Self-field quantum electrodynamics: The two-level atom

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We use a self-field approach to quantum electrodynamics (QED) to show how one may obtain spontaneous emission and the Lamb shift in a two-level atom without second quantization of the radiation field. In addition, we compare the self-field formalism to that of the neoclassical theory of electrodynamics advanced by Crisp and Jaynes [Phys. Rev. **179**, 1253 (1969)]. We show that the neoclassical model can be obtained from the self-field approach used here, but that the two are not equivalent. In particular, the self-field approach appears to give a more complete description of radiative processes. Finally, we show that the neoclassical theory's prediction of a nonexponential "chirruped" decay is most likely a mathematical artifact of the improper application of the superposition principle in a nonlinear model where such a principle does not hold. A correct treatment with self-field QED yields the usual exponential decay dynamics.

I. INTRODUCTION

In classical electrodynamics one realizes that the Lorentz equation of motion for a charge in an electromagnetic (EM) field is incomplete, inasmuch as it does not include radiation reaction. Considerations such as this have led to the Abraham-Dirac-Lorentz (ADL) equation of motion which, in covariant form, can be written as[1]

$$m\ddot{z}=F^{\text{ext}}\dot{z}+\frac{2\alpha}{3}(\dddot{z}+\dot{z}\ddot{z}^{2}) , \tag{1}$$

where $z=z_{\mu}$ is the coordinate of the charge $q=e$, the dots denote differentiation with respect to the propertime τ, and $\alpha=e^{2}/4\pi$. (We use throughout this paper the convention $c=\hbar=1$.) A covariant external force $F=F_{\mu}$ is allowed for also. In order to arrive at this equation (1), it is necessary that the entire problem be treated covariantly from the outset, with nonrelativistic approximations possible after the formula given above has been specified.

A very interesting derivation of Eq. (1) was given by Wheeler and Feynman using their action at a distance formulation of classical electrodynamics.[2] The idea goes back to a paper by Tetrode,[3], which shows that all of classical electrodynamics—Maxwell's equations and the ADL equation of motion—can be derived from a single unified action principle, if one demands that an accelerating charge produces a field which is symmetric in the retarded and advanced solutions to Maxwell's equations. In such a theory, the contributions of radiation reaction to the Lorentz equation of motion arise very naturally. It is well known that Wheeler and Feynman never produced a quantum version of this theory, although Süssman has presented a second quantized version.[4] The self-field approach to quantum electrodynamics (QED), as proposed by Barut and his co-workers, falls in between these two extremes. We replace the classical particle trajectory z_{μ} with either the scalar Schrödinger wave function ψ, the Pauli two component spinor ϕ, or finally the Dirac four-component spinor Ψ. With the Dirac spinor version the theory is fully covariant and may now be studied as a candidate for a complete theory of QED. At no point do we second quantize either the matter or the radiation field. Süssman, who does second quantize the fields in his quantum version of the Wheeler-Feynman approach, arrives at a correct explanation of spontaneous emission with the right Einstein A coefficient. This, we shall see, is also possible even at the atomic level where the particle is treated nonrelativistically as being described by a Schrödinger wave function, and nothing is second quantized. This result is understandable if we think of spontaneous emission as the quantum analog of the classical radiation reaction line broadening of an oscillating charge. Since the Wheeler-Feynman action accounts for radiation reaction naturally, spontaneous emission is a logical consequence of this approach, even if the EM field is not quantized. Notice that in our method there can be no EM vacuum field fluctuations since the field is not quantized. This precludes the notion of zero-point fluctuations as the physical cause of spontaneous emission in our picture.[5] If a semiclassical theory is defined as a theory which is not second quantized, then self-field QED has been quite a successful semiclassical theory (at least to order α) in accounting for quite an array of phenomena thought to require at least the second quantization of the radiation field for their explanation. Both relativistic and nonrelativistic accounts of spontaneous emission, the Lamb shift, and $g-2$ have been given.[6] Nonrelativistic calculations of cavity-induced changes to these effects have been carried out also, as well as a calculation of the Unruh effect (whereby an accelerating detector senses a bath of thermal radiation).[7] In the present paper we show how the theory can be used to treat the decay dy-

namics of a two-level atom, and that the self-field theory in some sense contains—but is not equivalent to—the neoclassical theory of Crisp and Jaynes.[8]

The paper is organized as follows. First, we present a review of self-field QED, emphasizing the point that the theory is a quantum generalization of the action at a distance approach to classical electrodynamics of Fokker, Schwarzshild, Tetrode, Wheeler, and Feynman. (Equivalently, it is a quantum theory of radiation reaction.) Second, we reduce the theory to that of a two-level atom and obtain the correct exponential spontaneous-emission decay law, as well as the Lamb shift contribution to the energy levels. Third, we indicate how in self-field QED we can arrive at the same erroneous decay law as that of the neoclassical theory if we assume that the superposition principle holds—which it does not in our nonlinear theory. Fourth, we review the neoclassical approach of Jaynes and show that it can lead to the same incorrect, chirruped exponential decay law. The conclusion is then that the same illegal use of superposition in the neoclassical theory—which is similarly nonlinear in the wave function ψ—leads to the wrong decay law. We will detail how the correct exponential decay can be recovered by avoiding recourse to the superposition principle in the neoclassical theory. We shall finally also indicate why we believe that the self-field approach to QED offers a more complete description of radiative corrections than does the neoclassical theory.

II. ACTION AT A DISTANCE ELECTRODYNAMICS

The action at a distance formulation of classical electrodynamics, as presented by Wheeler and Feynman,[2] presupposes an action principle used by Fokker, Schwarzschild and Tetrode.[1,3] Consider a number of charges e_i of mass m_i interacting by means of an action integral W, defined as

$$W=\sum_i \int d\tau\, m_i \dot{z}_i^{\,2} + \sum_i \sum_j e_i e_j \int\int d\tau\, dv\, \dot{z}_i(\tau)\cdot\dot{z}_j(v) D(z_i - z_j) , \tag{2}$$

where τ and v are proper times, $z_i = z_i^{\mu}$ is the four-position of the ith particle, and integration is over all space-time. (We are using standard four-vector notation: $z \equiv z_\mu$, $z^2 \equiv z_\mu z^\mu = z\cdot z$, etc.) The $D(x-y)$ is an electromagnetic Green's function. In order for the variational problem to have a solution, the Green's function $D(x)$ must be symmetric under particle interchange $i \leftrightarrow j$ and also in past and future.[2] These requirements lead to the two equations

$$D = \tfrac{1}{2}(D^{\text{advanced}} + D^{\text{retarded}}) , \tag{3a}$$

$$D(z_i - z_j) = D(z_j - z_i) . \tag{3b}$$

We note that the usual Feynman propagator of QED satisfies both of these conditions. When we extend the theory to the quantum domain, the choice of a symmetric Feynman boundary condition of the form of (3a) will be *required* in order for the variational problem to have a solution. Hence the choice of such a propagator will not be *ad hoc*, but will arise as a natural requirement of the theory.[2,3] With certain further assumptions concerning boundary conditions, it is well known that the variation of the action (2) with respect to z_μ yields, for the Euler-Lagrange equations of motion, the ADL equation (1). Hence we have a classical action principle which yields radiation reaction. If one considers the classical motion of a harmonically bound charge with radiation reaction included, one finds that there arises a level shift and a line broadening to the energy of the oscillator.[1,9] We shall see that these classical phenomena have as their natural quantum analog the Lamb shift and spontaneous emission. From the self-field point of view, all quantum electrodynamic, radiation reaction effects are viewed as the quantum extensions of such classical effects. The problem now is to pose an action principle such as Eq. (2) in a quantum-mechanical setting. To see how to proceed, let us relate the action principle of (2) to one which resembles that of the usual classical field theory. The electromagnetic four-potential $A_\mu^{(i)}(x)$ of the ith particle at the point $x = x_\mu$ is given by

$$A_\mu(x) = e\int d\tau\, D(x - z(\tau))\dot{z}_\mu(\tau) , \tag{4}$$

where the subscript i has now been suppressed. Such a potential gives rise to a field tensor $F_{\mu\nu}$, which is symmetric in retarded and advanced fields; so long as the Green's function satisfies expression (3a).[1] The field tensor also obeys Maxwell's equations, provided we take the current density of the ith particle as

$$j_\mu(x) = e\int d\tau\, \dot{z}_\mu \delta(x - z(\tau)) . \tag{5}$$

If we insert expression (4) into expression (2) for the action W, we obtain

$$W = \sum_i \int d\tau\, m_i \dot{z}_i^{\,2} + \sum_i \sum_j \int d\tau\, e_i \dot{z}_i(\tau) A_j(z_i(\tau)) , \tag{6}$$

which, by inspection of the current j of Eq. (5), we see is the usual classical field theoretical action with a $j\cdot A$ type interaction term. This procedure now give us a clue as to how to go about constructing quantum versions of the action principle embodied in expression (2).

(i) Write down the usual action for either the Schrödinger, Pauli, Or Dirac equation, with the $j\cdot A$ interaction.

(ii) Separate the EM potential according to $A_\mu = A_\mu^{\text{ext}} + A_\mu^{\text{self}}$, where A_μ^{ext} is some field arising from charges assumed to be at infinity and A_μ^{self} is the self-field of the charges in a localized interaction region.

(iii) Eliminate A_μ^{self} entirely from the total action by use of the Feynman Green's function $D(x)$ via the prescription

$$A_\mu^{\text{self}}(x) = e\int dx\, D_{\mu\nu}(x - y) j^\nu(y) , \tag{7}$$

which is a generalization of Eq. (4).

Here the j_μ are quantum electron density currents appropriate to the wave functions ψ, ϕ, or Ψ. When these steps are carried out, one is left with an action principle in which the self-field potential A_μ^{self} has been entirely el-

iminated. The external field $A_\mu^{\rm ext}$ still remains; however, it too could be eliminated if we took our interaction region to be the entire universe. In this case we would have a pure, quantum, action at a distance theory; in particular, it would be a generalization of the Wheeler-Feynman approach contained in Eq. (2). We would hope that the quantum versions of the theory now account for radiation reaction automatically, just as the classical version does so. We shall see, furthermore, that the classical radiation effects of line broadening and level shift have, as their expression in the quantum versions of the theory, spontaneous emission and the Lamb shift. This might be expected from correspondence principle grounds. We now formalize these motivational comments and remarks into a presentation of the self-field theory of quantum electrodynamics.

III. SELF-FIELD QUANTUM ELECTRODYNAMICS

Maxwell's equations and the quantum-mechanical (QM) equations of motion—including radiative or radiation reaction effects—arise from a single action principle, if we use a Feynman Green's function to relate the electromagnetic potential to the current that produces it, via Eq. (7). It is postulated that there are no EM fields independent of the sources that produce them, and hence no possibility of vacuum field fluctuations. It is assumed that the field surrounding a charge can be split as $A_\mu = A_\mu^{\rm ext} + A_\mu^{\rm self}$, where the external field has as its source charges at infinity, while the self-field is the field produced by a localized charge in some interaction region. With this *ansatz* the nonhomogeneous Maxwell equation

$$F^{\mu\nu}{}_{,\mu} = ej^\nu \tag{8}$$

has the general solution

$$A_\mu(x) = A_\mu^{\rm ext} + e\int dy\, D_{\mu\nu}(x-y)j^\nu(y)\ , \tag{9}$$

where $D_{\mu\nu}(x-y)$ *must* be symmetric in retarded and advanced Green's functions. Its precise form will depend on the gauge, and also the overall boundary conditions on the EM field. The total action W may be written as the four-dimensional integral of an action density w,

$$W = \int dx\, w[x;\psi, A_\mu]\ . \tag{10}$$

The action density w will have the general form

$$w(x) = w_0(x) + eA_\mu j^\mu + \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\ , \tag{11}$$

where the specific form of $w_0(x)$, the matter action, will depend on the extent to which a charge is treated nonclassically. If one uses integration by parts, the homogeneous Maxwell's equation, and the assumption that $A_\mu^{\rm self}$ is sufficiently localized, one obtains

$$\begin{aligned} w &= w_0 + eA_\mu^{\rm ext} j^\mu + \frac{e}{2}A_\mu^{\rm self} j^\mu \\ &= w_i + \frac{e}{2}A_\mu^{\rm self} j^\mu \\ &= w^{\rm ext} + w^{\rm self}\ . \end{aligned} \tag{12}$$

The interpretation is that $w^{\rm ext}$ is responsible for the usual electronic motion in an external field, while $w^{\rm self}$ contains radiation reaction effects or radiative corrections such as the Lamb shift and spontaneous emission, corresponding to level shifts and line broadening in the classical theory. With the definition (10), we find that the variation of W with respect to $A_\mu^{\rm self}$ yields, for the Euler Lagrange equations of motion,

$$\frac{\delta W}{\delta A_\nu^{\rm self}} - \partial_\mu \frac{\delta W}{\delta A_{\nu,\mu}^{\rm self}} = -ej^\nu + F^{\mu\nu}_{{\rm self},\mu} = 0\ , \tag{13}$$

which is the inhomogeneous Maxwell equation, provided we have identified

$$\frac{\delta W}{\delta A_\mu^{\rm self}} = -ej^\mu\ . \tag{14}$$

This development has thus far been independent of the choice of the action density w_0. We now summarize the action densities and their corresponding currents for the most important cases.

(i) *Classical action density and current*:

$$w_i = m\dot z^{\,2} - eA_\mu \dot z^{\,\mu}\ , \tag{15a}$$

$$j^\mu = \int d\tau e\dot z^{\,\mu}\delta(x - z(\tau))\ . \tag{15b}$$

(ii) *Schrödinger* action density and current:

$$w_i = \psi^*\left[\frac{1}{2m}(\overleftarrow{\nabla} + ie\,\mathbf{A})\cdot(\vec\nabla - ie\,\mathbf{A}) + eA_0 - i\frac{\partial}{\partial t}\right]\psi\ , \tag{16a}$$

$$j^\mu = \psi^*\left[1, \frac{1}{2mi}\overleftrightarrow{\nabla} - \frac{e}{m}\mathbf{A}\right]\psi\ . \tag{16b}$$

(iii) *Pauli action density and current*:

$$w_i = \phi^*\left[\frac{1}{2m}[(\overleftarrow{\nabla} + ie\,\mathbf{A})\cdot\sigma]\times[\sigma\cdot(\vec\nabla - ie\,\mathbf{A})] + eA_0 - i\frac{\partial}{\partial t}\right]\phi\ , \tag{17a}$$

$$j^\mu = \phi^*\left[1, \frac{1}{2mi}\overleftrightarrow{\nabla} + \frac{1}{2m}(\overleftarrow{\nabla}\times\sigma - \sigma\times\vec\nabla) - \frac{e}{m}\mathbf{A}\right]\phi\ . \tag{17b}$$

(iv) *Dirac action density and current*:

$$w_i = \bar\Psi[\gamma^\mu(i\partial_\mu - eA_\mu) - m]\Psi\ , \tag{18a}$$

$$j^\mu = \bar\Psi\gamma^\mu\Psi\ . \tag{18b}$$

Variation of W with respect to z_μ, ψ, ϕ, or Ψ yields, respectively, the ADL, Schrödinger, Pauli, or Dirac

equations of motion. It must be emphasized that the A_μ which appear in the equations above are not just the external field $A_\mu^{\rm ext}$ alone, but rather the sum of $A_\mu^{\rm ext}+A_\mu^{\rm self}$, as given in Eq. (9). So, unlike the usual semiclassical theory, the action densities listed above contain nonlinear and nonlocal terms of the general form

$$W^{\rm self}=\frac{e^2}{2}\int\int dx\,dy\,j^\mu(x)D_{\mu\nu}(x-y)j^\nu(y)\ , \qquad (19)$$

which are responsible for radiative corrections. [Note the similarity between this expression and the Wheeler-Feynman double integral of Eq. (2) for the classical action, in the self-interaction case where $i=j$.] In particular, we should notice that because the equations of motion are nonlinear *the superposition principle does not hold.* It is not possible to expand the exact solutions of the exact nonlinear equation as a superposition of solutions to the approximate, linear equation which does not contain $W^{\rm self}$. In addition, the terms of the form (19), which now appear in the action integral W, are not perturbations which can be turned on or off at will. They are an integral part of the entire action and their inclusion is always required in order to have a *complete* equation of motion which includes radiation reaction, in analogy to the classical ADL equation (1). For example, if $A_\mu^{\rm ext}=(-Z_e/r,0)$, the static Coulomb potential, then the usual hydrogenic wave functions ψ^0_{nlm} with eigenvalues E_n^0 are not—even in principle— solutions to the complete Schrödinger action, which contains now the nonlinear term $W^{\rm self}$ given in Eq. (19). The conclusion is that the hydrogen atom has no precisely defined sharp energy levels, other than the ground state.[5–7] The excited states cannot be stable, according to the self-field picture, due to radiation reaction. Hence they are never precise levels—but they always have a nonzero linewidth which manifests itself as spontaneous emission. Mathematically, the ψ^0_{nlm} form a complete set of states, and in the usual perturbation theory the solution to the perturbed eigenvalue problem can be expanded as a linear superposition of this complete set. Such an approach would not be correct here, since the principle of linear superposition does not hold for our equations of motion—they contain nonlinear current interaction terms of the form (19). One must be wary of blindly applying the machinery of QM to a problem without regard for the hypotheses upon which such an application is based. (We should point out that even in the standard approach to QED the radiative corrections are not really perturbations either. If one adheres to the notion that radiative effects have their origin in the vacuum field fluctuations, then such corrections form a necessary part of the problem, since the vacuum fluctuations can not be turned off—even in principle. Hence in standard QED the hydrogen atom cannot have exact eigenstates. All of the states, except for the ground state, will have a spread to them which cannot, under any circumstances, be eliminated.)

IV. SELF-FIELD QED FOR A TWO-LEVEL ATOM

We shall now derive the spontaneous-emission rate and Lamb shift for a two-level atom, and attribute them to be physical consequences of the covariant inclusion of the electron's self-field. The interpretation is that spontaneous emission and the Lamb shift are triggered by the electron's radiation reaction field—in complete analogy to the classical account of the line broadening and level shift of the energy of a harmonically bound charge.

It is sufficient for a two-level atom to consider a Schrödinger action principle. The total Schrödinger action can be obtained by inserting the expressions (16) into the action density (12) and then by integrating over all of space-time as per the definition given by Eq. (10). The result is

$$W=\int dx\,\psi^*(x)\left[-\frac{1}{2m}\nabla^2-i\frac{\partial}{\partial t}+\frac{ie}{m}\mathbf{A}^{\rm ext}\cdot\nabla+\frac{ie}{2m}\mathbf{A}^{\rm self}\cdot\nabla+\frac{e^2}{2m}(\mathbf{A}^{\rm ext})^2+\frac{e^2}{2m}\mathbf{A}^{\rm ext}\cdot\mathbf{A}^{\rm self}\right.$$
$$\left.+eA_0^{\rm ext}+\frac{e}{2}A_0^{\rm self}+\frac{ie}{2m}\nabla\cdot\mathbf{A}^{\rm ext}+\frac{ie}{4m}\nabla\cdot\mathbf{A}^{\rm self}\right]\psi(x)\ , \qquad (20)$$

where $dx\equiv d^4x$. If we were to take $A_\mu^{\rm self}=0$ in this expression, we would recover precisely what is usually called semiclassical electrodynamics. However, we can not set the self-field to zero and maintain a complete theory of electronic motion which includes radiative effects. This was first pointed out by Schrödinger.[10] Even formally $A_\mu^{\rm self}$ can never be zero, for it is always given by Eq. (9), which makes it proportional to the current. The only way $A_\mu^{\rm self}$ can be zero at all points in space-time is if the electron four-current is zero at all points in space-time—in which case we have no electron.

Variation of (20) with respect to ψ^* will yield the usual Schrödinger equation, augmented by new, nonlinear terms which contain $A_\mu^{\rm self}$. Using a few techniques from S-matrix theory, we can work directly with the total action (20) and extract radiative corrections to the usual Coulomb energies. Notice that $A_\mu^{\rm self}$ depends on j_μ, via Eq. (9), but that j_μ also depends on $A_\mu^{\rm self}$ via equation (16b). Hence we have a "feedback loop" in the equations, with each cycle of the loop contributing corrections of successively higher orders in the fine-structure constant α. (In this work we keep only corrections to first order in α.) To this order the $\mathbf{A}^{\rm ext}\cdot\mathbf{A}^{\rm self}$ term of Eq. (20) is negligible. For weak external fields $(\mathbf{A}^{\rm ext})^2$ is also negligible.

In fact, since we are interested in hydrogenic atoms, we may set $\mathbf{A}^{\rm ext}=0$ and $A_0^{\rm ext}=-Ze/r$. Finally, a choice of the Coulomb or radiation gauge will eliminate both $\nabla\cdot\mathbf{A}^{\rm ext}$ and $\nabla\cdot\mathbf{A}^{\rm self}$. With these observations the action W can be written

$$W=\int dx\ \psi^*(H_0+H_1+H_2)\psi$$
$$=W_0+W_1+W_2\ , \tag{21}$$

where

$$H_0=-\frac{1}{2m}\nabla^2+eA_0^{\rm ext}-i\frac{\partial}{\partial t}\ , \tag{22a}$$

$$H_1=\frac{ie}{2m}\mathbf{A}^{\rm self}\cdot\nabla\ , \tag{22b}$$

$$H_2=\frac{e}{2}A_0^{\rm self}\ . \tag{22c}$$

It turns out that H_0 is responsible for the usual Coulombic motion, H_1 gives rise to spontaneous emission and the Lamb shift, and finally, H_2 corresponds to a mass renormalization analogous to that which appears in the classical theory of radiation reaction.[1,5]

As we mentioned above, we are using the Coulomb gauge $\nabla\cdot\mathbf{A}=0$. In this gauge the components of the Green's function $D_{\mu\nu}(x-y)$ become

$$D_{ij}(x-y)=\frac{1}{(2\pi)^4}\int dk\frac{e^{-ik(x-y)}}{k^2+i\epsilon}(\delta_{ij}+\hat{k}_i\hat{k}_j)\ , \tag{23a}$$

$$D_{00}(x-y)=\frac{1}{(2\pi)^4}\int dk\frac{e^{-ik(x-y)}}{\lambda^2+i\epsilon}\ , \tag{23b}$$

$$D_{i0}(x-y)=D_{0i}(x-y)=0\ , \tag{23c}$$

where $\lambda\equiv|\mathbf{k}|^2$, $k^2\equiv k^\mu k_\mu$, and the $+i\epsilon$ in the denominator insures that the correct symmetry between retarded and advanced solutions to Maxwell's equations is obtained. With this choice of Green's function the equation (9) for the self-field can be written as

$$\mathbf{A}^{\rm self}=-\frac{e}{(2\pi)^4}\int\int dy\ dk\frac{e^{-ik(x-y)}}{k^2+i\epsilon}$$
$$\times[\mathbf{j}(y)-\hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\mathbf{j}(y))]\ , \tag{24a}$$

$$A_0^{\rm self}(x)=\frac{e}{(2\pi)^4}\int\int dy\ dk\frac{e^{-ik(x-y)}}{\lambda^2+i\epsilon}\rho(y)\ , \tag{24b}$$

where ρ and $\mathbf{j}$ are the time and space components of the current j_μ as given in Eq. (16b). In our notation above we use $dy\equiv d^4y$, $dk\equiv d^4k$, and $\hat{\mathbf{k}}\equiv\mathbf{k}/|\mathbf{k}|$.

If we now insert Eq. (16b) into the above expressions (24) and put these into the total action given in (21), we obtain

$$W_0=\int dx\ \psi^*(x)\left[-\frac{1}{2m}\nabla^2+eA_0^{\rm ext}-i\frac{\partial}{\partial t}\right]\psi(x)\ , \tag{25a}$$

$$W_1=-\frac{\alpha}{(2\pi)^3m^2}\int\int\int dx\ dy\ dk\frac{e^{-ik(x-y)}}{k^2+i\epsilon}[\psi^*(x)\nabla_x\psi(x)]\cdot\{\psi^*(y)[\nabla_y-\hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\nabla_y)]\psi(y)\}\ , \tag{25b}$$

$$W_2=\frac{\alpha}{(2\pi)^3}\int\int\int dx\ dy\ dk\frac{e^{-ik(x-y)}}{\lambda^2+i\epsilon}\rho(x)\rho(y)\ , \tag{25c}$$

where $\alpha=e^2/4\pi$.

The ψ which appears in these equations (25) is assumed to be the function which minimizes the total action $W=W_0+W_1+W_2$. Equivalently, they are solutions to the augmented, integro-differential Schrödinger equation one obtains from the Euler Lagrange equations of motion when W is varied with respect to ψ^*. The point is that the $\psi(x)$ are as of yet unknown functions of the space-time coordinate $x=x_\mu=(t,\mathbf{x})$. Now, without the self-field contributions of W_1 and W_2, the action integral W_0 alone is minimized by the usual Coulombic wave functions, which we shall denote ψ_n^0 (where $n\equiv nlm$ contains all three hydrogenic quantum numbers). Physically, we should expect that there are true solutions ψ_n with energies E_n that minimize the complete nonlinear action W, and that these functions are in some sense close to the Coulombic functions ψ_n^0, with eigenenergies E_n^0, which minimize the linear action W_0 alone.

It is usual in perturbation theory to assume that since the Coulombic wave functions ψ_n^0 form a complete set, any solution ψ to the perturbed equation of motion can be expanded as a linear superposition given by

$$\psi(\mathbf{x},t)=\sum_n C_n(t)\psi_n^0(\mathbf{x})\ . \tag{26}$$

As a further *ansatz*, one can assume that the rapid oscillations proportional to $\exp(iE_n^0t)$ can be separated from the more slowly changing time behavior. Hence in atomic physics—especially in the two-level atom model—one writes expression (26) in the form

$$\psi(\mathbf{x},t)=\sum_n C_n(t)e^{iE_n^0t}\psi_n^0\ , \tag{27}$$

where the $C_n(t)$ are presupposed to vary only slowly in time when compared to the exponential factors. The E_n^0 are the eigenenergies of the ψ_n^0. Conservation of charge requires the normalization condition

$$\int d^3x\,|\psi(\mathbf{x},t)|^2=\sum_n|C_n(t)|^2=1\ . \tag{28}$$

This entire mathematical apparatus requires that the perturbation to the Schrödinger equation be linear, and thus that the principle of linear superposition holds. In self-field QED the nonlinear expression (19) is not a perturbation, but rather a required part of the equation of motion for a complete theory—it cannot be turned off. If one did try to treat this as a perturbation, using Eqs. (26) or (27), one could not trust the result because (19) is nonlinear in ψ and hence the superposition principle needed for expansions (26) or (27) does not hold. We shall see later in this paper that the erroneous consequence of making such an attack on the problem is the chirruped exponential decay predicted by neoclassical theory. Clearly a different approach is needed here.

In the eigenfunction expansion (26) we physically are making the *ansatz* that the spatial behavior is described by the known functions $\psi_n^0(\mathbf{x})$, and then we use this knowledge to obtain information about the time behavior of the unknown $C_n(t)$. We propose now to reverse this procedure by making instead the *Fourier expansion*[6]

$$\psi(\mathbf{x},t)=\sum_n \psi_n(\mathbf{x})e^{-iE_n t}\ , \tag{29}$$

in which we assume that the time behavior is known, and of the form $\exp(-iE_n t)$. It is now information about the unknown wave functions ψ_n and the corresponding energies E_n that we are looking for. Physically we would expect that the ψ_n and the energies E_n are approximately equal to the Coulombic wave functions ψ_n^0 with energies E_n^0. The ψ_n in addition would have a complex phase factor $\exp(i\phi_n)$, which would make the Fourier expansion (29) convergent. However, in a two-level atom, we can to first order of iteration replace ψ_n with ψ_n^0, since $n\in\{1,2\}$, and then solve for E_n, assuming that it has the form $E_n=E_n^0+\delta E_n$.

Let us now insert the Fourier expansion (29) for the action integral $W=W_0+W_1+W_2$ found in Eq. (25). For W_0 we find

$$\begin{aligned} W_0&=\int\int d^3x\,dt\sum_{n,m}\psi_n^*(\mathbf{x})H_0\psi_m(\mathbf{x})e^{i\omega_{nm}t}\\ &=\sum_{n,m}\int dt\langle n|H_0|m\rangle e^{i\omega_{n,m}t}\\ &=2\pi\sum_{n,m}\langle n|H_0|m\rangle\delta(\omega_{nm})\ , \end{aligned} \tag{30}$$

where $\omega_{nm}\equiv E_n-E_m$.

If the ψ_n were Coulombic wave functions, this expression would be zero. The ψ_n^0 minimize W_0 alone. However, now the entire action W, of which W_0 is only one term, must be minimized as a whole. For our two-level atom discussion $n,m\in\{1,2\}$, but the general treatment holds for an atom with a complete set of levels.[5,6]

We now discuss the piece of the action W_1, which contains the Lamb shift and spontaneous emission. Inserting the expansion (29) into W_1 of (25b) we obtain, after carrying out the $x_0\equiv t$ and $y_0\equiv u$ time integrations,

$$W_1=-\frac{4\alpha}{3m^2}\sum_{n,m,p,q}\int d\lambda\,\lambda^2\frac{\delta(\omega_{nm}+\omega_{pq})}{\omega_{pq}^2-\lambda^2+i\epsilon}\times\langle n|\nabla|m\rangle\langle p|\nabla|q\rangle\ , \tag{31}$$

where we have used the dipole approximation (DA), i.e., $\exp[i(\mathbf{x}-\mathbf{y})]\approx 1$. The kets $|n\rangle$ are still exact solutions which we assume minimize the total action W. The δ function is satisfied by either of the two conditions

$$n=m,\quad p=q \tag{32a}$$

$$n=q,\quad m=p \tag{32b}$$

but one can show[6] that condition (32a) causes W_1 to vanish identically from parity considerations. This is because matrix elements of the form $\langle n|\nabla|n\rangle$ vanish for even azimuthal quantum numbers, and terms with opposite azimuthal quantum numbers will cancel in the summation in Eq. (31). Hence only the choice of (32b) gives any contribution to the total action. This then leaves us with the expression

$$W_1=\frac{4\alpha}{3}\sum_{n,m}\int d\lambda\,\lambda^2\frac{\omega_{nm}^2|\mathbf{x}_{nm}|^2}{\omega_{nm}^2-\lambda^2+i\epsilon}\ , \tag{33}$$

where we have used the relation $\langle n|\nabla|m\rangle\equiv\nabla_{nm}=-m_{\text{elec}}\omega_{nm}\mathbf{x}_{nm}$. Using the symmetry in the dummy indices n and m, we may write a partial fraction expansion

$$\frac{\lambda^2}{\omega_{nm}^2-\lambda^2}=\frac{\omega_{nm}}{\omega_{nm}-\lambda}-1\ , \tag{34}$$

where the equality sign is understood to hold under the double summation $\sum_{n,m}$. The -1 in Eq. (34) corresponds to an energy shift proportional to ∇^2, and hence to a change in the electron mass. This term may be eliminated by renormalizing the electron mass,[5] leaving only the first term on the right-hand side of Eq. (34). The implied contour integration embodied in the $+i\epsilon$ in the denominator of (33) may be carried out by the usual prescription of writing the integrand as a principal part P plus a residue, as per

$$\frac{1}{\omega_{nm}-\lambda}=\mathrm{P}\left[\frac{1}{\omega_{nm}-\lambda}\right]-i\pi\delta(\omega_{nm}-\lambda)\ , \tag{35}$$

which gives

$$W_1=\frac{4\alpha}{3}\sum_{n,m}\omega_{nm}^3|\mathbf{x}_{nm}|^2\int_0^\infty\frac{d\lambda}{\omega_{nm}-\lambda}-\frac{4\pi\alpha}{3}i\sum_{\substack{n,m\\ m<n}}\omega_{nm}^3|\mathbf{x}_{nm}|^2\ , \tag{36}$$

where the δ function of (35) contributes only if $m<n$. We now extract the contribution to level n alone, and convert to units of energy, via an S-matrix prescription.[11] For a bound-state problem, the total action W is related to the total invariant energy $\mathcal{E}$ of the system via

$$W_{fi}=2\pi\delta(E_f-E_i)\mathcal{E}\ , \tag{37}$$

and so the contribution from (36) to the total energy of level n is given by

$$\mathcal{E}_1^{(n)}=\frac{W_1^{(n)}}{2\pi}$$
$$=\frac{2\alpha}{3\pi}\sum_m \omega_{nm}^3|\mathbf{x}_{nm}|^2\int_0^\infty \frac{d\lambda}{\omega_{nm}-\lambda}$$
$$-\frac{2\alpha i}{3}\sum_{\substack{n,m\\ m<n}}\omega_{nm}^3|\mathbf{x}_{nm}|^2$$
$$\equiv\delta E_n - iA_n \ . \tag{38}$$

The real part of the energy shift $\mathrm{Re}\{\mathcal{E}_1^{(n)}\}$ is the nonrelativistic result for the Lamb shift, first obtained by Bethe.[12] For a two-level atom the sum runs over $m=1,2$. If we define $\omega_{nm}=\omega_{21}\equiv\omega_0$ as usual, we have

$$A_1=0 \ , \tag{39a}$$

$$A_2\equiv A=\frac{2\alpha}{3}\omega_0^3|\mathbf{x}_{21}|^2 \ . \tag{39b}$$

This shows us that the ground state ψ_1 is stable, but that the excited state ψ_2 decays with a time characterized by $\tau\equiv 1/A$, where A is the usual Einstein coefficient of spontaneous emission. If we could prepare the two-level system in level two at time $t=0$, Eq. (29) would become

$$\psi(\mathbf{x},t)=\psi_2(\mathbf{x})e^{-i(E_2^0+\delta E_2)t-At} \ , \tag{40}$$

where a factor of 2 should now be included in the definition (39b) of A to account for the two polarization degrees of freedom of the photon. Equation (40) contains the usual exponential decay dynamics for a two-level atom, as found in standard QED. We now see that, self-field QED, if treated correctly, does not predict any nonstandard dynamics, such as the chirruped decay profile predicted by the neoclassical theory of Jaynes.

It can be shown[5] that the corrections to the total action coming from H_2 give rise to a static shift which is the same for all levels and hence unobservable, and also a small level shift contribution which, in the relativistic version of the theory, corresponds to the vacuum polarization term of Wichmann and Kroll. This effect is negligible in our two-level model, when compared to the dominant contribution arising from the real part of Eq. (38), and we will not consider it further in this paper.

V. SELF-FIELD QED ASSUMING SUPERPOSITION

Let us now see what would have happened had we used the usual, but inadmissable, expansion (27), together with the conservation of electronic charge condition (28), all *instead* of the Fourier expansion (29). [Recall that an expansion such as (27) in terms of a superposition of known eigenstates is not valid because the solutions of the complete nonlinear Schrödinger equation are not known—and because there is no superposition principle for a nonlinear equation.] For a two-level atom the expansion (27) and condition (28) become

$$\psi(\mathbf{x},t)=C_1(t)\psi_1^0(\mathbf{x})e^{-iE_1^0t}+C_2(t)\psi_2^0(\mathbf{x})e^{-iE_2^0t} \ , \tag{41a}$$

$$|C_1(t)|^2+|C_2(t)|^2=1 \ . \tag{41b}$$

Following the usual development of the two-level atom model, we assume a Hamiltonian of the form

$$H=H_0+H' \ , \tag{42}$$

where

$$H_0\psi_n^0=E_n^0 \quad (n=1,2) \tag{43}$$

and

$$\rho=\langle\psi|\psi\rangle=|C_1|^2+|C_2|^2=1 \ , \tag{44}$$

with

$$H'_{nm}\equiv\langle n|H'|m\rangle \ . \tag{45}$$

We define as before $\omega_0\equiv E_2-E_1$. The equations of motion become, in all generality for two levels,[13]

$$i\dot{C}_1=C_1H'_{11}+C_2H'_{12}e^{-i\omega_0t}\equiv M_{11}+M_{12} \ , \tag{46a}$$

$$i\dot{C}_2=C_1H'_{21}e^{i\omega_0t}+C_2H'_{22}\equiv M_{21}+H_{22} \ . \tag{46b}$$

So far we have done nothing but summarize the theory of a dynamic two-level atom with a perturbation. In free space, however, the only possible candidate for a perturbation is the radiation reaction response of the atom to the electronic self-field. At a simple level it can be showed that the results of self-field QED arise as a self-induced Stark and Zeeman effect which arises when the electron cloud responds to its own electric field and magnetic fields. The Stark level shifts then are those of the usual Lamb shift.

We now take $H'\equiv H_1+H_2$ as our perturbation, where H_1 and H_2 are given in Eq. (22). (The self-field contribution from H' are not perturbations, we recall, and already at this stage we should not expect this procedure to yield correct results.) From parity considerations it is easy to show that[13]

$$H_{11}^{(2)}=H_{22}^{(1)}=0 \ , \tag{47a}$$

$$H_{12}^{(1)}=H_{21}^{(1)*} \ , \tag{47b}$$

$$H_{12}^{(2)}=H_{21}^{(2)}=0 \ . \tag{47c}$$

To calculate the nonzero matrix elements, we begin with H_1 in Eq. (22). Into this expression we insert the EM vector potential $\mathbf{A}^{\mathrm{self}}$ from (24a) with the current $\mathbf{j}$ taken from (16b). This yields the result

$$H_{12}^{(1)}=-\frac{\alpha}{3m^2\pi^2}\int\int\int d\lambda\, du\, d\omega\frac{\lambda^2 e^{-i\omega t}}{\omega^2-\lambda^2+i\epsilon}\nabla_{12}\cdot[C_2^*(u)C_1(u)\nabla_{21}e^{i(\omega+\omega_0)u}+C_1^*(u)C_2(u)\nabla_{12}e^{i(\omega-\omega_0)u}]\ , \tag{48}$$

where u is a dummy time integration variable. (We have used the dipole approximation.) Since the $C_i(t)$ are assumed to vary only slowly with time, we may replace $C_i(u)\approx C_i(t)$ in the integrand, allowing us to carry out the u integration. In the rotating-wave approximation we neglect the terms of expression (48), which contain the exponential $\exp[i(\omega+\omega_0)]$ and then the matrix elements $M_{ij}^{(1)}$ of Eq. (46), arising from the perturbation H_1, become

$$M_{12}^{(1)}=-\frac{2\alpha}{3\pi}\omega_0^2|\mathbf{x}_{21}|^2 I(\omega_0)|C_2|^2C_1\ , \tag{49a}$$

$$M_{21}^{(1)}=-\frac{2\alpha}{3\pi}\omega_0^2|\mathbf{x}_{21}|^2 I^*(\omega_0)|C_1|^2C_2\ , \tag{49b}$$

where we have defined

$$I(\omega_0)\equiv\int d\lambda\frac{\lambda^2}{\omega_0^2-\lambda^2+i\epsilon}\ , \tag{50}$$

which is identical to the integral which appears in W_1. Hence we expect $I(\omega_0)$ to contribute a level shift and a line broadening as before. Similarly, as in Eq. (34), in $I(\omega_0)$ we can renormalize away a linearly divergent mass term as Bethe does,[12] leaving a logarithmically divergent contribution to the Lamb shift, and a complex residue which mediates the decay of the atom, i.e.,

$$I(\omega_0)\rightarrow\omega_0\ln\left[\frac{\Lambda}{\omega_0}\right]-i\pi\omega_0\ , \tag{51}$$

where the cutoff Λ is usually taken as $\Lambda=m$.

We now calculate the matrix elements $M_{ij}^{(2)}$, which come from the perturbation H_2. Using the definition of H_2 from Eq. (22c), the charge density $j^0\equiv\rho$ from (16b), and the charge conservation condition (44); we obtain in the dipole approximation

$$H_{11}^{(2)}=H_{22}^{(2)}=\frac{\alpha}{\pi}\Lambda\ , \tag{52}$$

where Λ is the same photon cutoff parameter used in (51). This divergent energy shift is level independent, and hence unobservable. (It is the same for all levels, and hence can be subtracted off by rescaling the energy axis. This divergence is an artifact of the dipole approximation.) Combining the results of (49) and (52), we have for the time evolution equations (46)

$$i\dot{C}_1=\frac{\alpha\Lambda}{\pi}C_1-\frac{2\alpha}{3\pi}\omega_0^3|\mathbf{x}_{21}|^2\times\left[-\ln\left[\frac{\Lambda}{\omega_0}\right]-i\pi\omega_0\right]|C_2|^2C_1\ , \tag{53a}$$

$$i\dot{C}_2=\frac{\alpha\Lambda}{\pi}C_2-\frac{2\alpha}{3\pi}\omega_0^3|\mathbf{x}_{21}|^2\times\left[-\ln\left[\frac{\Lambda}{\omega_0}\right]+i\pi\omega_0\right]|C_1|^2C_2\ . \tag{53b}$$

Now take Eq. (53a) and multiply it by C_1^*; then take the complex conjugate of (53a) and multiply through by C_1. Add the resultant equations together, and make use of the charge conservation condition (44). The final equations that remain after these operations are

$$\frac{d}{dt}|C_1|^2=2A(1-|C_1|^2)|C_1|^2\ , \tag{54a}$$

$$\frac{d}{dt}|C_2|^2=-\frac{d}{dt}|C_1|^2\ , \tag{54b}$$

where A is the usual Einstein A coefficient for spontaneous emission, defined as

$$A\equiv\frac{2\alpha}{3}\omega_0^3|\mathbf{x}_{21}|^2\ . \tag{55}$$

Notice how the terms containing the cutoff Λ have dropped out. These terms correspond to level shifts, and do not affect the dynamics of the spontaneous decay of level two into level one. If we define $X\equiv|C_1|^2$ and $Y\equiv|C_2|^2$, Eqs. (54) may be integrated to give

$$X(t)=\frac{1}{Ke^{-2At}+1}\ , \tag{56a}$$

$$Y(t)=\frac{1}{Le^{+2At}+1}\ , \tag{56b}$$

which are the chirruped hyperbolic decay profiles predicted by the neoclassical theory of Jaynes. (The K and L are constants of integration, with $K=1/L$.)

Now, how is it possible that we have lost the purely exponential decay profile of Eq. (40)? The physical input to the theory has not changed—only the mathematical analysis leading to the final result given above in Eq. (56). Although the decay constant $\tau=1/A$ is correct, the functional form is not. It is our contention that the chirruped decay arises not from incorrect physics, but rather from the incorrect use of the superposition principle, as it appears in the form of Eq. (41a). The perturbation H' is nonlinear in ψ and hence the total wave function that solves the perturbed, nonlinear, equation of motion is not necessarily expandable as a linear combination of ψ_1 and ψ_2. Such a procedure can yield at most an approximately correct solution. And indeed we see that the solutions (56) decay with the correct time constant A, and exhibit the correct exponential decay asymptotically as $t\rightarrow\infty$. The decay, however, is not correct for short times.

VI. REVIEW OF NEOCLASSICAL ELECTRODYNAMICS

The neoclassical theory of Crisp and Jaynes[8] is essentially equivalent to an idea of Schrödinger[10] and Fermi:[14] to include some quantum analog of classical radiation reaction effects in the Schrödinger equation to account for spontaneous emission. In this sense the neoclassical theory is in the same spirit as self-field QED. Fermi's development is essentially the same as that of Crisp and Jaynes, and so we present primarily his methodology here. We will restrict ourselves to a two-level atom discussion, to maintain consistency with the previous presentation.

Consider a solution $\psi(\mathbf{x},t)$ to Schrödinger's equation

$$i\frac{\partial\psi}{\partial t}=H\psi\ , \tag{57a}$$

$$H\equiv H_0+H'\ , \tag{57b}$$

$$H_0\equiv-\frac{1}{2m}\nabla^2+V_0(\mathbf{x})\ , \tag{57c}$$

$$H'\equiv V'(\mathbf{x},t)\ , \tag{57d}$$

where V' is supposed to correspond to a radiation reaction potential of some sort. We suppose that the electron charge density is given as usual by $e\psi^*\psi$, and conservation of charge requires that $\int d^3x\rho=1$. The electric dipole moment can be written as

$$\mathbf{p}=\int d^3x\,\mathbf{x}\rho(\mathbf{x})\ , \tag{58}$$

which is equal to the classical expression. The solutions $\psi_n^0(\mathbf{x},t)$ to the unperturbed equation $H=H_0$ can be written

$$\psi_n^0(\mathbf{x},t)=\psi_n^0(\mathbf{x})e^{-iE_n^0t}\quad(n=1,2) \tag{59}$$

where the $\psi_n^0(\mathbf{x})$ are solutions to the stationary equation $H_0\psi_n^0=E_n^0\psi_n^0$, and are normalized as usual as $\langle n|m\rangle=\delta_{nm}$.

In order to get the neoclassical theory of spontaneous emission we make the following *ansatz* for the form of the potential $V'(\mathbf{x},t)$:

$$V'(\mathbf{x},t)=\frac{2}{3}\left[\frac{e}{2\pi}\right]\mathbf{x}\cdot\ddot{\mathbf{P}}\ , \tag{60}$$

which is taken directly from classical electrodynamics.[1] Expression (60) does not arise naturally in the theory of neoclassical EM, but is inserted in a rather *ad hoc* fashion in order to *make* the neoclassical theory. Notice that with the inclusion of (60) in the Hamiltonian, the resultant Schrödinger equation is nonlinear and nonlocal since V' depends on the three-dimensional space integral of $\mathbf{x}\psi^*\psi$. The situation is very similar to what was obtained in the self-field theory in Eq. (21), except that the neoclassical Schrödinger equation is not as complete. In addition to being a rather arbitrary prescription, as it stands it can account only for spontaneous emission and not the Lamb shift, vacuum polarization, $g-2$, etc.

To continue with the analysis in the context of neoclassical theory, one next assumes that the full solution of the perturbed equation (57a) can be expanded as a *linear superposition* of the eigenstates of the unperturbed equation. As we saw previously in the context of self-field QED, such an assumption is not correct due to the nonlinearity inherent in the perturbed Schrödinger equation (57a). This, we believe, is the erroneous step in neoclassical theory which gives the unphysical chirruped decay profile. To see that this is indeed so, we assume a linear superposition such as in Eq. (41). With this expansion, the potential V' becomes

$$V'(\mathbf{x},t)=\frac{2i\alpha}{3}\sum_{n,m}(\mathbf{x}\cdot\mathbf{x}_{nm})\omega_{nm}^3e^{i\omega_{nm}t}C_n^*C_m\ , \tag{61}$$

where $\omega_{nm}\equiv E_n^0-E_m^0$ and $\mathbf{x}_{nm}=\langle n|\mathbf{x}|m\rangle$, as before. If we insert Eq. (61) into (57), along with expansion (41), and then operate on the result expression with $\int d^3x\,\psi_k^*(\mathbf{x})$ and sum both sides, we arrive at the following time evolution equation:

$$\dot{C}_k=-\frac{2\alpha}{3}\sum_{l,n,m}C_lC_n^*C_m\omega_{nm}^3(\mathbf{x}_{kl}\cdot\mathbf{x}_{nm})e^{i(\omega_{nm}+\omega_{kl})t}\ , \tag{62}$$

where $n,m,l,k\in\{1,2\}$. Conservation of energy requires $\omega_{nm}+\omega_{kl}=0$, which can be satisfied by $k=m$ and $l=m$. Hence expression (62) reduces to

$$\dot{C}_k=-\frac{2\alpha}{3}\sum_{l=1}^{2}C_k|C_l|^2\omega_{kl}^3|\mathbf{x}_{kl}|^2 \tag{63}$$

or

$$\dot{C}_1=+AC_1|C_2|^2\ , \tag{64a}$$

$$\dot{C}_2=-AC_2|C_1|^2\ , \tag{64b}$$

where we have defined A as before in Eq. (55). Multiplying (64a) through by C_1^*, and the complex conjugate of (64a) by C_1 and adding these results give precisely the same nonlinear equation found in (54a). A similar calculation using C_2 gives (54b). Hence the $|C_i|^2$ obey the same time evolution equations (56) as before, exhibiting the chirruped hyperbolic profile, which is the trademark of the neoclassical account of the dynamics of spontaneous emission. This decay is not physical, but rather a result of the invalid application of the superposition principle that is assumed in expansion (41).

We now recalculate the spontaneous-emission decay rate in a different fashion, still within the context of neoclassical theory, and show that the theory does admit a correct exponential decay, so long as the expansion (41) is not used. Let us begin with the perturbed, time dependent Schrödinger equation (57). Instead of using the expansion (41), now we use the Fourier expansion (29). With this decomposition of the wave function, the perturbing radiation reaction potential V' of (57) becomes

$$V'(\mathbf{x},t)=\frac{2i\alpha}{3}\sum_{n,m}(\mathbf{x}\cdot\mathbf{x}_{nm})\omega_{nm}^3e^{i\omega_{nm}t}\ . \tag{65}$$

Notice the absence of the $C_i(t)$ here, as compared to Eq. (61). Ideally, the matrix elements $\mathbf{x}_{nm}$ are taken with respect to the exact solutions ψ_n to the entire nonlinear

equation (57). However, we may as before substitute the approximate Coulombic solutions ψ_n^0 so as to iterate the first-order energy correction to E_n^0. Suppose we have prepared a state ψ_n that is an exact solution which minimizes the entire exact nonlinear action W. Let us assume that this state can be written $\psi_n=\psi_n^0+\delta\psi$ with energy $E_n=E_n^0+\delta\psi$. Then the Schrödinger equation

$$H\psi_n=i\frac{\partial\psi_n}{\partial t}=E_n\psi_n$$

can be written as

$$(H_0+H')(\psi_n^0+\delta\psi_n)=(E_n^0+\delta E_n)(\psi_n^0+\delta\psi_n)\ , \tag{66}$$

which reduces to

$$V'|n\rangle=-\delta E_n|n\rangle e^{-iE_nt}\ , \tag{67}$$

where we have used the separation $\psi_n^0(\mathbf{x},t)=\psi_n^0(\mathbf{x})e^{-iE_n^0t}$. We have neglected $\delta\psi_n$, and here $|n\rangle\equiv\psi_n^0(\mathbf{x})$. Inserting expansion (65) for V', operating from the left with $\langle k|$, and performing an additional sum on both sides with respect to a dummy summation index, we arrive at

$$\delta E_k=-\frac{2i\alpha}{3}\sum_{l,n,m}\omega_{nm}^3(\mathbf{x}_{kl}\cdot\mathbf{x}_{nm})e^{i(\omega_{nm}+\omega_{kl})t}\ . \tag{68}$$

Now we may integrate both sides over all time t, and then divide by 2π to extract an energy shift of the kth level, as per the method of (37), to be

$$\delta E_k=-\frac{2i\alpha}{3}\sum_{l,n,m}\omega_{nm}^3(\mathbf{x}_{kl}\cdot\mathbf{x}_{nm})\delta(\omega_{nm}+\omega_{kl})\ . \tag{69}$$

The δ function expresses a conservation of energy condition, which can be satisfied by the choice $k=m$ and $l=n$. [See Eqs. (32).] This finally yields an imaginary energy shift given

$$\delta E_k=-\frac{2i\alpha}{3}\sum_l\omega_{lk}^3|\mathbf{x}_{lk}|^2\ , \tag{70}$$

which can be rewritten as

$$\delta E_1=-\frac{2i\alpha}{3}\omega_0^3|\mathbf{x}_{21}|^2=-iA\ , \tag{71a}$$

$$\delta E_2=-\frac{2i\alpha}{3}\omega_0^3|\mathbf{x}_{21}|^2=+iA\ , \tag{71b}$$

which, when inserted back into Eq. (66), gives

$$\psi_1(\mathbf{x},t)=\psi_1(\mathbf{x})e^{-iE_n^0t+At}\ , \tag{72a}$$

$$\psi_2(\mathbf{x},t)=\psi_2(\mathbf{x})e^{-iE_n^0t-At}\ . \tag{72b}$$

Hence the excited level shows the correct exponential decay, but the ground state exhibits a clearly nonphysical exponential growth. Intuitively, this is because, at the somewhat primitive level of neoclassical theory, the radiation reaction perturbation V' is just as likely to perturb the ground state as it is the excited state. It is tempting to compare (72b) with the so-called *runaway solutions* of classical radiation reaction theory. This problem of the decay of the ground state did not arise in the complete self-field treatment given earlier. In the self-field expression (36) spontaneous emission emerges as the residue of a contour integral. For the ground state there is no pole enclosed by the contour, and hence the residue is zero, and the ground state is stable.

VII. CONCLUSION

In this paper we discussed the self-field approach to QED, emphasizing the theory's origin in the action at a distance theory of classical electrodynamics of Wheeler and Feynman. We showed how the theory could be used to give a nonrelativistic account of spontaneous emission and the Lamb shift in a two-level atom, with the usual exponential decay profile found in the standard approach.

We then showed how misuse of the superposition principle could lead to an incorrect prediction of a chirruped decay profile, as predicted in the neoclassical theory of Crisp and Jaynes. Reviewing the neoclassical theory, we showed that the chirruped decay could be eliminated by avoiding the use of the superposition principle, and that an excited two-level atom decays in the proper exponential fashion. The neoclassical theory appears to predict a runaway solution for a gound-state electron. We believe that this is due to the rather *ad hoc* fashion in which the neoclassical theory accounts for the radiation reaction field.

The self-field approach apparently contains elements of the neoclassical theory, but is more comprehensive in scope. The covariant elimination of the self-field $A_\mu^{\rm self}$ of the electron through use of an EM Green's function takes into consideration the electron's radiation reaction field in a compelling and natural manner. In addition, self-field QED can make predictions about the Lamb shift, vacuum polarization, $g-2$, etc.—all of which seem to be beyond the scope of the original neoclassical theory. The theory of the general covariant self-field QED is given elsewhere.[15]

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