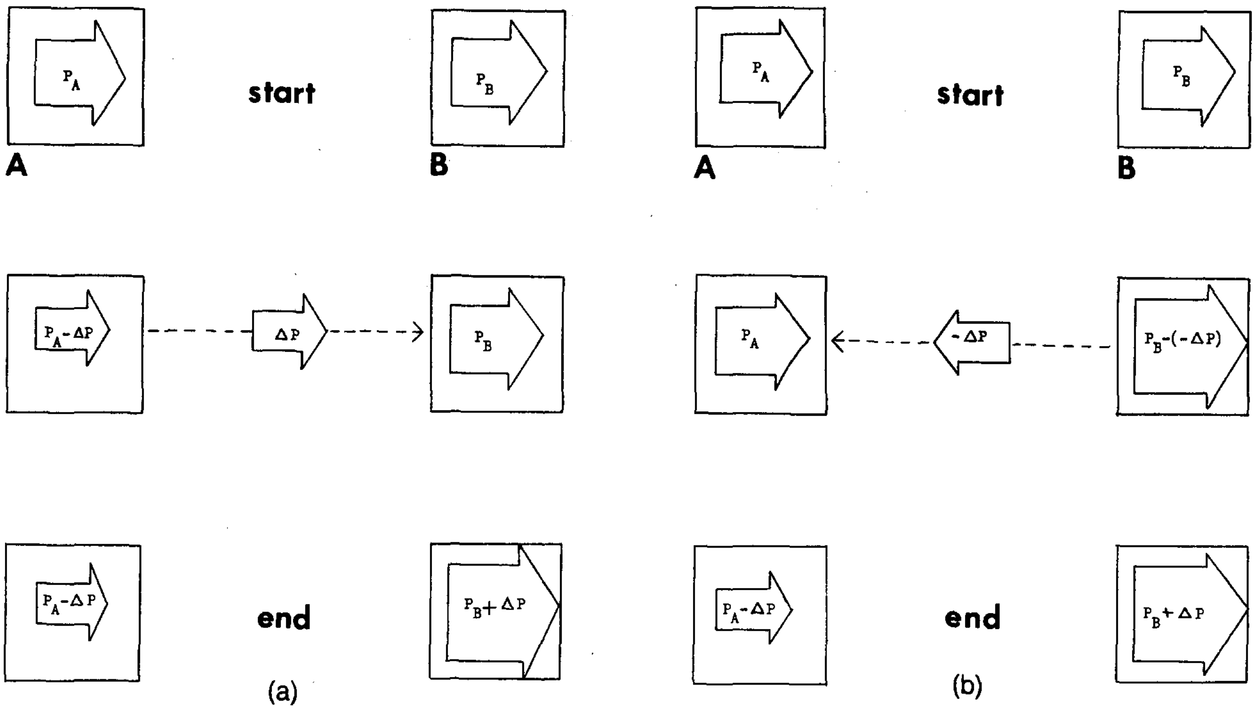

Fig. 3. (a) Transferring ΔP from A to B. (b) Transferring $-\Delta P$ from B to A.

than the already 10 times normal stress. It seems quite clear how a running horse can break a leg from a small stumble.

Similar reasoning implies that a bouncing ball must experience a force during bounce that is at least the ratio of (the time involved flying) by (the time involved in a bounce) greater than gravity.

D. Stress in a pressure tank

Consider a thin shelled (thickness T) spherical metal tank (of radius r) containing a gas (at pressure P). What is the stress in the tank?

Here one is confronted with a system in which momentum is circulating internally and the question is, how much? If we slice the tank in half then we see $PA = P\pi r^2$ worth of momentum pumped via gas pressure from one half of the system to the other. That must return via the tank over the surface of metal cut, $2\pi rT$. Hence the stress S (rate of flow per unit area) satisfies $2\pi rTS = P\pi r^2$, $S = Pr/2T$.

Incidentally, the image of particles of a gas flying symmetrically in all directions is a quite useful concrete image for momentum flow in a uniformly pressurized medium.

The pressure tank contains within it two very different modes of momentum flow: that via internal and contact forces and that via mass transfer (in the motion of molecules between collisions). The latter is of a very special type since the momentum flowing must clearly be in the direction of the flow! Thus it is a flow type which one can characterize as a push and never as a pull (momentum pointing opposite direction of flow) or a shear (momentum perpendicular to flow). This simple and obvious fact assures that pressure in a gas can never be negative (though it can be in a solid) and that, in fact, any pair of systems which communicate only by mass transfer can only push on one another.

E. Bernoulli's law

Bernoulli's law is "obviously" a question of energy because of the $(1/2)\rho v^2$ term in it. On the other hand ρv^2 is precisely the rate of momentum flow (per unit area) of a mass transfer. The following derivation of Bernoulli's law points out that, indeed, this ρv^2 term can be attributed to momentum flow.

For simplicity consider some continuous incompressible medium flowing within a pipe or within a surface of flow trajectories. We will be looking only at momentum in the direction of flow. Now, conservation of the medium implies Av is constant, where A is the cross-sectional area and v is the velocity of flow. The momentum flow through the medium is of two types, pressure (PA) and mass transfer ($\rho v^2 A$).

If one considers the change in momentum flow from one surface cutting the flow to an adjacent one, $d(PA + \rho v^2 A)$, this must be equal to the amount of momentum leaked through the walls. This, in turn, is just pressure times area perpendicular to flow, $P\,dA$ (see Fig. 4). The rest is just arithmetic:

$$d(PA + \rho v^2 A) = P\,dA,$$

$$d[PA] + d[(\rho Av)v] = P\,dA.$$

Now using the fact that (ρAv) is a constant:

$$P\,dA + A\,dP + (\rho Av)dv = P\,dA,$$

$$A(dP + \rho v\,dv) = 0,$$

$$dP + \rho v\,dv = 0,$$

$$P + (1/2)\rho v^2 = \text{const.}$$

F. Stress in a spinning rim

Suppose one asks what the tension T is in a spinning

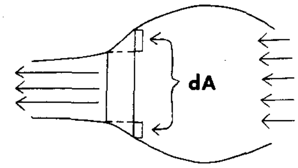

Fig. 4. Flow through two adjacent surfaces.

circular loop. If one looks through a fixed window at a section of the loop one always sees the same momentum. Thus one can consider the momentum at rest and the loop moving through it! In one revolution an imaginary cut in the loop sweeps through the whole rim's momentum, hence the rate of flow is mv/t, where t = period = $2\pi r/v$. So $T = mv^2/(2\pi r) = \rho v^2$, where ρ is mass per unit length.

This is a particularly unconventional analysis. Let us examine why that is so. First, the momentum view transforms a moving wheel into an entirely static picture of spatially stationary momentum. This violates one of the chief heuristics built up in approaching momentum through $F = ma = dp/dt$, that one should view momentum as attached to objects, moving with them. It is easy to imagine the heuristic being taught in paradigm solutions using $F = ma$ where m refers to a particle, hence the boundaries of matter determine system decomposition. In this case, that heuristic suggests moving with a chunk of wheel and requires one to take account of a *changing* momentum.

The second unconventionality is that this analysis *seems* difficult to rigorously reduce to basic principles. It is entirely possible to see this again as a problem of familiarity, culture, rather than intrinsic difficulty of rigorous justification. Making that case in detail is more than can be offered here, but a bit more can be shown of how the story might go with an analog in justifying $F = ma$ arguments. Consider the intuitive and natural use of the fact that position and velocity determine future trajectory in the $F = ma$ perspective. It is not that freshmen can prove the equivalent rigorous fact, the existence and uniqueness of solutions to second-order differential equations. Instead, they use the idea with confidence because it's part of the culture they experience. The "facts" which justify a momentum flow analysis such as this are not in the culture in the same way.

III. TWO COMPUTER PROJECTS

The two computer projects below use momentum flow in dealing with more than usually complex situations.

A. Simulating a sailboat

Consider the following first-order theory of a sailboat's motion. Wind moving toward a sail is a source of momentum. A sink of a similar kind is the "exhaust" wind flowing back off the sail. The sailboat must be left with $\mathbf{w} - \mathbf{e}$, where $\mathbf{w}$ and $\mathbf{e}$ represent the amount of momentum flow provided by wind and lost to the exhaust. For simplicity we can assume $\mathbf{e}$ has the same magnitude as $\mathbf{w}$ and the direction of $\mathbf{e}$ is parallel to the sail.

There is one other sink for momentum. The keel will see to it that most momentum acquired which is perpendicular to the line of the boat will be "dumped" into the ocean: $\mathbf{w} - \mathbf{e}$ must be projected onto the line of the boat to find net momentum gain of the boat. If we make the simplifying assumption that in this viscous situation the velocity of the boat is proportional to the net rate of acquisition of momentum, that translates what we have said so far into a moving boat. Finally we must realize that $\mathbf{w}$ is really dependent on the *relative* velocity of wind to boat, so we must take $\mathbf{w} = \mathbf{W} - \mathbf{v}$, where $\mathbf{W}$ is the given wind velocity and $\mathbf{v}$ the velocity of the boat.[2]

Making such a simulator is an excellent exercise in vectors. Playing with it exercises qualitative understanding of a relatively complex physical system and also the character of simplified modeling. Modifying the simulation (for example adding hysteresis by including the fact that the boat must accumulate significant momentum in acquiring speed), one can better and better approach the real situation.

B. Stress machine

Suppose one wishes to build a simulation of stress and strain in a simplified, but nearly real life extended body. Consider a truss and the following physically meaningful "relaxation" method. Rods are "momentum pumps" which move momentum between vertices at a rate proportional to fractional compression or extension (Hooke's Law). Vertices are places where momentum can collect, and at which we can inject external momentum from, say, a load. Special vertices, supports, dump as much momentum as enters them into the earth. To this we add a "viscous law of motion" for the vertices, that they move like the sailboat above, at a speed proportional to the rate of accumulation of momentum. Viscosity saves us from waves and vibrations persisting in the body.

So we start an unstressed truss working. Momentum from the load flows into vertices and therefore cause them to move. In response, rods are stretched or compressed causing more momentum to begin to circulate. Thus vertices receive more or less momentum and move accordingly, and so on. In the end an equilibrium flow is established (or the bridge collapses) which, in net, carries momentum from the load to the supports.[3]

Even the concept of such a program does a great deal to bridge the usual schism between simple physical principles which are often only exemplified in simple problems and the real pervasive power of those principles to illuminate the hidden inner workings of the complexities of everyday life. A student could select structures and loads and watch (with some appropriately animated and color-coded display) the distribution of momentum flow change. Design a bridge: How does shape reflect function? What are the criteria for maximum "efficiency?" Suppose one used different materials in the same structure, some with high tensile strength, some with high compressive strength: What techniques make for good synergy? How does prestressing function? If a structure is to be made solely of one material, how can its shape characteristics vary as the character of the material's strength is assumed to change?

IV. INTUITIVE ASPECTS OF MOMENTUM FLOW

This section turns to the question of "learnability." Simply put, is momentum flow simple enough to teach to elementary physics students? The following remarks point to some factors relevant in answering this question.

Some students' gross impression of what is happening in certain circumstances is really much closer to momentum flow than to $F = ma$. Consider a uniformly accelerating train on a level track. One spontaneous explanation of what is going on involves each car "absorbing" some of the force exerted on it, leaving a lesser force to act on the car to the rear. Phrased literally in those terms, of course, that is nonsense. But if one spoke of momentum rather than force, one can appreciate how the insight really relates to physics. Momentum flow can serve to engage and refine such intuitions.

There are other aspects of a momentum flow approach which link well with abilities novice students have or can develop outside of the category of "prerequisite technical knowledge." Visual reasoning is one example. The ability merely to draw pictures or conjure up images is useful in posing questions and expressing answers. This is particularly important in gaining contact with the continuous topology one encounters in situations involving extended bodies. Again, oppose this to the discrete topology dealt with by $F = ma$ in free-body diagrams.

Closely related to the visual reasoning above is the "input-outgo" mode of reasoning supported by descriptions of the "X receives I and loses O" sort. This is natural and expeditious language to be used at the stage in problem solving immediately preceeding writing down equations. Particularly if augmented by "packets and transfer" images (see Fig. 3), qualitative reasoning at this level is quite simple.

Momentum flow reasoning also naturally engages "flow heuristics" which afford important insight on the issue of selecting a physical system for analysis and of how that system relates to the rest of the world. By flow heuristic I mean pieces of advice to aid in making an analysis of flow. Consider: "Look for sources, sinks, leaks and accumulations." "You may measure anywhere along a steady conserved flow to find the amount of flow." Even the image of declaring invisible boundaries between parts of a system, just as necessary for application of $F = ma$, has a natural logic as the fundamental flow measurement device. (Compare student complaints, "How can you just declare that collection of atoms a system!" to the naively richer notion of deciding where to put a meter to measure a flow.) Furthermore, consider the following advice for effectively making such cuts: "Try to make a cut perpendicular or parallel to flow." "Respect the symmetry of the situation in making the cut." "To make integration trivial, try to make a cut across which flow is uniform rather than varying in intensity or direction." These are, in words, as simple as the equivalent force pronouncements, but are, in images, more like "the obvious thing to do." In view of these concerns, compare the two equivalent questions regarding the spherical pressure tank: "What is the system for which I need write $F = ma$?" "How do I measure the flow circulating in this thing?"

Another advantage of a momentum flow approach is that it engages (not to say solves) in a direct and profitable way some of the standard intuition faults one finds in nearly all students. The two elephants breaking a rope problem and the lack of understanding, indeed utter neglect, of mechanisms for distributing forces inside bodies are two examples. As well, many of the issues involved with flow generally arise with a vengeance when students get to electromagnetic theory. Consult the flow heuristics above and think of their application in using Gauss's Law. Indeed, momentum flow might be an ideal interpolation between a more usual mechanics course and electromagnetic theory.

In fairness there are a few obvious difficulties with teaching momentum flow. First, a subtle point, the causality involved with momentum flow is more refined than needed in considering a force causing an acceleration. Answering the question, "What causes momentum to flow?" is not trivial for someone expecting some kind of pump mechanism, as a novice may well expect. On the other hand a refined notion of causality is generally necessary in understanding Newtonian mechanics,[5] and momentum flow simply forces the issue sooner than $F = ma$.

A second difficulty arises from the ambiguity of direction of flow and choice of what kind of momentum is flowing. This is a peculiar ontology for "stuff which flows." For most, this kind of arbitrariness in representation will not have been encountered before, and one can argue plausibly that discomfort with it could interfere with the carry over of the visual reasoning and flow heuristics which make momentum flow intuitively accessible in the first place. A similar ontological difficulty with the flow metaphor is that momentum may flow through a rod without being present in it. Again it is not wise to make too much of this kind of difficulty as notions like force and acceleration must undergo similar transformations from naive connotations.

Finally, and this is the most serious criticism, a formalism which treats momentum flow completely and analytically must involve something like tensor fields. Though any similar point of view ($F = ma$ included) allows students to phrase more complex questions than they are capable of dealing with analytically or even conceptually, momentum flow is particularly conducive to expressing complex questions. Problem solving power and simplicity aside, students may balk at a point of view which allows (perhaps even prompts) them to deal with the world at such a level of detail.

[1]T. S. Kuhn, *The Structure of Scientific Revolutions* (University of Chicago, Chicago, IL, 1970).

[2]To be technically more correct if **w** is a velocity, momentum acquired should be in the direction of **w** but with magnitude proportional to w^2.

[3]There are parallel processing languages (Smalltalk, Actors) which in principle could enact this model very simply. Since such languages are not widespread, most will have to resort to having "professionals" or at least paid students implement the program. We have had high school students implement such programs together with graphic output and a front end which allows truss shape to be defined at will (see Ref. 4).

[4]H. Abelson and A. diSessa, *Student Science Training Program in Math, Physics and Computer Science: Final Report to the NSF,* MIT AI Memo 393 (MIT Artificial Intelligence Laboratory, Cambridge, MA, 1976).

[5]A. diSessa, *On Learnable Representations of Knowledge: A Meaning for the Computational Metaphor,* in *Cognitive Process Instruction,* edited by J. Lochhead and J. Clement (Franklin Institute, Philadelphia, PA, 1979).