Aesthetics for the Working Mathematician

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Beauty and the Mathematical Beast: Mathematics and Aesthetics
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William Blake

"And I made a rural pen,
And I stained the water clear,
And I wrote my happy songs,
Every child may joy to hear."

From *Songs of Innocence and Experience*

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1 Introduction

"If my teachers had begun by telling me that mathematics was pure play with presuppositions, and wholly in the air, I might have become a good mathematician. But they were overworked drudges, and I was largely inattentive, and inclined lazily to attribute to incapacity in myself or to a literary temperament that dullness which perhaps was due simply to lack of initiation." (George Santayana[1])

Most research mathematicians neither think deeply about nor are terribly concerned about either pedagogy or the philosophy of mathematics. Nonetheless, as I hope to indicate, aesthetic notions have always permeated (pure and applied) mathematics. And the top researchers have always been driven by an aesthetic imperative:

"We all believe that mathematics is an art. The author of a book, the lecturer in a classroom tries to convey the structural beauty of mathematics to his readers, to his listeners. In this attempt, he must always fail. Mathematics is logical to be sure, each conclusion is drawn from previously derived statements. Yet the whole of it, the real piece of art, is not linear; worse than that, its perception should be instantaneous. We have all experienced on some rare occasions the feeling of elation in realizing that we have enabled our listeners to see at a moment's glance the whole architecture and all its ramifications." (Emil Artin, 1898-1962[2]):

I shall similarly argue for aesthetics before utility. Through a suite of examples[3], I aim to illustrate how and what that means at the research mine face. I also will argue that the opportunities to tie research and teaching to aesthetics are almost boundless — at all levels of the curriculum. [4] This is in part due to the increasing power and sophistication of visualization, geometry, algebra and other mathematical software.

1.1 Aesthetics(s) according to Webster

Let us finish this introduction by recording what one dictionary says:

aesthetic, adj 1. pertaining to a sense of the beautiful or to the science of aesthetics.

[1] "Persons and Places," 1945, 238–9.

[2] Quoted by Ram Murty in *Mathematical Conversations, Selections from The Mathematical Intelligencer*, compiled by Robin Wilson and Jeremy Gray, Springer-Verlag, New York, 2000.

[3] The transparencies, and other resources, expanding the presentation that this paper is based on are available at
`www.cecm.sfu.ca/personal/jborwein/talks.html`,
`www.cecm.sfu.ca/personal/jborwein/mathcamp00.html` and
`www.cecm.sfu.ca/ersonal/loki/Papers/Numbers/`.

[4] An excellent middle school illustration is described in Nathalie Sinclair's "The aesthetics is relevant," *for the learning of mathematics*, **21** (2001), 25-32.

2. having a sense of the beautiful; characterized by a love of beauty.

3. pertaining to, involving, or concerned with pure emotion and sensation as opposed to pure intellectuality.

4. a philosophical *theory or idea of what is aesthetically valid at a given time and place*: the clean lines, bare surfaces, and sense of space that bespeak the machine-age aesthetic.

5. aesthetics.

6. Archaic. the study of the nature of sensation.

Also, esthetic. Syn 2. discriminating, cultivated, refined.

aesthetics, noun 1. the branch of philosophy dealing with such notions as the beautiful, the ugly, the sublime, the comic, etc., as applicable to the fine arts, with a view to establishing the meaning and validity of critical judgments concerning works of art, and the principles underlying or justifying such judgments.

2. *the study of the mind and emotions in relation to the sense of beauty.*

Personally, I would require (unexpected) simplicity or organization in apparent complexity or chaos, consistent with views of Dewey, Santayana and others. We need to integrate this aesthetic into mathematics education so as to capture minds not only for utilitarian reasons. I do believe detachment is an important component of the aesthetic experience, thus it is important to provide some curtains, stages, scaffolds and picture frames and their mathematical equivalents. Fear of mathematics does not hasten an aesthetic response.

2 Gauss, Hadamard and Hardy

Three of my personal mathematical heroes, very different men from different times, all testify interestingly on the aesthetic and the nature of mathematics.

2.1 Gauss

Carl Friedrich Gauss (1777-1855) once confessed[5],

> *"I have the result, but I do not yet know how to get it."*

One of Gauss's greatest discoveries, in 1799, was the relationship between the lemniscate sine function and the arithmetic-geometric mean iteration. This was based on a purely computational observation. The young Gauss wrote in his diary that the result *"will surely open up a whole new field of analysis."*

He was right, as it pried open the whole vista of nineteenth century elliptic and modular function theory. Gauss's specific discovery, based on tables of integrals provided by Stirling (1692-1770), was that the reciprocal of the integral

$$\frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{1-t^4}}$$

[5]See "Isaac Asimov's Book of Science and Nature Quotations," Isaac Asimov and J. A. Shulman (eds.), Weidenfield and Nicolson", New York (1988), 115.

agreed numerically with the limit of the rapidly convergent iteration given by $a_0 := 1,\ b_0 := \sqrt{2}$ and computing

$$a_{n+1} \quad := \quad \frac{a_n + b_n}{2}, \qquad b_{n+1} \ := \ \sqrt{a_n b_n}$$

The sequences a_n, b_n have a common limit 1.1981402347355922074....

Which object, the integral or the iteration, is more familiar, which is more elegant — then and now? Aesthetic criteria change: 'closed forms' have yielded centre stage to 'recursion' much as biological and computational metaphors (even 'biology envy') have replaced Newtonian mental images with Richard Dawkin's 'the blind watchmaker'.

2.2 Hadamard

A constructivist, experimental and aesthetic driven rationale for mathematics could hardly do better than to start with:

> *The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it.*
> (J. Hadamard[6])

Jacques Hadamard (1865-1963) was perhaps the greatest mathematician to think deeply and seriously about cognition in mathematics[7]. He is quoted as saying *"... in arithmetic, until the seventh grade, I was last or nearly last"* which should give encouragement to many young students.

Hadamard was both the author of "The psychology of invention in the mathematical field" (1945), a book that still rewards close inspection, and co-prover of the Prime Number Theorem (1896):

> *"The number of primes less than n tends to ∞ as does $\frac{n}{\log n}$."*

This was one of the culminating results of 19th century mathematics and one that relied on much preliminary computation and experimentation.

2.3 Hardy's Apology

Correspondingly G. H. Hardy (1877-1947), the leading British analyst of the first half of the twentieth century was also a stylish author who wrote compellingly in defense of pure mathematics. He noted that

> *"All physicists and a good many quite respectable mathematicians are contemptuous about proof."*

[6] In E. Borel, "Lecons sur la theorie des fonctions," 1928, quoted by George Polya in *Mathematical discovery: On understanding, learning, and teaching problem solving* (Combined Edition), New York, John Wiley (1981), pp. 2-126.

[7] Other than Poincaré?

in his apologia, "A Mathematician's Apology". The Apology is a spirited defense of beauty over utility:

> *"Beauty is the first test. There is no permanent place in the world for ugly mathematics."*

That said, his comment that

> *"Real mathematics ... is almost wholly 'useless'."*

has been over-played and is now to my mind very dated, given the importance of cryptography and other pieces of algebra and number theory devolving from very pure study. But he does acknowledge that

> *"If the theory of numbers could be employed for any practical and obviously honourable purpose, ..."*

even Gauss would be persuaded.

The Apology is one of Amazon's best sellers. And the existence of Amazon, or Google, means that I can be less than thorough with my bibliographic details without derailing a reader who wishes to find the source.

Hardy, in his tribute to Ramanujan entitled "Ramanujan, Twelve Lectures ...," page 15, gives the so-called 'Skewes number' as a *"striking example of a false conjecture"*. The integral

$$\operatorname{li} x = \int_0^x \frac{dt}{\log t}$$

is a very good approximation to $\pi(x)$, the number of primes not exceeding x. Thus, $\operatorname{li} 10^8 = 5,761,455$ while $\pi(10^8) = 5,762,209$.

It was conjectured that

$$\operatorname{li} x > \pi(x)$$

holds for all x and indeed it so for many x. Skewes in 1933 showed the first explicit crossing at $10^{10^{10^{34}}}$. This has by now been now reduced to a relatively tiny number, a mere 10^{1167}, still vastly beyond direct computational reach.

Such examples show forcibly the limits on numeric experimentation, at least of a naive variety. Many will be familiar with the 'Law of large numbers' in statistics. Here we see what some number theorists call the 'Law of small numbers': *all small numbers are special*, many are primes and direct experience is a poor guide. And sadly or happily depending on one's attitude even 10^{1166} may be a small number.

We shall meet Ramanujan again in the sequel.

3 Research motivations and goals

As a computational and experimental pure mathematician my main goal is: *insight*. Insight demands speed and increasingly parallelism as described in an article I recently coauthored on challenges for mathematical computing.[8] Speed and enough space is a prerequisite:

- For rapid verification.
- For validation and falsification; proofs *and* refutations.

What is 'easy' is changing and we see an exciting merging of disciplines, levels and collaborators. We are more and more able to:

- Marry theory & practice, history & philosophy, proofs & experiments.
- Match elegance and balance to utility and economy.
- Inform all mathematical modalities computationally: analytic, algebraic, geometric & topological.

This is leading us towards an *Experimental M***a***thodology* as a philosophy and in practice.[9] It is based on:

- Meshing computation and mathematics — intuition is acquired. Blake's innocent may become the shepherd.
- Visualization — three is a lot of dimensions. Nowadays we can exploit pictures, sounds and other haptic stimuli.
- 'Caging' and 'Monster-barring' (in Imre Lakatos' words). Two particularly useful components are:
  - graphic checks: comparing $2\sqrt{y} - y$ and $\sqrt{y}\ln(y)$, $0 < y < 1$ pictorially is a much more rapid way to divine which is larger that traditional analytic methods.
  - randomized checks: of equations, linear algebra, or primality can provide enormously secure knowledge or counter-examples when deterministic methods are doomed.

My own methodology depends heavily on:

1. (*High Precision*) computation of object(s) for subsequent examination.
2. *Pattern Recognition* of *Real Numbers* (e.g., using CECM's Inverse Calculator and 'RevEng')[10], or *Sequences* (e'g., using Salvy & Zimmermann's 'gfun' or Sloane and Plouffe's Online Encyclopedia).

[8] J.M. Borwein and P.B. Borwein, "Challenges for Mathematical Computing," *Computing in Science & Engineering*, May/June **3** (2001), 48-53. [CECM Preprint 00:1605]

[9] Jonathan M. Borwein and Robert Corless, "Emerging tools for experimental mathematics," *American Mathematical Monthly*, **106** (1999), 889-909. [CECM Preprint 98:110].

[10] ISC space limits have changed from 10Mb being a constraint in 1985 to 10Gb being 'easily available' today.

3. Extensive use of *Integer Relation Methods*: *PSLQ* & *LLL* and FFT.[11] Exclusion bounds are especially useful and such methods provide a great test bed for 'Experimental Mathematics'.

4. Some automated theorem proving (using methods of *Wilf-Zeilberger* etc).

All these tools are accessible through the listed CECM websites.

3.1 Pictures and symbols

> *"If I can give an abstract proof of something, I'm reasonably happy. But if I can get a concrete, computational proof and actually produce numbers I'm much happier. I'm rather an addict of doing things on the computer, because that gives you an explicit criterion of what's going on. I have a visual way of thinking, and I'm happy if I can see a picture of what I'm working with."* (John Milnor[12])

Let us consider the following images of zeroes of 0/1 polynomials that are manipulatable at `www.cecm.sfu.ca/interfaces/`. These images are also shown and described in my recent survey paper.[13] In this case graphic output allows insight no amount of numbers could.

We have been building educational software with these precepts embedded, such as LetsDoMath.[14] The intent is to challenge students honestly (e.g., through allowing subtle exploration within the 'Game of Life') while making things tangible (e.g., Platonic solids offer virtual manipulables that are more robust and expressive that the standard classroom solids!).

But symbols are often more reliable than pictures. The following picture purports to be evidence that a solid can fail to be polyhedral at only one point. It is the steps up to pixel level of inscribing a regular 2^{n+1}-gon at height 2^{1-n}. But ultimately such a construction fails and produces a right circular cone. The false evidence in this picture held back a research project for several days!

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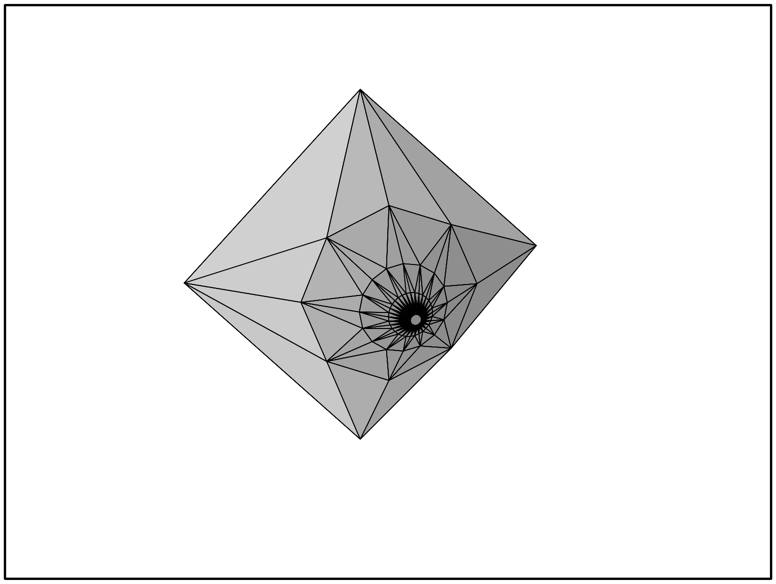
[11] Described as one of the top ten "Algorithm's for the Ages," Random Samples, Science, Feb. 4, 2000.

[12] Quoted in *Who got Einstein's Office?* by Ed Regis – a delightful 1986 history of the Institute for Advanced Study.

[13] D.H. Bailey and J.M. Borwein, "Experimental Mathematics: Recent Developments and Future Outlook," in *Mathematics Unlimited* — 2∞1 *and Beyond,* B. Engquist and W. Schmid (Eds.), Springer-Verlag, 2000, ISBN 3-540-66913-2. [CECM Preprint 99:143]

[14] See `www.mathresources.com`.



3.2 Four kinds of experiment

Medawar usefully distinguishes four forms of scientific experiment.

1. The **Kantian** example: generating "the classical non-Euclidean geometries (hyperbolic, elliptic) by replacing Euclid's axiom of parallels (or something equivalent to it) with alternative forms."
2. The **Baconian** experiment is a contrived as opposed to a natural happening, it "is the consequence of 'trying things out' or even of merely messing about."
3. **Aristotelian** demonstrations: "apply electrodes to a frog's sciatic nerve, and lo, the leg kicks; always precede the presentation of the dog's dinner with the ringing of a bell, and lo, the bell alone will soon make the dog dribble."
4. The most important is **Galilean**: "a critical experiment – one that discriminates between possibilities and, in doing so, either gives us confidence in the view we are taking or makes us think it in need of correction."

The first three forms are common in mathematics, the fourth is not. It is also the only one of the four forms which has the promise to make Experimental Mathematics into a serious replicable scientific enterprise.[15][16]

4 Two things about $\sqrt{2}$...

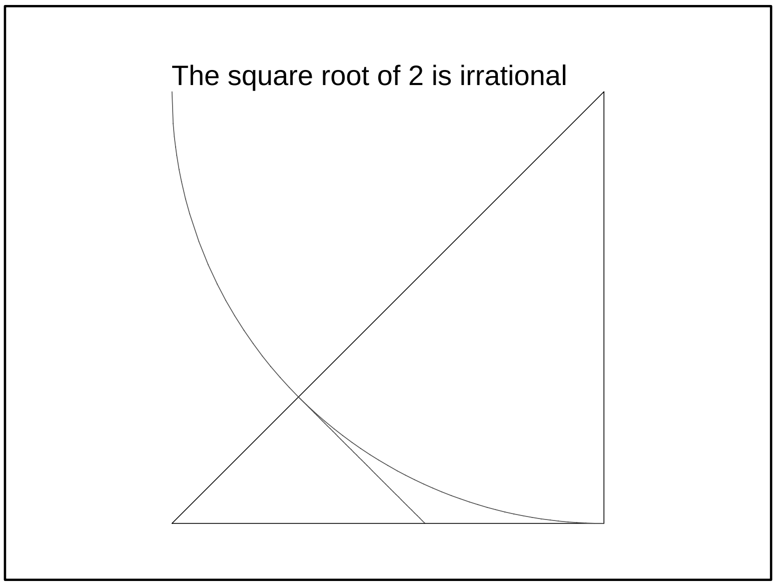
Remarkably one can still find new insights in the oldest areas:

[15]From Peter Medawar's wonderful *Advice to a Young Scientist*, Harper (1979).
[16]See also: D.H. Bailey and J.M. Borwein, "Experimental Mathematics: Recent Developments and Future Outlook."

4.1 Irrationality

We present graphically, Tom Apostol's lovely new geometric proof[17] of the irrationality of $\sqrt{2}$.

PROOF. Consider the *smallest* right-angled isoceles integral with integer sides. Circumscribe a circle of length the vertical side and construct the tangent on the hypotenuse.



The *smaller* isoceles triangle is again integral $\cdots$

4.2 Rationality

$\sqrt{2}$ also makes things rational:

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\left(\sqrt{2}\cdot\sqrt{2}\right)} = \quad \sqrt{2}^{2} = 2.$$

Hence by *the principle of the excluded middle*:

$$\text{Either } \sqrt{2}^{\sqrt{2}} \in \mathbb{Q} \text{ or } \sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}.$$

In either case we can deduce that there are irrational numbers α and β with α^β rational. But how do we know which ones? One may build a whole mathematical philosophy project around this. Compare the assertion that

$$\alpha := \sqrt{2} \text{ and } \beta := 2\ln_2(3) \text{ yield } \alpha^\beta = 3$$

[17] *MAA Monthly*, November 2000, 241-242.

as Maple confirms. This illustrates nicely that verification is often easier than discovery (similarly the fact multiplication is easier than factorization is at the base of secure encryption schemes for e-commerce). There are eight possible (ir)rational triples:

$$\alpha^\beta = \gamma,$$

and finding examples of all cases is now a fine student project.

4.3 ... and two integrals

Even Maple knows $\pi \neq \frac{22}{7}$ since

$$0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\, dx = \frac{22}{7} - \pi,$$

though it would be prudent to ask 'why' it can perform the evaluation and 'whether' to trust it? In contrast, Maple struggles with the following *sophomore's dream*:

$$\int_0^1 \frac{1}{x^x}\, dx = \sum_{n=1}^{\infty} \frac{1}{n^n},$$

and students asked to confirm this typically mistake numerical validation for symbolic proof.

Again we see that computing adds reality, making concrete the abstract, and makes some hard things simple. This is strikingly the case in Pascal's Triangle: `www.cecm.sfu.ca/interfaces/` affords an emphatic example where deep fractal structure is exhibited in the elementary binomial coefficients. Berlinski writes

> *"The computer has in turn changed the very nature of mathematical experience, suggesting for the first time that mathematics, like physics, may yet become an empirical discipline, a place where things are discovered because they are seen."*

and continues

> *" The body of mathematics to which the calculus gives rise embodies a certain swashbuckling style of thinking, at once bold and dramatic, given over to large intellectual gestures and indifferent, in large measure, to any very detailed description of the world. It is a style that has shaped the physical but not the biological sciences, and its success in Newtonian mechanics, general relativity and quantum mechanics is among the miracles of mankind.*
>
> But the era in thought that the calculus made possible is coming to an end. *Everyone feels this is so and everyone is right."* (David Berlinski[18])

[18]Two quotes I agree with from Berlinski's "A Tour of the Calculus," Pantheon Books, 1995

5 π and friends

My research with my brother on π also offers aesthetic and empirical opportunities. The next algorithm grew out of work of Ramanujan.

5.1 *A quartic algorithm* (Borwein & Borwein 1984)

Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$y_{k+1} = \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}} \tag{1}$$

$$a_{k+1} = a_k(1 + y_{k+1})^4 b - 2^{2k+3} y_{k+1}(1 + y_{k+1} + y_{k+1}^2) \tag{2}$$

Then a_k converges *quartically* to $1/\pi$.

We have exhibited 19 pairs of simple algebraic equations (1, 2) that written out in full still *fit on one page* and differ from π (the most celebrated transcendental number) only after 700 billion digits. After 17 years, this still gives me an aesthetic buzz!

This iteration has been used since 1986, with the Salamin-Brent scheme, by Bailey (Lawrence Berkeley Labs) and by Kanada (Tokyo). In 1997, Kanada computed over 51 billion digits on a Hitachi supercomputer (18 iterations, 25 hrs on 2^{10} cpu's). His present world record is 2^{36} digits in April 1999. A billion (2^{30}) digit computation has been performed on a single Pentium II PC in under 9 days.

The 50 billionth decimal digit of π *or* of $\frac{1}{\pi}$ is $04\underline{2}$! And after 18 billion digits, 0123456789 has finally appeared and Brouwer's famous intuitionist example *now* converges![19]

5.1.1 A further taste of Ramanujan

G. N. Watson, discussing his response to such formulae of the wonderful Indian mathematical genius Ramanujan (1887-1920), describes:

> *"a thrill which is indistinguishable from the thrill I feel when I enter the Sagrestia Nuovo of the Capella Medici and see before me the austere beauty of the four statues representing 'Day,' 'Night,' 'Evening,' and 'Dawn' which Michelangelo has set over the tomb of Guiliano de'Medici and Lorenzo de'Medici."* (G. N. Watson, 1886-1965)

One of these is Ramanujan's remarkable formula, based upon the elliptic and modular function theory initiated by Gauss,

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!\,(1103 + 26390k)}{(k!)^4\, 396^{4k}}.$$

[19] Details about π are at `www.cecm.sfu.ca/personal/jborwein/pi_cover.html`.

Each term of this series produces an additional *eight* correct digits in the result — and only the ultimate $\sqrt{2}$ is not a rational operation. Bill Gosper used this formula to compute 17 million terms of the continued fraction for π in 1985. This is of interest because we still can not prove that the continued fraction for π is unbounded. Again everyone *knows* this is true.

That said, Ramanujan prefers related explicit forms such as

$$\frac{\log(640320^3)}{\sqrt{163}} = 3.1415926535897930\underline{164} \approx \pi,$$

correct until the underlined places.

The number e^{π} is the easiest transcendental to fast compute (by elliptic methods). One 'differentiates' $e^{-t\pi}$ to obtain algorithms such as above for π, via the (AGM).

5.2 Integer relation detection

We make a brief digression to describe what integer relation detection methods do.[20] We then apply them to π.[21]

5.2.1 The uses of LLL and PSLQ

DEFINITION: A vector $(x_1, x_2, \cdots, x_n)$ of reals *possesses an integer relation* if there are integers a_i not all zero with

$$\boxed{0 = a_1x_1 + a_2x_2 + \cdots + a_nx_n.}$$

PROBLEM: Find a_i if such exist. If not, obtain lower 'exclusion' bounds on the size of possible a_i.

SOLUTION: For $n = 2$, *Euclid's algorithm* gives a solution. For $n \geq 3$, Euler, Jacobi, Poincaré, Minkowski, Perron, and many others sought methods. The *first general algorithm* was found in 1977 by Ferguson & Forcade. Since '77 one has many variants: **LLL** (also in Maple and Mathematica), HJLS, PSOS, **PSLQ** ('91, *parallelized* '99).

Integer Relation Detection was recently ranked among "the 10 algorithms with the greatest influence on the development and practice of science and engineering in the 20th century," by J. Dongarra and F. Sullivan in *Computing in Science & Engineering*, **2** (2000), 22-23. Also listed were: Monte Carlo, Simplex, Krylov Subspace, QR Decomposition, Quicksort, ..., FFT, Fast Multipole Method.

[20] These may be tried at `www.cecm.sfu.ca/projects/IntegerRelations/`.

[21] See also J.M. Borwein and P. Lisoněk, "Applications of Integer Relation Algorithms," *Discrete Mathematics*, **217** (2000), 65-82. [CECM Research Report 97:104]

5.2.2 Algebraic numbers

Asking about algebraicity is handled by computing α to sufficiently high precision ($\mathrm{O}(n = N^2)$) and apply LLL or PSLQ to the vector

$$(1, \alpha, \alpha^2, \cdots, \alpha^{N-1}).$$

- Solution integers, a_i, are coefficients of a polynomial likely satisfied by α. If one has computed α to $n+m$ digits and run LLL using n of them, one has m digits to heuristically confirm the result. I have never seen this return an honest 'false positive' for $m > 20$ say.

- If no relation is found, exclusion bounds are obtained, saying for example that any polynomial of degree less than N must have the Euclidean norm of its coefficients in excess of L (often astronomical).

5.2.3 Finalizing formulae

If we know or suspect an identity exists integer relations methods are very powerful.

- (*Machin's Formula*). We try Maple's `lin_dep` function on

  $$[\arctan(1), \arctan(1/5), \arctan(1/239)]$$

  and 'recover' `[1, -4, 1]`. That is,

  $$\frac{\pi}{4} = 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239}).$$

  Machin's formula was used on all serious computations of π from 1706 (100 digits) to 1973 (1 million digits). After 1980, the methods described above started to be used instead.

- (*Dase's Formula*). We try `lin_dep` on

  $$[\pi/4, \arctan(1/2), \arctan(1/5), \arctan(1/8)]$$

  and recover `[-1, 1, 1, 1]`. That is,

  $$\frac{\pi}{4} = \arctan(\frac{1}{2}) + \arctan(\frac{1}{5}) + \arctan(\frac{1}{8}).$$

  This was used by Dase to compute 200 digits of π in his head in perhaps the greatest feat of mental arithmetic ever — ' 1/8' is apparently better than '1/239' for this purpose.

5.3 Johann Martin Zacharias Dase

Another burgeoning component of modern research and teaching life is access to excellent (and dubious) databases such as the MacTutor History Archive maintained at: `www-history.mcs.st-andrews.ac.uk`. One may find details there on almost all the mathematicians appearing in this article. We illustrate its value by showing verbatim what it says about Dase.

> "Zacharias Dase (1824-1861) had incredible calculating skills but little mathematical ability. He gave exhibitions of his calculating powers in Germany, Austria and England. While in Vienna in 1840 he was urged to use his powers for scientific purposes and he discussed projects with Gauss and others.
>
> Dase used his calculating ability to calculate to 200 places in 1844. This was published in Crelle's Journal for 1844. Dase also constructed 7 figure log tables and produced a table of factors of all numbers between 7 000 000 and 10 000 000.
>
> Gauss requested that the Hamburg Academy of Sciences allow Dase to devote himself full-time to his mathematical work but, although they agreed to this, Dase died before he was able to do much more work."

5.4 *'Pentium farming'* for binary digits.

Bailey, P. Borwein and Plouffe (1996) discovered a series for π (and corresponding ones for some other *polylogarithmic constants*) which somewhat disconcertingly allows one to compute hexadecimal digits of π *without* computing prior digits. The algorithm needs very little memory and no multiple precision. The running time grows only slightly faster than linearly in the order of the digit being computed.

The key, found by 'PSLQ', as described above, is:

$$\pi = \sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^k \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$$

Knowing an algorithm would follow they spent several months hunting by computer for such a formula. Once found, it is easy to prove in Mathematica, in Maple or by hand — and provides a very nice calculus exercise.

This was a most successful case of

REVERSE MATHEMATICAL ENGINEERING

This is entirely practicable, God reaches her hand deep into π: in September 1997 Fabrice Bellard (INRIA) used a variant of this formula to compute 152 binary digits of π, starting at the *trillionth position* (10^{12}). This took 12 days on 20 work-stations working in parallel over the Internet.

5.4.1 Percival on the web

In August 1998 Colin Percival (SFU, age 17) finished a similar naturally or "embarrassingly parallel" computation of the *five trillionth bit* (using 25 machines at about 10 times the speed of Bellard). In *hexadecimal notation* he obtained:

$$\underline{0}7E45733CC790B5B5979.$$

The corresponding binary digits of π starting at the 40 trillionth place are

$$\underline{0}0000111110011111.$$

By September 2000, the quadrillionth bit had been found to be '0' (using 250 cpu years on 1734 machines from 56 countries). Starting at the $999,999,999,999,997$th bit of π one has:

$$111\underline{0}00110001000010110101100000110.$$

6 Solid and discrete geometry

6.1 De Morgan

Augustus De Morgan, one of the most influential educators of his period, wrote:

> *"Considerable obstacles generally present themselves to the beginner, in studying the elements of Solid Geometry, from the practice which has hitherto uniformly prevailed in this country, of never submitting to the eye of the student, the figures on whose properties he is reasoning, but of drawing perspective representations of them upon a plane. ... I hope that I shall never be obliged to have recourse to a perspective drawing of any figure whose parts are not in the same plane."*
> (Augustus De Morgan, 1806-71, First London Mathematical Society President.[22])

I imagine that De Morgan would have been happier using JavaViewLib: `www.cecm.sfu.ca/interfaces/`. This is Konrad Polthier's modern version of Felix Klein's (1840-1928) famous geometric models. Correspondingly, a modern interactive version of Euclid is provided by Cinderella.de, which is illustrated at `personal/jborwein/circle.html`, and is largely comparable to *Geometer's Sketchpad* which is discussed in detail in other papers in this volume. Klein, like DeMorgan, was equally influential as an educator and as a researcher.

[22]From Adrian Rice, "What Makes a Great Mathematics Teacher?" MAA Monthly, June 1999, p. 540.

6.2 Sylvester's theorem

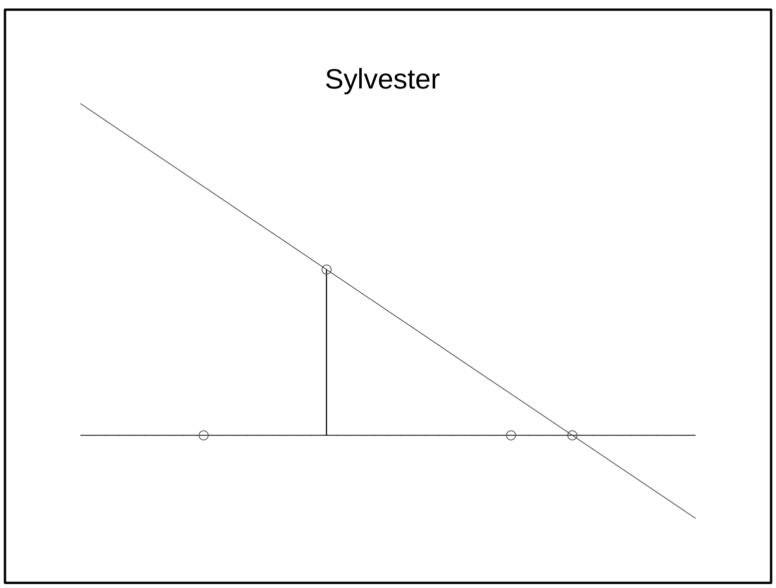
> *"The early study of Euclid made me a hater of geometry."*
> (James Joseph Sylvester, 1814-97, Second London Mathematical Society President.[23])

But discrete geometry (now much in fashion as 'computational geometry' and another example of very useful pure mathematics) was different:

THEOREM. Given N non-collinear points in the plane there is a *proper* line through only two points.[24]

Sylvester's conjecture was it seems forgotten for 50 years. It was first established —"badly" in the sense that the proof is much more complicated — by Gallai (1943) and also by Paul Erdos who named 'the Book' in which God keeps aesthetically perfect proofs. Erdos was an atheist. Kelly's proof was actually published by Donald Coxeter in the MAA Monthly in 1948! A fine example of how the archival record may get obscured.

6.3 Kelly's "Proof from 'The Book' "



PROOF. Consider the point *closest* to a line it is not on and suppose that line has three points on it (the horizontal line).

The middle of those three points is clearly *closer* to the other line!

[23]In D. MacHale, "Comic Sections" (1993).
[24]Posed in *The Educational Times,* **59** (1893).

• As with our proof of the irrationality of $\sqrt{2}$ we see the power of the right *minimal configuration*.

Two more examples that belong in 'the Book' are afforded by:

- Niven's marvellous half page 1947 proof that π *is irrational* (See `www.cecm.sfu.ca/personal/jborwein/pi.pdf`); and
- *Snell's law* — does one use the Calculus to establish the Physics, or use physical intuition to teach students how to avoid tedious calculations?

7 Partitions and patterns

Another subject that can be made highly accessible is additive number theory, especially *partition theory*. The number of *additive partitions* of n, $p(n)$, is generated by

$$P(q) := \prod_{n\geq 1}(1-q^n)^{-1}.$$

Thus $p(5) = 7$ since

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1$$

$$= 2+1+1+1 = 1+1+1+1+1.$$

QUESTION: How hard is $p(n)$ to compute — in 1900 (for MacMahon) and in 2000 (for Maple)?

ANSWER: Seconds for Maple, months for MacMahon. It is interesting to ask if development of the beautiful asymptotic analysis of partitions, by Hardy, Ramanujan and others, would have been helped or impeded by such facile computation?

Ex post facto algorithmic analysis can be used to facilitate independent student discovery of *Euler's pentagonal number theorem*:

$$\boxed{\prod_{n\geq 1}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{(3n+1)n/2}.}$$

Ramanujan used MacMahon's table of $p(n)$ to intuit remarkable and deep congruences such as

$$p(5n+4) \equiv 0 \mod 5$$

$$p(7n+5) \equiv 0 \mod 7$$

and

$$p(11n+6) \equiv 0 \mod 11,$$

from data like

$$\begin{aligned}P(q) = 1 \quad &+ \quad q + 2\,q^2 + 3\,q^3 + 5\,q^4 + 7\,q^5 + 11\,q^6 + 15\,q^7 + 22\,q^8 + 30\,q^9 \\ &+ \quad 42\,q^{10} + 56\,q^{11} + 77\,q^{12} + 101\,q^{13} + 135\,q^{14} + 176\,q^{15} + 231\,q^{16} \\ &+ \quad 297\,q^{17} + 385\,q^{18} + 490\,q^{19} + 627\,q^{20}b + 792\,q^{21}b + 1002\,q^{22} + 1255\,q^{23} + \cdots\end{aligned}$$

If introspection fails, we can recognize the *pentagonal numbers* occurring above in *Sloane* and Plouffe's on-line 'Encyclopedia of Integer Sequences': `www.research.att.com/personal/njas/sequences/eisonline.html`. Here we see a very fine example of *Mathematics: the science of patterns* as is the title of Keith Devlin's 1997 book. And much more may similarly be done.

8 Some concluding discussion

8.1 George Lakoff & Rafael E. Nunez

> "*Recent Discoveries about the Nature of Mind.*
>
> In recent years, there have been revolutionary advances in cognitive science — advances that have a profound bearing on our understanding of mathematics.[25] Perhaps the most profound of these new insights are the following:
>
> *1. The embodiment of mind.* The detailed nature of our bodies, our brains and our everyday functioning in the world structures human concepts and human reason. This includes mathematical concepts and mathematical reason.
>
> *2. The cognitive unconscious.* Most thought is unconscious — not repressed in the Freudian sense but simply inaccessible to direct conscious introspection. We cannot look directly at our conceptual systems and at our low-level thought processes. This includes most mathematical thought.
>
> *3. Metaphorical thought.* For the most part, human beings conceptualize abstract concepts in concrete terms, using ideas and modes of reasoning grounded in sensory-motor systems. The mechanism by which the abstract is comprehended in terms of the concept is called *conceptual metaphor*. Mathematical thought also makes use of conceptual metaphor, as when we conceptualize numbers as points on a line."[26]

They later observe:

[25]More serious curricular insights should come from neuro-biology (Dehaene et al., "Sources of Mathematical Thinking: Behavioral and Brain-Imaging Evidence," *Science*, May 7, 1999).

[26]From "Where Mathematics Comes From," Basic Books, 2000, p. 5.

"What is particularly ironic about this is that it follows from the empirical study of numbers as a product of mind that it is natural for people to believe that numbers are not a product of mind!"[27]

I find their general mathematical schema persuasive but their specific accounting of mathematics forced and unconvincing. Compare a more traditional view which I also espouse:

"The price of metaphor is eternal vigilance."
(Arturo Rosenblueth and Norbert Wiener[28])

8.2 Form follows function

"The waves of the sea, the little ripples on the shore, the sweeping curve of the sandy bay between the headlands, the outline of the hills, the shape of the clouds, all these are so many riddles of form, so many problems of morphology, and all of them the physicist can more or less easily read and adequately solve." (D'Arcy Thompson, "On Growth and Form" 1917)[29])

A century after biology started to think physically, how will mathematical thought patterns change?

"The idea that we could make biology mathematical, I think, perhaps is not working, but what is happening, strangely enough, is that maybe mathematics will become biological!" (Greg Chaitin, Interview, 2000)

Consider the metaphorical or actual origin of the present 'hot topics': simulated annealing ('protein folding'); genetic algorithms ('scheduling problems'); neural networks ('training computers'); DNA computation ('traveling salesman problems'); and quantum computing ('sorting algorithms').

8.3 Kuhn and Planck

Much of what I have described in detail or in passing involves changing set modes of thinking. Many profound thinkers view such changes as difficult:

"The issue of paradigm choice can never be unequivocally settled by logic and experiment alone.

...

in these matters neither proof nor error is at issue. The transfer of allegiance from paradigm to paradigm is a conversion experience that cannot be forced." (Thomas Kuhn[30])

[27] Lakoff and Nunez, p. 81.

[28] Quoted by R. C. Leowontin in *Science,* p. 1264, Feb 16, 2001. (The *Human Genome* Issue.)

[29] In Philip Ball's "The Self-Made Tapestry: Pattern Formation in Nature," `http://scoop.crosswinds.net/books/tapestry.html`.

[30] In *Who got Einstein's Office?*

and

> *"... a new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents die and a new generation grows up that's familiar with it."*
> (Albert Einstein quoting Max Planck[31])

8.4 Hersh's humanist philosophy

However hard such paradigm shifts and whatever the outcome of these discourses, mathematics is and will remain a uniquely human undertaking. Indeed Reuben Hersh's arguments for a humanist philosophy of mathematics, as paraphrased below, become more convincing in our setting:

> *1. Mathematics is human.* It is part of and fits into human culture. It does not match Frege's concept of an abstract, timeless, tenseless, objective reality.
>
> *2. Mathematical knowledge is fallible.* As in science, mathematics can advance by making mistakes and then correcting or even re-correcting them. The "fallibilism" of mathematics is brilliantly argued in Lakatos' *Proofs and Refutations.*
>
> *3. There are different versions of proof or rigor.* Standards of rigor can vary depending on time, place, and other things. The use of computers in formal proofs, exemplified by the computer-assisted proof of the four color theorem in 1977, is just one example of an emerging nontraditional standard of rigor.
>
> *4. Empirical evidence, numerical experimentation and probabilistic proof all can help us decide what to believe in mathematics.* Aristotelian logic isn't necessarily always the best way of deciding.
>
> *5. Mathematical objects are a special variety of a social-cultural-historical object.* Contrary to the assertions of certain post-modern detractors, mathematics cannot be dismissed as merely a new form of literature or religion. Nevertheless, many mathematical objects can be seen as shared ideas, like Moby Dick in literature, or the Immaculate Conception in religion. [32]

The recognition that "quasi-intuitive" methods may be used to gain mathematical insight can dramatically assist in the learning and discovery of mathematics. Aesthetic and intuitive impulses are shot through our subject, and honest mathematicians will acknowledge their role.

[31] From "The Quantum Beat," by F.G. Major, Springer, 1998.

[32] From "Fresh Breezes in the Philosophy of Mathematics," *American Mathematical Monthly*, August-September 1995, 589–594.

8.5 Santayana

> *"When we have before us a fine map, in which the line of the coast, now rocky, now sandy, is clearly indicated, together with the winding of the rivers, the elevations of the land, and the distribution of the population, we have the simultaneous suggestion of so many facts, the sense of mastery over so much reality, that we gaze at it with delight, and need no practical motive to keep us studying it, perhaps for hours altogether. A map is not naturally thought of as an aesthetic object ...*

This was my earliest, and still favourite, encounter with aesthetic philosophy. It may be old fashioned and undeconstructed but to me it rings true:

> *And yet, let the tints of it be a little subtle, let the lines be a little delicate, and the masses of the land and sea somewhat balanced, and we really have a beautiful thing; a thing the charm of which consists almost entirely in its meaning, but which nevertheless pleases us in the same way as a picture or a graphic symbol might please. Give the symbol a little intrinsic worth of form, line and color, and it attracts like a magnet all the values of things it is known to symbolize. It becomes beautiful in its expressiveness."* (George Santayana[33])

To avoid accusations of mawkishness, I finish by quoting Jerry Fodor[34]:

> *"... it is no doubt important to attend to the eternally beautiful and true. But it is more important not to be eaten."*

8.6 A few final observations

- Draw your own — perhaps literally $\cdots$!
- While proofs are often out of reach to students or indeed lie beyond present mathematics, understanding, even certainty, is not.
- Good software packages can make difficult concepts accessible (e.g., Mathematica and Sketchpad).
- Progress is made 'one funeral at a time' (this harsher version of Planck's comment is sometimes attribute to Niels Bohr).
- 'We are Pleistocene People' (Kieran Egan).
- 'You can't go home again' (Thomas Wolfe).

[33] From "The Sense of Beauty," 1896.

[34] In Kieran Egan's book *Getting it Wrong from the Beginning*, in press.

Frontpiece of William Blake's *Songs of Innocence and Experience*
(Combined (1825) edition)