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ARTICLES

Hamilton, Rodrigues, and the Quaternion Scandal

*What went wrong with one of the major mathematical discoveries of the nineteenth century**

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Some of the best minds of the nineteenth century—and this was the century that saw the birth of modern mathematical physics—hailed the discovery of quaternions as just about the best thing since the invention of sliced bread. Thus James Clerk Maxwell, [**31**, p. 226], the discoverer of electromagnetic theory, wrote:

> The invention of the calculus of quaternions is a step towards the knowledge of quantities related to space which can only be compared, for its importance, with the invention of triple coördinates by Descartes. The ideas of this calculus, as distinguished from its operations and symbols, are fitted to be of the greatest use in all parts of science.

Not everybody, alas, was of the same mind, and some of the things said were pretty nasty:

> Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way, including Clerk Maxwell. (Lord Kelvin, letter to Hayward, 1892; see [**38**, vol. II, p. 1070].)

Such robust language as Lord Kelvin's may now be largely forgotten, but the fact remains that the man in the street is strangely averse to using quaternions. Side by side with matrices and vectors, now the *lingua franca* of all physical scientists, quaternions appear to exude an air of nineteenth-century decay, as a rather unsuccessful species in the struggle-for-life of mathematical ideas. Mathematicians, admittedly, still keep a warm place in their hearts for the remarkable algebraic properties of quaternions, but such enthusiasm means little to the harder-headed physical scientist.

This article will attempt to highlight certain problems of interpretation as regards quaternions which may seriously have affected their progress, and which might explain their present parlous status. For claims were made for quaternions which quaternions could not possibly fulfil, and this made it difficult to grasp what quaternions are excellent at, which is handling rotations and double groups. It is

*This article follows closely material from Chapters 1 and 12 of *Rotations, Quaternions, and Double groups*, by Simon L. Altmann, Clarendon Press, Oxford, 1986.

essentially the relation between quaternions and rotations that will be explored in this paper and the reader interested in double groups will find this question fully discussed in my recent book [**1**].

The Men Involved: Hamilton and Rodrigues

It is not possible to understand the quaternions' strange passage from glory to decay unless we look a little into the history of the subject, and the history of quaternions, more perhaps than that of any other nineteenth-century mathematical subject, is dominated by the extraordinary contrast of two personalities, the inventor of quaternions, Sir William Rowan Hamilton, Astronomer Royal of Ireland, and Olinde Rodrigues, one-time director of the Caisse Hypothécaire (a bank dedicated to lending money on mortgages) at the Rue Neuve-Saint-Augustin in Paris [**4**, p. 107].

Hamilton was a very great man indeed; his life is documented in minute detail in the three volumes of Graves [**15**]; and a whole issue in his honour was published in 1943, the centenary of quaternions, in the *Proceedings of the Royal Irish Academy*, vol. A50 and, in 1944 in *Scripta Mathematica*, vol. 10. There are also two excellent new biographies [**23**], [**33**] and numerous individual articles (see, e.g. [**26**]). Of Hamilton, we know the very minute of his birth, precisely midnight, between 3 and 4 August, 1805, in Dublin. Of Olinde Rodrigues, despite the excellent one-and-only published article on him by Jeremy Gray [**16**], we know next to nothing. He is given a mere one-page entry in the Michaud *Biographie Universelle* [**32**] as an 'economist and French reformer'. So little is he known, indeed, that Cartan [**6**, p. 57] invented a nonexistent collaborator of Rodrigues by the surname of Olinde, a mistake repeated by Temple [**37**, p. 68]. Booth [**4**] calls him *Rodrigue* throughout his book, and Wilson [**41**, p. 100] spells his name as *Rodriques*.

Nothing that Rodrigues did on the rotation group—and he did more than any man before him, or than any one would do for several decades afterwards— brought him undivided credit; and for much of his work he received no credit at all. This Invisible Man of the rotation group was probably born in Bordeaux on 16 October 1794, the son of a Jewish banker, and he was named Benjamin Olinde, although he never used his first name in later life. The family is often said to have been of Spanish origin, but the spelling of the family name rather suggests Portuguese descent (as indeed asserted by the *Enciclopedia Universal Illustrada Espasa-Calpe*). He studied mathematics at the École Normale, the École Polytechnique not being accessible to him owing to his Jewish extraction. He took his doctorate at the new University of Paris in 1816 with a thesis which contains the famous 'Rodrigues formula' for Legendre polynomials, for which he is mainly known [**14**].

The next 24 years or so until, out of the blue, he wrote the paper on rotations which we shall discuss later, are largely a blank as far as Rodrigues's mathematics is concerned. But he did lots of other things. The little that we know about Rodrigues relates to him mainly as a paranymph of Saint-Simon, the charismatic Utopian Socialist, whom he met in May, 1823, two months after Saint-Simon's attempted suicide. So, we read [**40**, p. 30] that the banker Rodrigues helped the poor victim in his illness and destitution, and supported him financially until his death in 1825. That Rodrigues must have been very well off we can surmise from Weill's reference to him as belonging to high banking circles, on a par with the wealthy Laffittes [**40**, p. 238]. After Saint-Simon died, with Rodrigues by his bedside, the latter shared the headship of the movement with Prosper Enfantin, an old friend and disciple of Saint-Simon. Thus he became *Père Olinde* for the acolytes. But the union did not last very long: in

1832 Rodrigues repudiated Enfantin's extreme views of sexual freedom and he proclaimed himself the apostle of Saint-Simonism. In August of that year he was charged with taking part in illegal meetings and outraging public morality, and was fined fifty francs [4]. Neither of the two early historians of Saint-Simonism, Booth and Weill, even mention that Rodrigues was a mathematician: the single reference to this is that in 1813 he was Enfantin's tutor in mathematics at the École Polytechnique. Indeed, all that we know about him in the year 1840 when he published his fundamental paper on the rotation group is that he was 'speculating at the Bourse' [4, p. 216].

Besides his extensive writings on social and political matters, Rodrigues published several pamphlets on the theory of banking and was influential in the development of the French railways. He died in Paris almost forgotten, however [32]. Even the date of his death is uncertain: 26 December 1850, according to the *Biographie Universelle* [32], or 17 December 1851, according to Larousse [27]. Sébastien Charléty [9, pp. 26,294], although hardly touching upon Rodrigues in his authoritative history of Saint-Simonism, gives 1851 as the year of Rodrigues's death, a date which most modern references seem to favour.

Hamilton survived Rodrigues by fourteen years and had the pleasure, three months before his death in 1865, to see his name ranked as that of the greatest living scientist in the roll of the newly created Foreign Associates of the American National Academy of Sciences. And quite rightly so: his achievements had been immense by any standards. In comparison with Rodrigues, alas, he had been born with no more than a silver-plated spoon in his mouth: and the plating was tarnishing. When he was three the family had to park various children with relatives and William was sent to his uncle, the Rev. James Hamilton, who ran the diocesan school at Trim. That was an intellectually explosive association of child prodigy and eccentric pedant: at three William was scribbling in Hebrew and at seven he was said by an expert at Trinity College, Dublin, to have surpassed the standard in this language of many Fellowship candidates. At ten, he had mastered ten oriental languages, Chaldee, Syriac, and Sanscrit amongst them plus, of course, Latin and Greek and various European languages. This is, at least, the received wisdom on Hamilton and it may contain an element of legend: it is pretty clear, e.g., that his knowledge of German was not all that strong in later life and the veracity of the reports on these linguistic feats is disputed by O'Donnell [33]. Mathematics—if one does not count mental arithmetic, at which he was prodigious—came late but with a bang when, at seventeen, reading on his own Laplace's *Mécanique Céleste*, he found a mistake in it which he communicated to the President of the Irish Academy. His mathematical career was already set in 1823 when, still seventeen, he read a seminal paper on caustics before the Royal Irish Academy.

From then on Hamilton's career was meteoric: Astronomer Royal of Ireland at 22, when he still had to take two quarterly examinations as an undergraduate, knight at 30. Like Oersted, the Copenhagen pharmacist who had stirred the world in 1820 with his discovery of the electromagnetic interaction, Hamilton was a Kantian and a follower of the *Naturphilosophie* movement then popular in Central Europe. For Hamilton 'The design of physical science is... to learn the language and to interpret the oracles of the universe' (Lecture on Astronomy, 1831, see [15, vol. I, p. 501]). He discusses in 1835, prophetically (because of the later application of quaternions in relativity theory), "Algebra as the Science of Pure Time". He writes copiously both in prose and in stilted verse, engages in a life-long friendship with Wordsworth, and goes to Highgate in the spring of 1832 to meet Coleridge, whom he visits and with whom he corresponds regularly in the next few years, the poet praising him for his understanding 'that Science...needs a Baptism, a regeneration in Philosophy' or Theosophy [15, vol. I, p. 546].

The Discovery of Quaternions

Hamilton had been interested in complex numbers since the early 1830's and he was the first to show, in 1833, that they form an algebra of couples. (See [22, vol. III].) I shall review briefly his ideas so as to lead the way to quaternions, but, here and hereafter, I shall use my own notation in order to avoid ambiguities. First, we define imaginary units, 1 and i with the well-known multiplication rules in TABLE 1. Then the elements of the algebra are the complex numbers $\mathbb{A} = a1 + A\mathsf{i}$, with a and A real.

TABLE 1. Multiplication table of the imaginary units.

	1	i
1	1	i
i	i	-1

Of course, to say that they form an algebra merely means that the formal rules of arithmetical operations are valid for the objects so defined. Thus, given $\mathbb{A}$ and a similarly defined $\mathbb{B}$, their product is

$$\mathbb{A}\mathbb{B} = ab - AB + \mathsf{i}(aB + bA). \tag{1}$$

We can now write the complex numbers $\mathbb{A}$ and $\mathbb{B}$ as *couples* (or *ordered pairs*)

$$\mathbb{A} = [\![a, A]\!], \qquad \mathbb{B} = [\![b, B]\!], \tag{2}$$

and their product is also a couple:

$$\mathbb{A}\mathbb{B} = [\![ab - AB, aB + bA]\!]. \tag{3}$$

Hamilton also recognized that the real number a can be written as the complex couple

$$a = [\![a, 0]\!]. \tag{4}$$

For the next ten years Hamilton's mind was occupied, if not obsessed, with two problems. On the one hand, Hamilton tried to extend the concept of the complex number as a couple in order to define a triple, with one real and two imaginary units. This however, not even he could do. On the other hand, the concept of a vector was beginning to form in his mind. It must be remembered that in the 1830's not even the word *vector* existed, although people were playing about, in describing forces and such other quantities, with concepts that we would recognize today as at least vector like. It is pretty clear that during this gestation period, as a result of which Hamilton would eventually invent the notion of vector, he had built up in his mind a picture of the addition and of some form of multiplication of vectors, but there was an operation which baffled him in the extreme: coming down the stairs for breakfast, Hamilton often could hear his elder son asking: 'Father, have you now learned how to divide vectors?'. Out of this preoccupation Hamilton was to invent the most beautiful algebra of the century, but he was also to feed the fever that eventually led him to corrupt his own invention.

We must now come to Monday, 16 October 1843, one of the best documented days in the history of mathematics and which, by one of those ironies of fate, happened to be the 49th birthday of Olinde Rodrigues, whose work, however ignored, was to give a new meaning to Hamilton's creation. Hamilton's letter to his youngest son, of 5 August 1865 [**15**, vol. II, p. 434], is almost too well known, but bears brief repetition. The morning of that day Hamilton, accompanied by Lady Hamilton, was walking along the Royal Canal in Dublin towards the Royal Irish Academy, where Hamilton was to preside at a meeting. As he was walking past Broome Bridge (referred to as Brougham Bridge by Hamilton and called by this name ever since), Hamilton, in a flash of inspiration, realized that three, rather than two, imaginary units were needed, with the following properties:

$$\mathsf{i}^2=\mathsf{j}^2=\mathsf{k}^2=-1, \qquad \mathsf{ij}=\mathsf{k}, \qquad \mathsf{ji}=-\mathsf{k}, \tag{5}$$

and cyclic permutation. As everyone knows, and de Valera was to do almost one century later on his prison wall, Hamilton carved these formulae on the stone of the bridge: poor Lady Hamilton had to wait. Armed now with four units, Hamilton called the number

$$\mathbb{A}=a1+A_x\mathsf{i}+A_y\mathsf{j}+A_z\mathsf{k}, \tag{6}$$

where the coefficients here are all real, a *quaternion*. Thus were quaternions born and baptized: it was entered on the Council Books of the Academy for that day that Mr. W. R. Hamilton was given leave to read a paper on quaternions at the First General Meeting of the Session, 13 November 1843.

One of the various falsehoods which have to be dispelled about quaternions is the origin of their name, since entirely unsupported sources are often quoted, in particular Milton, *Paradise Lost*, vol. 181 [28, p. 70] and the *Vulgate*, Acts 12:4 [**37**, p. 46]. Of course, we know that Milton was a favourite poet of Hamilton at 24 [**15**, vol. I, p. 321], and to suggest that he was not aware of *Acts* and the apprehension of Peter by a quaternion of soldiers would be absurd. (These references appear in fact in Dr. Johnson's *Dictionary*, which was familiar to every schoolboy of the time.) But no one with the slightest acquaintance with Hamilton's thought would accept the obvious when the recondite will do. In his *Elements of Quaternions* [**21**, p. 114]) we find our first clue: 'As to the *mere word*, *quaternion*, it signifies primarily (as is well known), like its Latin original, "Quaternio" or the Greek noun τετραχτυς, a Set of Four'. The key word here is 'tetractys,' and there is evidence for this coming from Hamilton's closest, perhaps his only real pupil, P. G. Tait, who, writing in the *Encyclopaedia Britannica* (see article on Quaternions in the XIth edition) says: 'Sir W. R. Hamilton was probably influenced by the recollection of its Greek equivalent the Pythagorean Tetractys..., the mystic source of all things...'. That Tait very much believed in this is supported by the unattributed epigraph in Greek in the title page of his own treatise on quaternions [**36**]: these are verses 47 and 48 of *Carmen Aureum* (*Golden Song*), a Hellenistic Pythagorean poem much in vogue in the Augustan era, the full text of which appears in Diehl [**11**, p. 45]. Of course, the concept of the tetractys embodying, as we shall see, multiple layers of meaning in a single word, must have attracted Hamilton: for Pythagoras, having discovered that the intervals of Greek music are given by the ratios 1:2. 3:2. 4:3 made it appear that *kosmos*, that is, order and beauty, flow from the first four digits, 1, 2, 3, 4, the sum of which gives the perfect number 10, and is symbolized by the sacred symbol, the tetractys:

0
0 0
0 0 0
0 0 0 0

(A famous depiction of the tetractys can be seen in the *School of Athens*, the fresco by Raphael at the Vatican where, anachronistically, the sacred symbol is given in Latin numerals in the figure held in front of Pythagoras.) The Pythagoreans used to take an oath by the tetractys, as recorded by Sextus Empiricus (see [24, p. 233]): 'The Pythagoreans are accustomed sometimes to say "All things are like number" and sometimes to swear this most potent oath: 'Nay, by him that gave to us the *tetractys*, which contains the fount and root of ever-flowing nature.' That the tetractys exercised the imagination of Hamilton, there is no doubt: besides the cryptic footnote in the *Elements*, already quoted, we find Augustus DeMorgan (with whom Hamilton entertained a very copious correspondence) acknowledging on 27 December 1851 a sonnet from Hamilton (apparently lost) on the tetractys. It is tempting to speculate that Hamilton might have been introduced to the tetractys by Coleridge, who called it 'the adorable tetractys, or tetrad' (see [2, p. 252]) and who referred to it many times.

In Praise of Hamilton: the Algebra of Quaternions

In comparison with the binary form (2) of a complex number, the quaternion (6) can also be written as a couple of a real number a and a vector $\mathbf{A}$ of components A_x, A_y, A_z, (as already said, we use modern rather than historical notation):

$$\mathbb{A} = [\![a, \mathbf{A}]\!], \quad \mathbf{A} = (A_x, A_y, A_z). \tag{7}$$

Just as for the complex numbers, in order to multiply two such objects we need the multiplication table of the quaternion units, which follows at once from eqn (5) and it is given in Table 2. Consider now a second quaternion $\mathbb{B}$, write both $\mathbb{A}$ and $\mathbb{B}$ as in eqn (6) and, on using the table, their product follows at once in the same manner as that in (3):

$$\mathbb{A}\mathbb{B} = [\![ab - \mathbf{A}.\mathbf{B}, a\mathbf{B} + b\mathbf{A} + \mathbf{A} \times \mathbf{B}]\!]. \tag{8}$$

Although Hamilton did not give names or symbols for these operations, it is here that the scalar and vector products of two vectors appear for the first time in history. We can now go back to TABLE 2 and reflect a little about why Hamilton made the product $\mathbf{ij}$ noncommutative. Not only was this the first time that a noncommutative product appeared in mathematics, but this was a true stroke of genius. Remember that Hamilton wanted to divide vectors: he never really achieved this (neither was it worth trying) but the point is that, because of this, he was after a *division algebra*, i.e., one in which the quotient of an element of the algebra by any other nonnull element always exists. Now, a necessary condition for a division algebra is this: the product of two elements of the algebra must vanish if and only if one of the factors is the null element. TABLE 2 is designed so that this happens, as can easily be verified from the resulting multiplication rule, given by (8). A counterexample will be instructive. Suppose we take $\mathbf{ij}$ and $\mathbf{ji}$ as equal in TABLE 2, and, similarly, for the other products. Then, under the new multiplication rules, it is very easy to verify that the product of the two nonnull elements

$$\mathbb{A} = \mathsf{i} + \mathsf{j}, \qquad \mathbb{B} = -\mathsf{i} + \mathsf{j}, \tag{9}$$

vanishes.

TABLE 2. Multiplication table of the quaternion units.

	1	i	j	k
1	1	i	j	k
i	i	−1	k	−j
j	j	−k	−1	i
k	k	j	−i	−1

Hamilton's everlasting monument (see [**30**]) is his construction of objects which, except for commutativity, obey the same algebra as that of the real and complex numbers and which therefore, like them, form a division algebra: and Hamilton was aware of this—although he could not foresee that his brain-child was going to receive at the hands of Frobenius in 1878 the supreme accolade of being proved to be the only possible algebra, in addition to the real and complex numbers, with this property.

The Trouble Starts

Now back to 16 October 1843. That same evening Hamilton wrote a long and detailed draft of a letter to his friend John Graves, first published by A. J. McConnell [**29**] and included in Hamilton [**22**, vol. III]. Next day a final letter was written and sent, later published in the *Philosophical Magazine* [**17**]. The November report to the Irish Academy was published almost at the same time [**18**]. We can thus follow almost hour by hour Hamilton's first thoughts on quaternions. Although in the morning of the glorious day he had been led to the discovery through the algebra of the quaternions, by the evening (and in this he acknowledges the influence of Warren [**39**]), he had been able to recognize a relation between quaternions and what we now call rotations. And in this, sadly, we cannot but see the germ of the canker that eventually consumed the quaternion body. Three separate themes, ever present in Hamilton's mind, contributed to this infirmity.

As regards the first theme: as in (4), Hamilton identified a real number with a real quaternion:

$$a = [\![a, \mathbf{0}]\!]. \tag{10}$$

Nothing wrong here, but it invited Hamilton to go on and identify a *pure quaternion* (a quaternion with a null scalar part) with a *vector*, a word which he invented for this purpose in 1846 [**19**, p. 54]:

$$\mathsf{A} = [\![0, \mathbf{A}]\!]. \tag{11}$$

As the inventor of the vector he was entitled to call this object anything he wanted but the problem is that by this time people were already thinking about forces and such like objects very much as we think of vectors today and that the identification of Hamilton's 'vectors' with what they had in mind created a great deal of confusion. The apparently innocent convention (11) entails in fact two serious problems. That something here was a serious worry must have been evident for decades, as Klein [**25**, p. 186], one of the leading nineteenth-century geometers, implied himself. Yet, the first explicit statement to the effect that something is wrong here which I have been able to find is as recent as 1958 by Marcel Riesz [**34**, p. 21]: 'Hamilton and his school professed that the quaternions make the study of vectors in three-space unnecessary since every vector can be considered as the vectorial part...of a quaternion...this interpretation is grossly incorrect since the vectorial part of a quaternion behaves with respect to coordinate transformations like a bivector or "axial" vector and not like an ordinary or "polar" vector.' However damning this statement is, it is only half the story, since the pure quaternion (11) is not anything like a vector at all: we shall see that it is a *binary rotation*, that is a rotation by π. The left-hand side of (11) should be written as $\mathbb{A}$ and carefully distinguished from the vector **A**. The fact that neither Hamilton, nor his successors to the present day, introduced any notational distinction between these two objects is the source of extraordinary confusion, as we shall soon witness ourselves.

Hamilton's second theme was closely connected to his first and has already been mentioned: he wanted to find a definition of the quotient of two vectors and however grateful we must be for this obsession, which has given us the last possible division algebra, we shall soon see that it led Hamilton to an interpretation of quaternions and of their operations which is not right.

Hamilton's capacious mind could not be at rest until he understood not just the formalities of his work but also what went on behind the scenes, and he had to understand the physical or geometrical meaning of equating the square of the imaginary or quaternion unit, $\mathbf{i}^2$, with -1. This was his third everlasting theme, for which he took a cue from Argand, who had observed in 1806 that the imaginary unit $\mathbf{i}$ rotates what we would now call a vector in the Argand plane by $\pi/2$, which made it possible to visualize the relation in question. From that point of view, in fact, $\mathbf{i}^2$ should be a rotation by π which, duly enough, multiplies each vector of the plane by the factor -1. For this reason, Hamilton always identified the quaternion units with *quadrantal rotations*, as he called the rotations by $\pi/2$ (see [**20**, p. 64, art. 71]). Clifford [**10**, p. 351] associates himself with this interpretation which he presents with beautiful clarity. The sad truth is that, however appealing this argument is, to identify the quaternion units with rotations by $\pi/2$ is not only not right, but it is entirely unacceptable in the study of the rotation group: we shall see, in fact, that they are nothing else except binary rotations.

Quaternions and Rotations: the First Steps

Already during the first day of his creation Hamilton knew what he had to do in order to define rotations. Since rotations must leave the lengths of vectors invariant, and since for Hamilton a vector was a particular case of a quaternion, the first thing we had to do is to define the *norm* or length $|\mathbb{A}|$ of a quaternion. He defined for this purpose the *conjugate quaternion*

$$[\![a,\mathbf{A}]\!]^* = [\![a,-\mathbf{A}]\!]. \tag{12}$$

The norm is now defined as follows:

$$\mathbb{A}\mathbb{A}^* = [\![a,\mathbf{A}]\!][\![a,-\mathbf{A}]\!] = [\![a^2+A^2,0]\!] = a^2+A^2 = |\mathbb{A}|^2. \tag{13}$$

A quaternion of unit norm is called *a normalized quaternion* and, although Hamilton also considered more general quaternions, these will be the only quaternions which we shall need for the purposes of this article. It was easy for Hamilton to prove that *the product of two normalized quaternions is a normalized quaternion.* (See, e.g. [**1**, p. 208].) We are now ready to accompany Hamilton in performing an extraordinary piece of legerdemain.

An Optical Illusion: the Rectangular Rotation

A rotation is an operation which transforms a unit, that is normalized, position vector (a vector with its tail at the origin), **r**, into another unit position vector **r**′. If we go along with Hamilton and identify the vector **r** with the pure quaternion $\mathbb{R}$ equal to $[\![0, \mathbf{r}]\!]$, the latter is clearly normalized. In order to achieve a rotation and keep the normalization requirement of $\mathbb{R}$, all that we need is to act on $[\![0, \mathbf{r}]\!]$ with a normalized quaternion. (See the italicized statement in the previous paragraph.) Let us, therefore, choose for this purpose the quaternion

$$\mathbb{A} = [\![\cos\alpha, \sin\alpha\mathbf{n}]\!], \qquad |\mathbf{n}| = 1, \tag{14}$$

which is clearly normalized. (Here **n** is a unit position vector.) This is not the end of the story, however, because we must require that the product $\mathbb{A}\mathbb{R}$ be not only normalized, but also a pure quaternion $\mathbb{R}'$ of the form $[\![0, \mathbf{r}']\!]$ which Hamilton would identify with the rotated position vector **r**′. This is what Hamilton envisioned on the same evening of Creation Day, and he also realized that for this idea to work it was necessary that the vector **n**, which he called the *axis of the quaternion*, be normal to the vector **r**. (This is why this is called the *rectangular transformation*.) To verify that this works is child's play on using the quaternion multiplication rule (8). Given that $\mathbf{r}, \mathbf{n} = 0$, then

$$\begin{aligned}\mathbb{A}\mathbb{R} &= [\![\cos\alpha, \sin\alpha\mathbf{n}]\!][\![0, \mathbf{r}]\!] \\ &= [\![0, \cos\alpha\mathbf{r} + \sin\alpha(\mathbf{n}\times\mathbf{r})]\!] = [\![0, \mathbf{r}']\!] = \mathbb{R}'.\end{aligned} \tag{15}$$

If we briefly avert our gaze while Hamilton rewrites this equation as

$$\mathbb{A}\mathbf{r} = \mathbf{r}', \tag{16}$$

then the job is done, that is, the quaternion $\mathbb{A}$ transforms the unit position vector **r** into another unit position vector **r**′ and, therefore, has rotated **r** into **r**′. What is more: it is clear from FIGURE 1 that the angle of rotation is α. Thus Hamilton identified the

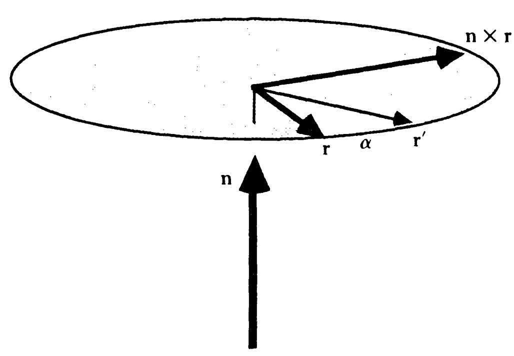

FIGURE 1
The rectangular transformation. The vector r′ is defined in eqn (15).

quaternion (14) with a rotation around the axis **n** of the quaternion by the angle α of rotation of the quaternion.

All this is so wonderfully convincing that it is difficult to believe that there is anything wrong here. Moreover, Hamilton immediately obtained confirmation that one of his themes was coming out all right, for it was clear to him that the quaternion units in this picture are quadrantal rotations, as it is immediate from (14): on comparing this equation with, say, the quaternion unit $\mathbb{k}$, given by $[\![0, \mathbf{k}]\!]$, one can see at once that the rotation angle to be associated with $\mathbb{k}$ must be $\pi/2$. His other theme was also coming out well here, since from (16) the quaternion $\mathbb{A}$ can be considered as the quotient of the vector $\mathbf{r}'$ by the vector $\mathbf{r}$. This picture of quaternions was thus so near Hamilton's heart that in his *Lectures on Quaternions* [**20**, p. 122], and ever after, the primary definition of a quaternion which he used was 'The quotient of two vectors, or the operator which changes one vector into another,' as later adopted by the *Oxford English Dictionary*, and this definition became the core of the quaternion dogma, thus causing endless damage. We shall see, in fact, that a quaternion can never operate on a vector, as (16) implies, and that this equation must always be understood as the quaternion product in (15).

The Comical Transformation (this heading contains a misprint)

The conical transformation was the means by which nature began to make its protest against Hamilton. Even accepting that a pure quaternion *is* a vector, we must ensure, in order to have a rotation, that the transform of a normalized pure quaternion is another normalized pure quaternion. Thus, a general rotation cannot be written as $\mathbb{A}\mathbb{R}$ because this product is not a pure quaternion unless, as we have seen, the axis of $\mathbb{A}$ is normal to **r**. Hamilton and his colleagues, therefore, searched for a quaternion transformation of a pure quaternion $\mathbb{R}$ under a normalized quaternion $\mathbb{A}$ which would always produce a normalized pure quaternion $\mathbb{R}'$. The result of this search was the following transformation:

$$\mathbb{A}\mathbb{R}\mathbb{A}^* = \mathbb{R}'. \tag{17}$$

It is, in fact, quite easy by means of (8) to verify that the left-hand side of this equation is normalized and pure. With a little bit of geometry (see [**1**, p. 214]), and assuming that $\mathbb{A}$ is given by (14), it can be proved that **r**, **n**, and $\mathbf{r}'$ are related as shown in FIGURE 2, i.e., that the vector **r** is rotated around **n** by the angle 2α.

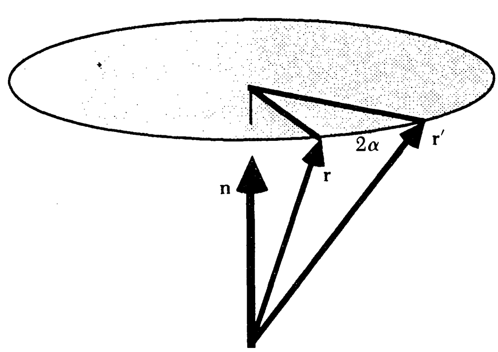

FIGURE 2
The conical transformation.

There are two problems here: first, the form of (17) has nothing at all to do with that of (15), so that it is no longer possible to say that the quaternion operates on a vector transforming it into another vector; even less that it is the quotient of two vectors. The second problem is this: which is the angle of rotation to be associated with a quaternion (14), α, as in the rectangular transformation, or 2α, as it turns out in the case of a general rotation? It is, perhaps, significant that Hamilton obtained (17) (writing the conjugate quaternion as the reciprocal, as it is valid for a normalized quaternion) but did not publish it for some time. Cayley [7] was the first to go into print, although in his collected papers [8, vol. I, p. 586, note 20] he concedes priority to Hamilton. Cayley notices that the components of $\mathbf{r}'$ are 'precisely those given for such transformations by M. Olinde Rodrigues.... It would be an interesting question to account, *a priori* for the appearance of these coefficients here'. Let us see what Hamilton has to say about this: 'The SYMBOL OF OPERATION $q(\ )q^{-1}$, where q may be called (as before) the operator quaternion, while the symbol (suppose r) of the operand quaternion is conceived to occupy the place marked by the parentheses... 'can be regarded as' a conical transformation of the operand round the axis of the operator, through *double the angle thereof.*' [**20**, p. 271, my italics]. It is clear that Hamilton, rather than accepting the result of the more general transformation (17) to recognize that the angle of rotation of the quaternion (14) is 2α, gives greater weight to the transformation (16) and keeps talking of the angle α in (14) as the angle of rotation. Naturally, whereas (16) had the shape that he expected, the form of (17), as Cayley stated, could not be explained. It is, perhaps, because of this that, although FIGURE 2 is nothing else than the most general rotation of a vector, Hamilton refers to it with the *ad hoc* name of *conical rotation*, as if it were a particular case of the transformation of a vector. It had, instead, been Rodrigues who had recognized, three years before Hamilton's invention of quaternions, that the angle α in (14) is not the rotation angle but only half of it. But his paper, which had puzzled Cayley, was almost certainly never read by Hamilton and it was never again quoted by any of the major quaternionists. As for Cayley's question, it was probably never answered until 1986. (See [1, p. 214].)

The Rodrigues Programme

Hamilton constructed quaternions as an algebra, whence the elements of the algebra were given a dual role as operators (rotations) and operands (vectors). This was very lucidly explained by Clifford [**10**], but it must be clearly appreciated that, as we have already asserted, the status of vectors in this scheme is highly dubious, of which more later. Be that as it may, in Hamilton's approach rotations become subservient to the algebra, which opens the door to a variety of misinterpretations.

Historically, however, a treatment of rotations and quaternions had been going on for some years before 1843, quite independently of Hamilton and taking a diametrically opposed view to his. This treatment was entirely geometrical, and because it tried to do a simple job in a simple way it was clear and precise and it was entirely successful; but it was largely ignored by everyone.

Let us consider rotations of a unit sphere with fixed centre about various axes. The first problem which arises is whether, if we apply one rotation after another, the net result is a rotation of the sphere around some unique axis by some unique angle. Euler [**12**] proved algebraically that this is so, but he did not provide either a geometric or a constructive solution (i.e., a solution in which the axis and angle of the resultant rotation are determined geometrically or algebraically). It was the paper by Rodrigues in 1840 [35] which solved all these aspects of the problem. In its § 8 he describes most clearly, without a figure, a geometrical construction which, given the angles and axes of two successive rotations, determines the orientation of the resultant axis of rotation and the geometrical value of the angle of rotation. This construction is usually called in the literature the *Euler construction*, although Euler had nothing to do with it. Not only was this construction ignored by the quaternionists but it is not even mentioned in modern books on the rotation group. Although Hamilton himself rediscovered, geometrically [20, p. 328], the results of the Rodrigues construction, this is not a theorem to which either he or his commentators paid much attention (see [1, pp. 19–20]).

Let us represent a rotation around the axis $\mathbf{p}$ by the angle ϕ with the symbol $R(\phi\mathbf{p})$. Then, if we use the Rodrigues construction for the following product of rotations,

$$R(\alpha\mathbf{l})R(\beta\mathbf{m}) = R(\gamma\mathbf{n}), \tag{18}$$

it turns out that the axes $\mathbf{l}$, $\mathbf{m}$, and $\mathbf{n}$ form a spherical triangle with the angles shown in FIGURE 3. (Remember that in the left-hand side of (18) the rotation around $\mathbf{m}$ is applied first and it is followed by that around $\mathbf{l}$: this is the usual convention for reading operators.)

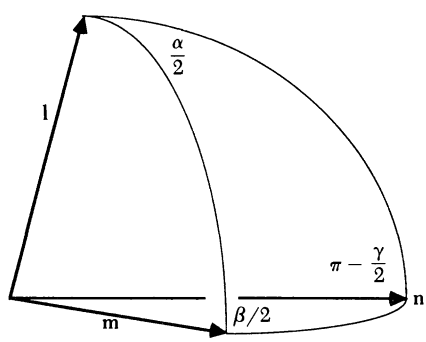

FIGURE 3
The product of the rotations $R(\alpha\mathbf{l})$ and $R(\beta\mathbf{m})$ is the rotation $R(\gamma\mathbf{n})$.

What is very remarkable about this very simple triangle is that the angles of the rotations appear in it as *half-angles*, and this is the first time that half-angles occur in the study of rotations. Their importance is absolutely crucial, as we shall see, and yet they were ignored by Euler and were never considered by Hamilton or his followers: it took more than forty years before their significance was appreciated. One would shirk nowadays at the solution of the spherical triangle in FIGURE 3 (see [1, p. 157]) but mathematical training in France on surveying and such was very good and Rodrigues was able to obtain quite easily expressions for the angle and axis of the resultant rotation, that is, the one on the right-hand side of (18), in terms of those of the factors which appear on its left. The following are Rodrigues's formulae exactly as he gave them, except that I have introduced vector notation, which was nonexistent in his time:

$$\cos\frac{\gamma}{2} = \cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\mathbf{l}.\mathbf{m}, \tag{19}$$

$$\sin\frac{\gamma}{2}\mathbf{n} = \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\mathbf{l} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\mathbf{m} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\mathbf{l}\times\mathbf{m}. \tag{20}$$

These formulae immediately suggest that a rotation $R(\alpha\mathbf{l})$ can be represented by a couple of a scalar and a vector (although this notation was not used by Rodrigues)

$$R(\alpha\mathbf{l}) = [\![\cos\frac{\alpha}{2}, \sin\frac{\alpha}{2}\mathbf{l}]\!], \tag{21}$$

so that the product (18) is written as follows

$$[\![\cos\frac{\alpha}{2}, \sin\frac{\alpha}{2}\mathbf{l}]\!][\![\cos\frac{\beta}{2}, \sin\frac{\beta}{2}\mathbf{m}]\!] = [\![\cos\frac{\gamma}{2}, \sin\frac{\gamma}{2}\mathbf{n}]\!], \tag{22}$$

with the parameters on the right-hand side of this equation being given by (19) and (20). It can immediately be seen that the multiplication rule for the couples so defined is identical with the multiplication rule (8) of Hamilton's quaternions! Rodrigues's couples are, therefore, quaternions, but the difference in parametrization between (21) and (14) is profound. We see at once that the conical transformation, which gives the angle of rotation as twice the angle which appears in the quaternion was right, and that Hamilton committed a serious error of judgement in basing his parametrization on the special case of the rectangular transformation.

Simple as the distinction is, the consequences are dramatic, and never more so than when we consider pure quaternions. From (21), it is clear that for a quaternion to be pure the angle of rotation must be π, that is, a pure quaternion is nothing other than a binary rotation:

$$[\![0, \mathbf{r}]\!] = R(\pi\mathbf{r}). \tag{23}$$

Thus, it is entirely wrong ever to identify a pure quaternion with a vector, as Hamilton had done in (11). This simple fact will exorcise all the demons so far lurking into our story. Before we do this, we must mention that the (four) rotation parameters in (21) are called the Euler-Rodrigues parameters in the literature. The reasons for this are entirely disreputable (see [**1**, p. 20]), since Euler never came near them: in particular, he never used half-angles which, as demonstrated by Rodrigues, are an essential feature of the parametrization of rotations.

The Resolution of the Paradoxes

Although quaternions are always rotations and never vectors, they allow us to mark points in space very much as a position vector does. Consider the unit sphere centered and fixed at the origin. A rotation of it determines a single point of the sphere which is called the *pole of the rotation*. This is the point of the sphere which is left invariant by the rotation and such that from outside it the rotation is seen as positive (counterclockwise). If we want to mark a point in space by means of rotation poles, it is sensible to use always binary rotations for this purpose, since, as it follows from (23) these are the nearest things to vectors that we can get within the quaternion algebra. (It should be stressed that this is purely a matter of convenience: the pole of any arbitrary rotation is just as good to denote a point of the unit sphere and thus to masquerade as a vector.) Let us now look again at (15) with the quaternion $\mathbb{A}$ in it given by

$$\mathbb{A}\mathbb{R} = \mathbb{R}', \qquad \mathbb{A} = [\![\cos\alpha, \sin\alpha\mathbf{n}]\!]. \tag{24}$$

If we compare the left-hand side of this line with (18) and (21) it says this: the product of a rotation by 2α around the axis $\mathbf{n}$, times a binary rotation around the axis

r, is a binary rotation around the axis **r**′:

$$R(2\alpha\mathbf{n})R(\pi\mathbf{r}) = R(\pi\mathbf{r}'). \tag{25}$$

The angle of rotation, which we already know, and the orientation of the axis **r**′ can, of course, be obtained from the quaternion multiplication rules, as given, e.g., in (19) and (20), but it will be instructive to obtain an independent geometrical verification, since this will show up the paradox involved in Hamilton's interpretation of the rectangular transformation. We do this in FIGURE 4. In order to multiply rotations we transform the unit sphere, whose intersection with the plane of the drawing is shown in FIGURE 4. The rotation axis **n** is perpendicular to and above the plane of the drawing. The rotation axis **r** is in the plane of the drawing and thus, as it must be in the rectangular transformation, is normal to **n**. A point above the plane of the drawing is represented with a cross and those below with a circle. In order to multiply the two rotations on the left of (25) we start with point 1 above the plane of the drawing. The first rotation to act on it (remember to read the left-hand side of (25) from right to left) is a rotation by π around **r** which takes it into the point 2 below the plane of the drawing. The rotation around **n** by 2α takes 2 to 3. Thus, the two combined operations take the point 1 above the drawing to the point 3 below the drawing, which is the effect of a binary rotation around the axis **r**′. Notice that the angle between the axes **r** and **r**′ is α and that this angle *is not the angle of rotation*. We can now see how Hamilton's optical illusion was performed. If in (24) we identify the quaternions $\mathbb{R}$ and $\mathbb{R}'$ with their corresponding vectors **r** and **r**′, FIGURE 4 now reads as the rotation of **r** into **r**′ by α. Incidentally, the correct reading of FIGURE 4 as stating that a rotation axis by 2α and a perpendicular binary axis determine another perpendicular binary axis at an angle α to the first one is so fundamental in crystallography that the whole of this science would collapse like a pack of cards if it were not true.

What about the conical transformation? I cannot go into all the details of the theory but a sketch will suffice. We must accept the following result (see [1, p. 215]). If we take a pole of a binary rotation **r** and we rotate this pole about an axis **n** by an angle α, the new pole thus obtained, **r**′, is the pole of another binary rotation given in the following form:

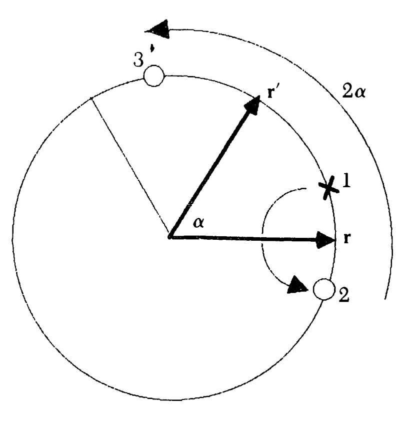

FIGURE 4
Product of a rotation by 2α around the axis **n** normal to the plane of the drawing, with a binary rotation around the axis **r**.

$$R(\alpha\mathbf{n})R(\pi\mathbf{r})R(-\alpha\mathbf{n}) = R(\pi\mathbf{r}'). \tag{26}$$

Because the corresponding quaternions must multiply in the same manner, we get immediately (17), since the inverse and conjugate quaternions are identical in our case. We must remember, though, to use the Rodrigues parametrization of the quaternion, as in (21), and not Hamilton's (14):

$$\mathbb{A}\mathbb{R}\mathbb{A}^* = \mathbb{R}', \qquad \mathbb{A} = [\![\cos\frac{\alpha}{2}, \sin\frac{\alpha}{2}\mathbf{n}]\!] \tag{27}$$

In this case it is heuristically possible to substitute $\mathbf{r}$ and $\mathbf{r}'$ for the quaternions $\mathbb{R}$ and $\mathbb{R}'$, with the poles of the binary rotations masquerading successfully as position vectors. This substitution, however, must never be done anywhere else. In particular, one must never attempt to operate with a quaternion on a vector, as is shown by the disastrous results of the crude interpretation of the rectangular transformation.

We must now discuss again the significance of the quaternion units. Because they are pure quaternions they must now be identified with binary rotations (rotations by π). This, for Hamilton, must have been absurd: the relation $\mathbf{i}^2$ equal to -1 must still be satisfied. But the product of two rotations by π about the same axis is a rotation by 2π. This is clearly the identity operation, i.e., one which does not change any vector, whereas we are now saying that it is equal to -1, i.e., that it changes the sign of all vectors in space. I believe that this is the reason Hamilton was forced to accept his parametrization, since this agreed with his picture of quaternion units as quadrantal rotations. Rodrigues, practical man as bankers must be, knew better than to worry about this strange result of his geometry—he did not carry, like Hamilton, all the world's problems on his shoulders. Nature and history, alas, were playing games with Hamilton. How was he to know that Cartan was going to discover in 1913 [**5**] objects (spinors) which are indeed multiplied by -1 under a rotation by 2π, exactly as Rodrigues's parametrization requires? Moreover, when the topology of the rotation group became understood in the 1920s through the work of Hermann Weyl, it became natural to accept that the square of a binary rotation multiplies the identity by -1 and thus behaves like the quaternion units. Though this should have shown the enormous importance of quaternions in the rotation group, they were by that time somewhat discredited, so that other much less effective parametrizations of the rotation group were in universal use.

It must be stressed that the Rodrigues approach to rotations, by emphasizing their multiplication rules and by regarding them entirely as operators, fully reveals the group properties of the set of all orthogonal rotations, the full orthogonal group $SO(3)$, as it is now called. The set of all normalized quaternions (in the Rodrigues parametrization) is a group homomorphic to $SO(3)$ and it is its *covering group*. Although I cannot go into the mathematical significance of this statement, its practical importance in quantum mechanics, e.g., can be easily understood: it permits the study of the transformation properties of the wave functions of the electron spin. It is for this purpose that quaternions are superb, because their use in dealing with rotations makes the work not only simpler but also more precise than with any other method.

The Decline

Hamilton was still under forty when he discovered quaternions, but he had more than twenty years of very productive research past him and was already showing the signs of having passed his prime. Financial and even sentimental worries are often mentioned, as well as overwork and an increasing consumption of alcohol [**33**]. I am inclined to believe that a major factor was that Hamilton himself was in some way aware of the internal contradictions of his doctrine and that he could not rest until he could peel off all layers of reality one by one to reach to the core. This was always beyond his power, since he was not prepared to renounce the, for him, essential picture of the quaternion units as quadrantal rotations. Be it as it may, his writing became more and more obscure: even his supporters found his books unreadable. And he himself became more isolated and eccentric.

E. T. Bell [3, Ch. 19] labelled the last twenty years of Hamilton 'The Irish Tragedy. Lanczos [**26**] compares them with Einstein's fruitless search for a unified field theory in his own last two decades. The truth is probably somewhere between these two views. For Hamilton suffered the weight of his own greatness: it was not enough for him to have an algebra, it was not enough to have a geometry, he had to 'interpret the oracles of the Universe' and the oracles trailed in front of him false clues that no one was to unravel for another three-score years after his death. If only he had known about spinors! The result, however, was that Hamilton, and to a much greater degree his followers, became dogmatic and intolerant (see [**25**, p. 182]) and that a great deal of sterile discussion ensued.

The last years of Hamilton, despite his immense fame, were not without worries: Continentals were spreading rumours that the great Gauss had actually discovered quaternions but had never bothered to publish. (They were right, as shown by Gauss's notes from 1819, published in 1900; see [**13**, vol. VIII, pp. 357–362].) In letters to De Morgan of January 1852 [**15**, vol. II, p. 490, vol. III, p. 330), Hamilton attacks these allegations. Curiously enough, of Rodrigues, who in 1840 not only had invented quaternions bar their name, but also published his formulae, there was never a word. Who would pay attention to a Socialist banker in matters mathematical?

After Hamilton's death his work began to give fruits but not in the direction which he had expected. His ideas of vectors and of their scalar and vector products were much too important so that people began to try and graft a new skin onto them in order to make these concepts usable. Grassmann in Germany and Heaviside in Britain moved some way in this direction, but one must admit that they were not much more transparent than Hamilton himself. It was left to Willard Gibbs of Yale to produce not only the first coherent picture of vectors and of their operations but also a good and successful working notation. This hardened the response of Hamilton's followers, who adopted a truly Byzantine posture, intent on stopping the flood of rebellion from across the Atlantic. Thus P. G. Tait [**36**, p. vi]:

> Even Prof. Willard Gibbs must be ranked as one of the retarders of quaternion progress, in virtue of his pamphlet on *Vector Analysis*, a sort of hermaphrodite monster, compounded of the notations of Hamilton and of Grassmann.

The kiss of life for quaternions, alas, much too late, came with the foundation in 1895 of an International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics: an acknowledgement that quaternions were a corpse in need of resuscitation. Alexander Macfarlane, who taught at Texas, became the leading force of the Association, which actually published a Bulletin from 1900 to 1923. The influence of this group extended as far as Japan, where Kimura in 1907 became one of the major influences of the Association. Nothing that they did, however, succeeded in preventing the rise of vectors and the consequent decline of quaternions.

A number of applications of quaternions went on appearing from time to time (see

[1, p. 18].) Ironically, however, by the time in the late twenties when quantum mechanics made the study of the rotation group crucial, thus giving the quaternions their real *raison d'être*, they had been submerged for much too long in the murky waters of their battle against vectors to be able to come to the surface again. They are much too useful in this context, though, for their time not to return.

There is a moral to this story: Rodrigues's applied mathematics yields a more accurate picture of the quaternions than that afforded by the pure mathematics of their inventor: it is probably a myth that pure mathematics is either born or can stand entirely on its own, although the aesthetic appeal of pure mathematics makes us often think otherwise.

Epilogue

After this article was communicated, a book was announced which contains new information about Rodrigues. This is the *Dictionnaire du Judaïsme Bordelais aux XVIII^e et XIX^e Siècles*, by Jean Cavignac (Archives Départementales de La Gironde, Bordeaux, 1987). This book contains a family tree of Rodrigues, which shows that his great-grandfather, Isaac Rodrigues-Henriques, was born in Spain, around 1689–91 and died in Bordeaux in 1767. He was indeed a banker but, contrary to previous belief, Olinde's father was an accountant. Surprisingly, the so-far universally accepted date of birth of Rodrigues is not right (so that the extraordinary coincidence with the day of the discovery of quaternions becomes a second-order effect). The correct date is 6 October 1795, and this is now unimpeachable, since his birth certificate is fully transcribed in a paper on Rodrigues by Paul Courteault (*Un Bordelais Saint-Simonien*) which I, like most people so far, had missed, since it was published in an obscure journal (*Revue Philomatique de Bordeaux*, Octobre–Décembre 1925, pp. 151–166). In accordance to this certificate, the date of birth was 14 *vendémiare* in the year IV of the Republic, at 1 p.m. Courteault (and Cavignac), instead, both agree with Michaud's date of death, 26.12.1850. Courteault also gives evidence that, although Olinde tried to enter the École Normale, he did not succeed in so doing, being prevented by his religion, so that how he learnt his advanced mathematics remains an unsolved mystery. Even worse, it appears that Rodrigues did not even attend the local secondary school (*Lycée*) at Bordeaux, so that we do not yet know anything at all about his formative years. It is of some value, however, that some of the traditional wisdom about this period, as repeated, I am afraid, in my paper, is now known to be worthless.

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