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Book Reviews

BOOK REVIEWS

Nick Trefethen's featured review of Elias Wegert's *Visual Complex Functions* is both very enthusiastic and well illustrated. One knows that we must move away from Ahlfors, when we compare the tables of Abramovitz and Stegen and the richness found in its successor, the NIST Digital Library of Mathematical Functions.

The other reviews in the issue treat combinatorics of permutations, quasi-stationary distributions, two-timing for Markov chains, risk analysis, contact problems, differential-algebraic equations, the history of weather forecasting, mathematical modeling for biologists, and the divergence theorem from the perspective of geometric measure theory.

We'll expect most of our readers will want to buy one or more of these recommended new publications.

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Book Reviews

Edited by Robert E. O'Malley, Jr.

Featured Review: Visual Complex Functions: An Introduction with Phase Portraits. *By Elias Wegert.* Birkhäuser/Springer Basel AG, Basel, 2012. $59.95. xiv+360 pp., softcover. ISBN 978-3-0348-0179-9.

I don't read many mathematics books cover to cover, but I just finished Elias Wegert's *Visual Complex Functions.* Four young colleagues and I studied it section by section over a period of a few months. What an experience! Anyone who works with complex variables should read this book.

Wegert's theme is that to understand functions in the complex plane, we should *look at them.* Now it is not news that complex analysis is a geometric subject, and in fact, Riemann's investigations in this area played a role in the creation of the field of topology. But when it comes to actually plotting a complex function $f(z)$, the tradition is sparse. Part of the problem is that since the domain and range spaces are each two-dimensional, the graph of $f(z)$ is four-dimensional. So what exactly can we plot?

There are some classic answers, like the beautiful figures in *Tables of Functions with Formulae and Curves* by Jahnke and Emde in 1909, which became the very first book republished by Dover Publications after they set up operations in 1941. Figure 1 shows Jahnke and Emde's iconic image of the absolute value of the gamma function in the complex plane, produced in an era of hand calculations and expert technical artists.

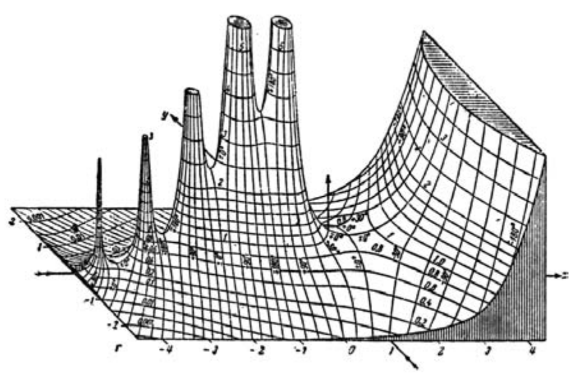

Fig. 1 *Gamma function from Jahnke and Emde.*

Times have changed, and computers with omnipotent color graphics are everywhere. Wegert's book is built on the idea of what he calls a *phase portrait* of a function. (He is not the first to work with such plots, but has carried the idea the

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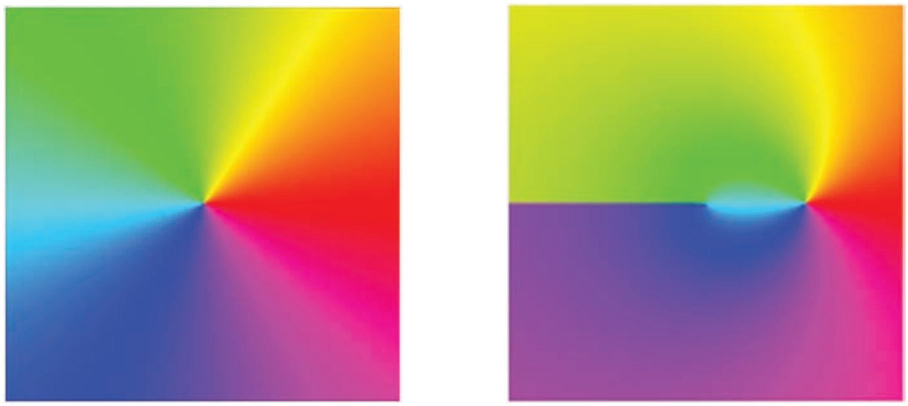

Fig. 2 *Phase portraits of z and* $\log(z)$.

furthest.) This is a 2D plot in which the color at each point z shows the phase of $f(z)$, that is, the number θ for which $f(z) = r\exp(i\theta)$ with $r > 0$. To explain the code, Figure 2(left) depicts a phase portrait of $f(z) = z$. Red corresponds to values of $f(z)$ that are positive real, light green to positive imaginary, cyan to negative real, and violet to negative imaginary.

In Figure 2(right), we see the function $f(z) = \log(z)$. Note the zero at $z = 1$, with the same sequence of colors around it as for $f(z) = z$ at $z = 0$. To the right the color is red, since $\log(z)$ is positive real for $z > 1$, and to the left it is cyan, since $\log(z)$ is negative real for $0 < z < 1$. Further left we see the branch cut along $(-\infty, 0]$. Above the branch cut, the predominant color is green, indicating a phase in the upper half-plane, and below it is violet, showing a phase in the lower half-plane.

After this warmup we can begin to make sense of more complicated functions, and my colleagues and I soon grew happily skilled at this. Consider, for example, the function f plotted in Figure 3(left). In the upper middle of the frame, f has a triple zero, that is, a zero of multiplicity 3. One sees this because the colors appear three times in sequence as one traces around the center point, with yellow to the left of red. In the right part of the frame it has a double pole, a pole of multiplicity 2. This is apparent because the colors are traced out twice in the reverse order, with yellow to the right of red. Near the bottom, it has an essential singularity, an isolated point singularity that is not a pole. The function shown in Figure 3(right) is $f(z) = \exp(1/z)$, the simplest example of a function with an essential singularity. I remember as a graduate student sensing a deep mystery in Picard's theorem, which asserts that all complex values except possibly one are attained in every neighborhood of an essential singularity. When you study figures like these, Picard's theorem begins to seem less mysterious.

Phase portraits don't show magnitudes directly, but they show them indirectly. We know that if yellow is to the left of red, this signifies a zero, implying that $|f|$ grows as one moves away from the zero. This happens on the left of the origin in the plot of $\exp(1/z)$, indicating the growth of this function there from values near zero as z moves leftward. To the right of the origin, yellow is to the right of red, indicating the decay of the function from values near infinity. Sometimes one might prefer to see the modulus directly, but Wegert makes a compelling case that there is usually more precise information in a phase portrait.

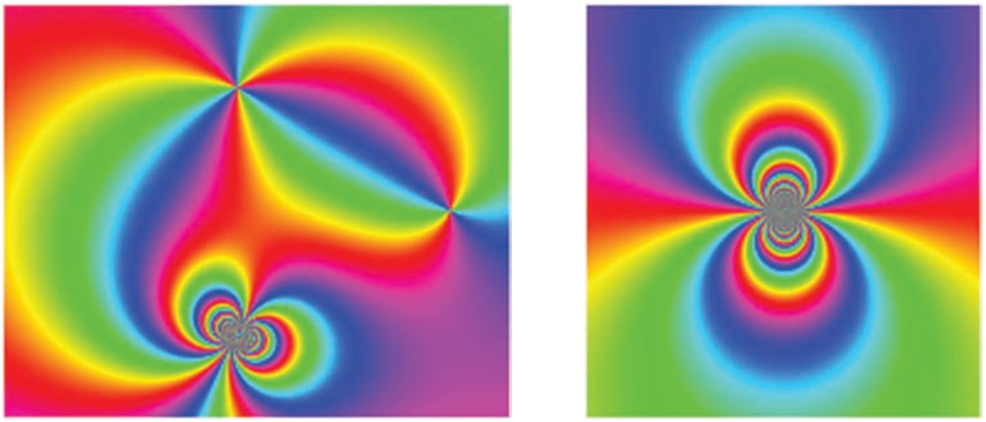

Fig. 3 *Poles, zeros, and essential singularities.*

In Figure 4(left), we see $f(z) = \sin(z)$. (This image is what Wegert calls an "enhanced phase portrait," with additional shading to further mark phase and amplitude.) The 2π-periodicity in the complex plane is evident, and one is reminded that $\sin(z)$ maps half-strips to half-planes. The point $z = 0$ at the center of the plot, for example, lies at the bottom of a half-strip that maps to the upper half-plane, whose predominant colors are orange, yellow, and green. The experienced eye can also infer from the colors the exponential increase of $|\sin(z)|$ as one moves away from the real axis, since yellow lies to the left of red in each parallel channel.

The right image in Figure 4 shows the Riemann zeta function $\zeta(z)$ in the region $-40 < \mathrm{Re} z < 10$, $-2 < \mathrm{Im} z < 48$. Notice that yellow is to the left of red at every rainbow point except one: these are the trivial zeros on the negative real axis and the nontrivial ones with $\mathrm{Re} z = \frac{1}{2}$. The exceptional point has yellow to the right of red, and this is the simple pole of $\zeta(z)$ at $z = 1$. In the left half-plane we see exponential growth in modulus as one moves leftward, since yellow again lies to the left of red in each channel.

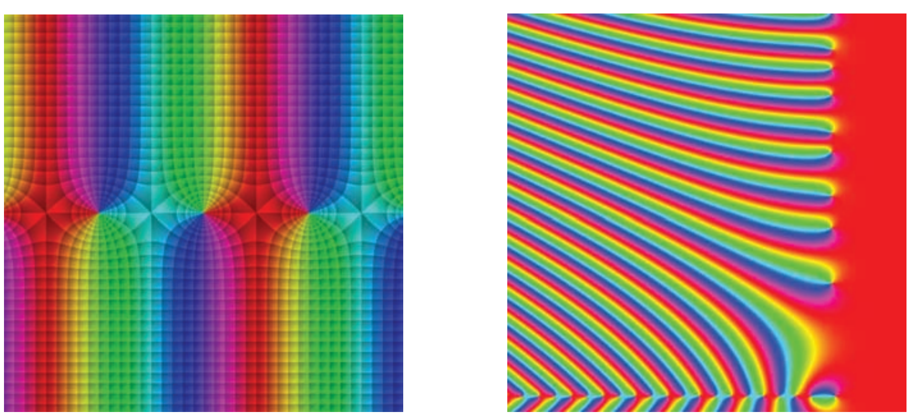

Fig. 4 $\sin(z)$ *and* $\zeta(z)$.

Portraits like these can give added depth to our understanding of many of the functions that have been studied by mathematicians in the past 200 years. Some of the functions explored in Wegert's book, at his Phase Plot Gallery at www.visual.wegert.com, and in his annual calendars of "Complex Beauties" coauthored with Gunter Semmler, include Bessel, Neumann, and Airy functions; theta, Bernoulli, and Painlevé functions; Fresnel integrals, elliptic functions, and continued fractions.

But Wegert's purpose is more than just the exploration of particular functions, for phase portraits can teach us fundamental principles of complex analysis. The two images shown in Figure 5 explore the idea of multivaluedness, on the left by following a chain of function elements twice around a square root singularity, and on the right with a 3D plot of the Riemann surface for the cube root. If you circle around twice on the left, or three times on the right, you return to your starting color.

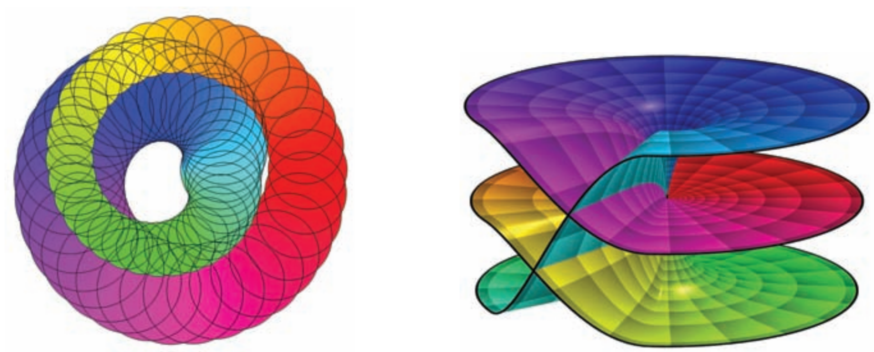

Fig. 5 *Multivalued functions $z^{1/2}$ and $z^{1/3}$.*

The plot in Figure 6(left) shows an analytic function f in the unit disk with a countably infinite set of zeros lying along a curve that spirals out to the unit circle. This picture makes it easy to understand that the circle is a natural boundary for this function, meaning that f cannot be analytically continued anywhere outside. This function happens to be an infinite Blaschke product, with constant modulus

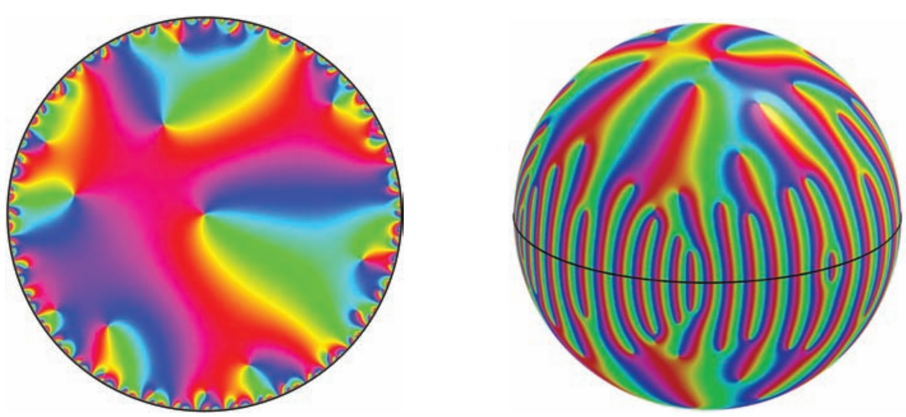

Fig. 6 *Blaschke products in the disk and on the Riemann sphere.*

in the limit as $|z| \to 1$, as is reflected in the color boundaries meeting the circle at right angles. In Figure 6(right), a simpler Blaschke product is shown, a finite one, analytically continuable to a meromorphic function on the whole Riemann sphere, i.e., a rational function. One sees an elegant symmetry of zeros in one hemisphere, poles symmetrically located in the other.

Phase portraits reveal fascinating properties of Taylor series as they converge to analytic functions. In Figure 7(left), $f(z)$ is the degree 60 Taylor polynomial for $1/(1-z)$. Inside the unit circle, the series closely represents the limit function, but outside it becomes a circus tent growing at the rate $|z|^{60}$ (yellow to the left of red in each channel). The transition from one behavior to the other is marked by a line of zeros that converges to the circle of convergence; this is *Jentzsch's theorem*. In Figure 7(right), the middle of the image looks like $\exp(z)$: the stripes are $2\pi i$-periodic and the growth is exactly exponential as one moves to the right, since the stripes are parallel with yellow to the left of red. Outside a certain curve of zeros, however, the behavior changes completely, and in fact, what is plotted here is not $\exp(z)$ but its Taylor polynomial of degree 60. Since $\exp(z)$ is entire, Jentzsch's theorem is inapplicable, and the curve in question moves out to ∞ as the degree increases. This curve was analyzed by Szegő and later generalized for Padé rational approximations of $\exp(z)$ by Saff and Varga.

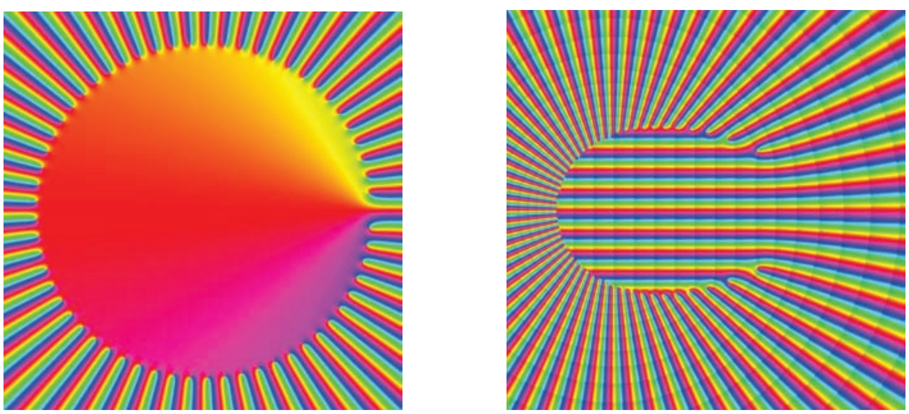

Fig. 7 *Taylor polynomials of* $1/(1-z)$ *and* $\exp(z)$.

One of the central techniques of complex variables is the construction of functions via Cauchy integrals. For example, suppose we consider a Cauchy integral defined by integrating the constant function 1 over the unit interval $[-1, 1]$. The resulting function $f(z)$, plotted in Figure 8(left), is a branch of $\log((z-1)/(z+1))/2\pi i$ with a branch cut along $[-1, 1]$; the Plemelj–Sokhotsky jump relation tells us that f jumps by exactly 1 across the interval. What if the same function 1 is used along a different contour? We get a less standard branch of $\log((z-1)/(z+1))/2\pi i$, as shown in Figure 8(right). Until I saw such pictures, Cauchy integrals never seemed so simple.

For a numerical analyst like me, these ideas become especially intriguing when Cauchy integrals are combined with discretization. The plot shown in Figure 9(left) shows a certain Cauchy integral approximated by the 200-point trapezoid rule over a contour that winds around twice. The resulting function $f(z)$ is a rational approximation to the function defined by the exact Cauchy integral, with 200 poles along the contour and corresponding zeros lining up nearby outside. In Figure 9(right),

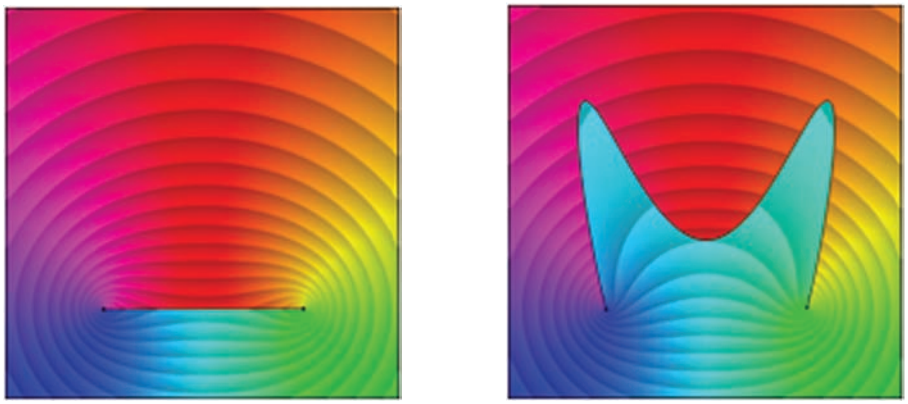

Fig. 8 *Two branches of* $\log((z-1)/(z+1))/2\pi i$.

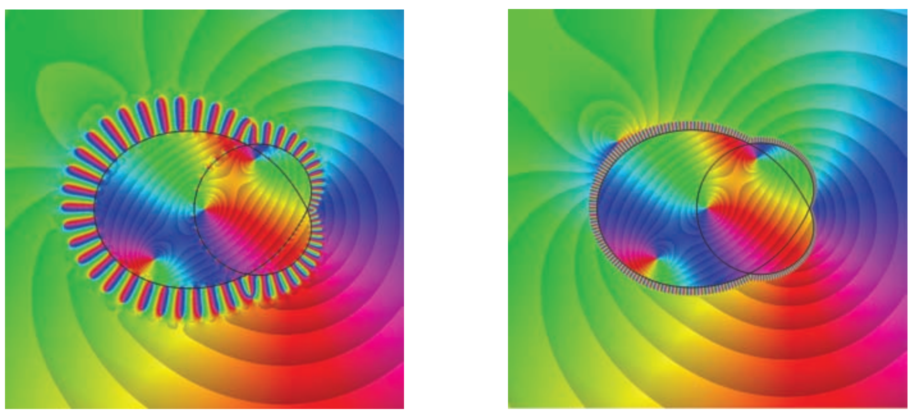

Fig. 9 *Discretized Cauchy integrals.*

the number of quadrature points has increased to 800, and the pole-zero pairs have come closer together. In the limit they define a layer potential of the kind familiar in the field of integral equations, and pictures like these reveal connections between integral equations, complex variables, quadrature theory, rational functions, potential theory, digital filtering, the theory of hyperfunctions, ... it is hard to know where to terminate the list!

Visual Complex Functions is a beautiful and careful presentation of an entire advanced introduction to complex analysis based on phase portraits and, where appropriate, other kinds of computer-generated pictures. The book is mathematically complete, with theorems and proofs as well as pictures. It treats series, products, singularities, analytic continuation, conformal mapping, Riemann surfaces, and many other topics always from a fresh and compelling point of view. My understanding of many ideas and phenomena deepened through reading this book.

Although Wegert's aim was to produce a volume that could be used as a main or companion text for an introductory course, my colleagues and I felt that this aim is not realized. It is hard to imagine learning complex variables for the first time with this book, for the treatment—however enlightening and exciting—is just too idiosyncratic.

A few of the definitions (for example, of paths, curves, arcs, and traces), while carefully thought through, seem not quite standard and potentially confusing for a beginner; occasionally one finds a term imperfectly translated from German, like *ring* instead of *annulus* or *neutral element* instead of *identity*. But for anyone who has already had a first exposure to complex variables, this book offers a thrilling second journey, or a third. Remarkably, despite 360 pages of beautiful glossy color, it sells for less than \$50. An epilogue shows how to draw your own phase portraits, and MATLAB-based software is available at www.visual.wegert.com.

I would like to thank my young colleagues for their contributions to this review: Anthony Austin, Mohsin Javed, Georges Klein, and Alex Townsend.

When I went off to graduate school in 1977, a good fraction of the way back to the era of Jahnke and Emde, I loved complex variables and wanted to work with them on the computer. Sometime during my first semester a friend from undergraduate days wrote and asked, "Are you skiing on the complex functions yet?" With Wegert's extraordinary book, at last I am skiing on the complex functions.

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Combinatorics of Permutations. Second Edition. *By Miklós Bóna.* CRC Press, Boca Raton, FL, 2012. \$89.95. xiv+458 pp., hardcover. ISBN 978-1-4398-5051-0.

In a first combinatorics or discrete mathematics course a student will learn the formula $n!$ for the number of permutations of n objects. Aside from using this as an annoying correction factor in counting unordered arrangements of various types, the next and last exposure to the combinatorics of permutations will often be to the formula for the number of derangements presented as an illustration of the use of the principle of inclusion-exclusion. This is a shame because the combinatorial theory of permutations contains a wealth of interesting and accessible results using methods from all the drawers of the combinatorial tool chest, and often suggesting surprising connections and relationships between apparently unrelated structures.

This book attempts to fill in some of those gaps while acknowledging that the area as a whole is still so broad that complete coverage is impossible. Of course this means that the selection of topics is somewhat according to the author's tastes, but certainly Bóna has done an excellent job of choosing a variety of areas to discuss. The introduction suggests that the material was developed for a graduate class in combinatorics, but it could just as easily be adapted to an advanced undergraduate course, or for independent study by a motivated student.

Chapters 1 and 2 deal with permutations thought of as sequences from the ordered set $1 < 2 < \cdots < n$. First, descents, i.e., adjacent inversions are discussed, and then inversions in general are considered. The results discussed are largely enumerative, but the mix of techniques (surjections and bijections, generating functions, recurrences) already illustrates one of the main themes of the book. Chapter 3 turns to the familiar decomposition of permutations as products of cycles and investigates various questions one might ask about properties of such decompositions, or the permutations whose representations are of a particular type. In particular we return to the derangements, but in a much more general setting which permits one to see striking results like Example 3.58, that the exponential generating function for permutations whose cycle decomposition consists of cycles of odd length only is $\sqrt{(1+x)/(1-x)}$.

Chapters 4 and 5 (and the reviewer must declare an interest) turn to the less familiar area of pattern avoidance. This again deals with permutations thought of as sequences from a linearly ordered set. In that context, any subsequence can also be thought of as a permutation in its own right based simply on the relative order of its elements. For instance, the subsequence 351 of the permutation 35421 contains the pattern 231 since it begins with its middle element (by value), then its greatest element, and then its smallest element. This relationship gives a partial order on permutations, and the (initial) objects of study are the collections of permutations which avoid (i.e., do not contain) certain patterns. Beginning with a basic discussion of these ideas, and some early results, Chapter 4 culminates with the proof by Marcus and Tardos of the Stanley–Wilf conjecture, which states that there is an exponential bound on the number of permutations avoiding any single pattern. This proof, if not from "the book," is from a very nearby shelf and illustrates beautifully how the right change of perspective, together with extremely clever but elementary arguments, can cut through some seemingly intractable combinatorial problems.

Chapters 6 and 7 deal with random permutations and the algebraic combinatorics of permutations. As the author indicates in the preface these are two enormous areas of study, and to some extent these chapters can only skim the surface of some selected topics. The somewhat lighter Chapter 8 turns to algorithms for generating and sorting permutations (which has connections with the material on pattern avoidance). Finally Chapter 9 turns to an "application" of the combinatorics of permutations in certain (rather constrained and somewhat artificial) models of genome rearrangements.

There are a few criticisms that can be made of this book. First, on the nonmathematical side, despite the fact that it is a second edition, a worryingly large number of typographical and similar errors remain which should have been caught by careful proofreading. The typographical ones are easily spotted by an experienced mathematician, but would be more confusing for a student. Others, such as the repetition of the definition of northeastern lattice path on page 17, are just annoying. The subject is one where proofs and examples are usefully supplemented by key diagrams, but in this book the quality of the typesetting of the diagrams is quite poor—they look rather out of place in comparison with the main body of text. In a somewhat similar vein, the overall layout of the pages is very dense, and the frequent use of in-line formulas leads to some rather choppy line spacings and attendant difficulties in reading fluidly. Finally, the index is rather sketchy (though on the other hand the references are excellent!). As regards the actual content, the only criticism is that there is a certain lack of flow and development, either in terms of topics or difficulty. To some extent this is simply a consequence of the survey nature of the book as a whole, but already parts of Chapter 1 are quite dense and technical—this sort of take-no-prisoners approach might discourage a student. Beginning the book with the rather lighter material of the final two chapters might have made a difference here. Of course this is something that an instructor can easily take into account as well.

On the other hand, there are many, many positive things to be said. Its principle contribution is to provide an overview of (parts of) this rich area of combinatorial knowledge which has not been previously available in textbook form. The topics chosen form a good representation of starting points for further research in the area, and to some extent illustrate the interconnections between them. There is a rich supply of interesting and well-chosen exercises. Of particular note are the "Problems Plus" sections, which provide a wealth of further topics to explore—often linking directly to the research literature. For an interested student, or as a source for ideas for presentations or essays (as opposed to the usual weekly exercise sheets), these are invaluable.

Overall, I think this is an excellent book and well worth considering for an appropriate course, or as a recommendation to a student for independent study. It's a shame that some rather minor technical issues detract from its attractiveness, but on the whole its positive aspects far outweigh the negative ones.

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Quasi-Stationary Distributions: Markov Chains, Diffusions and Dynamical Systems. *By Pierre Collet, Servet Martinez, and Jaime San Martín.* Springer, New York, 2013. $129.00. xvi+280 pp., hardcover. ISBN 978-3-642-33130-5.

This book is concerned with Markov processes and dynamical systems whose orbits get *killed* (or terminated) once they hit a *trap*. In the area of Markov process this topic has deep roots and goes back to Kolmogorov's work in 1938 (see the book under review and references therein). While in the area of dynamical systems, research on this topic started in the late 1970s [4]. In both areas this topic is currently one of the most active branches of research. This is in part due to its interesting applications in other sciences. In mathematical biology Markov processes and dynamical systems with a trap has applications in birth-death processes (see the book under review and references therein). In physics, among a vast literature on applications, it found applications in transport in heat conduction [2]. It also plays an important role in astronomy [5]. More recently, this topic has provided insights into the study of *metastable dynamical systems* which behave approximately like a collection of dynamical systems with traps: the infrequent transitions between almost invariant regions in a metastable system are similar to infrequent escapes from associated systems with traps [3, 1].

Over the last 20 years, the authors of the book have made remarkable contributions to this topic of research, and this book cultivates their work to date. The book starts by introducing the main tools needed to study the long-term behavior of Markov process and dynamical systems with traps. This is done by introducing the definition of the so-called *quasi-invariant distribution* (QSD) and its associated exponential rate of survival, which are indeed the main tools needed to describe the long-term statistics of a Markov process (and for dynamical systems) with a trap. Then the book presents results on the problem of existence of QSD and its associated exponential rate of survival in discrete-time Markov chains on a countable state space, N-dimensional Brownian motion, although, in the latter, more emphasis is put on the 1-dimensional case. Moreover, it briefly touches on dynamical systems, such as iterated maps and flows, where the trap is often called in the dynamical systems literature a *hole*.

The book under review is well written and to a certain extent is self-contained. The authors of the book are leading experts in the area, and they made all the effort to make the book accessible to everyone, from graduate students to researchers, working in this area. Chapter 8 is the only place where I would have liked more material on dynamical systems with holes. Nevertheless, this is an important, unique book. To the best of my knowledge I have never seen a book which is devoted only to the study of QSD and which covers both Markov process and dynamical systems.

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Continuous-Time Markov Chains and Applications: A Two-Time-Scale Approach. Second Edition. *By G. George Yin and Qing Zhang.* Springer, New York, 2013. $129.00. xxii+427 pp., hardcover. ISBN 978-1-4614-4345-2.

The book is devoted to a study of continuous time singularly perturbed (SP) Markov chains and their applications in problems of control and optimization.

Singularly perturbed systems arise in modeling "real life" systems appearing in a variety of applications (physical, biological, and management sciences, electrical, chemical, and mechanical engineering, and others).

Due to its importance for applications and due to the mathematical challenges it is presenting, singular perturbations theory has been studied intensively in the literature. Numerous papers and a good number of books have been devoted to this topic over the years; see, e.g., Bensoussan [1], Dontchev [2], Kabanov and Pergamenshchikov [3], Kokotovic, Khalil, and O'Reilly [4], Kushner [5], O'Malley [6], Pervozvanski and Gaitsgory [7], Plotnikov, Plotnikov, and Vituk [8], Vasil'eva and Butuzov [9], and Yin and Zhang [10].

The distinctive feature of the book under review is that it is devoted exclusively to the in-depth study of SP continuous time Markov chains and their applications. This makes the book complementary to the existing literature.

SP Markov chains are characterized by the fact that their generators Q^ϵ depend on a small parameter $\epsilon > 0$ in a "singular" way. Namely,

$$Q^\epsilon = \frac{1}{\epsilon}\tilde{Q} + \hat{Q},$$

where $\tilde{Q}$ and $\hat{Q}$ are generators themselves. The dynamics of the state probabilities vector $p(t)$ is described by the differential equation

$$\frac{dp(t)}{dt} = (Q^\epsilon)^T p(t),$$

with the presence of the small parameter implying that some linear combinations of the state probabilities vector may change their values with the rates of the order $O(\frac{1}{\epsilon})$, while the rates of changes of some other such combinations are of the order $O(1)$. The decomposition of the state probabilities dynamics into fast (of the order $O(\frac{1}{\epsilon})$) and slow (of the order $O(1)$) parts is most distinctive in the case in which $\tilde{Q}$ has a block-diagonal structure,

$$\tilde{Q} = \mathrm{diag}(\tilde{Q}^1, \tilde{Q}^2, \ldots, \tilde{Q}^l),$$

where Q^k, $k = 1, \ldots, l$, are irreducible generators of suitable Markov chains. The block-diagonal structure of $\tilde{Q}$ implies that the set M of states of the corresponding Markov chain is decomposed into l subsets,

$$M = M_1 \cup \cdots \cup M_l.$$

The fast dynamics in this case is represented by frequent transitions between the states within each subset M_i. These transitions are governed by the Markov chain with the generator $\frac{1}{\epsilon}\tilde{Q}^i + O(1)$, and are resulted in that the probability distribution within M_i becomes approximately equal to the stationary distribution of the Markov chain with the generator $\tilde{Q}^i$ $(i = 1, \ldots, l)$. The slow part of the dynamics is associated with "rare" transitions between the states belonging to different subsets. The process of moving from subset to subset is approximately described by the "aggregated" Markov chain, the elements of the generator of the latter being obtained via summing up appropriate elements of $\hat{Q}$ with weights defined as stationary distribution within the subsets.

The book contains a number of examples illustrating possible applications of SP Markov chains in control and optimization problems. One of these is the problem of minimization of the expected cost,

$$\min_{u(\cdot)} E \int_0^T G(x(t), u(t), \alpha^\epsilon(t))dt,$$

considered on the solutions of the system

$$\frac{dx(t)}{dt} = f(x(t), u(t), \alpha^\epsilon(t)), \quad x(0) = x_0,$$

where $x(t)$ is the state variable (or vector of state variables), $u(t)$ is the control function, and $\alpha^\epsilon(t)$ represents random disturbances, the latter being modeled by an SP Markov chain. Assuming that the generator of this Markov chain has a perturbed diagonal structure (as in the example above), it is natural to conjecture that a near optimal control can be found via solving an averaged optimal control problem, in which the SP Markov chain is replaced by the Markov chain with adequately aggregated states. The fact that the number of aggregated states is generally much less than the number of states in the original SP chain has a potential to greatly simplify the analysis.

The verification of this and similar conjectures about dynamical systems driven by SP Markov chains is based on in-depth understanding of asymptotic properties of such chains, the methodical study of which is the main focal point of the book.

The book consists of three parts. Part I provides background material. It exposes, in particular, a number of models and examples of Markov chains (including birth and death processes, queueing systems with finite capacities, optimal control of jump linear systems, and others). Part II is the largest of the three, and it is devoted to a thorough analysis of various asymptotic properties of SP Markov chains. Among the topics studied here are SP chains with recurrent states, SP chains having absorbing states, SP chains containing transient states, and SP chains with perturbed block-diagonal structure (the consideration including the case when the number of blocks is infinite). Part III deals with some applications including Markov decision processes, near optimal control of stochastic dynamical systems, numerical solutions of control and optimization of Markov chains, and hybrid two-time-scale LQG problems.

The first edition of the book was very well received by the research community. It has provided a valuable reference and stimulated research in the area. There is no doubt that this second, revised and updated edition will be also well received by the intended audience (researchers and graduate students in applied mathematics and control engineering interested in modeling and optimization of complex stochastic systems) and will excite further interest to the topic.

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Improving Risk Analysis. *By Louis Anthony Cox, Jr.* Springer, New York, 2013. $179.00. xxvi+386 pp., hardcover. ISBN 978-1-4614-6057-2.

This excellent book presents up-to-date material, various portions of which are sure to be of great interest to several distinct audiences. The primary focus is on statistical issues and causality, with a secondary emphasis on decision theory. But even teachers of game theory and ordinary differential equations are likely to find interesting examples of methods and contexts that might fit into their syllabi or make good case studies for their students to take on as projects. This diversity of subject matter should not be surprising since the wide range of societal decision problems involving risk includes fields as diverse as epidemiology, finance, industrial safety, defense, and others. Three unifying themes permeate the text: how statistical methods can be used and abused, how nonstatistical methods can complement statistics in understanding causality, and how analytical decision support methods, if used correctly, can enhance human decision making. The book is full of thorough but accessible examples and case studies with extensive references to the literature.

First, let's look at the big picture. We make individual and collective decisions constantly that involve considerations of risk: what to eat, where to invest, what to regulate, when to seek help, how strong to build buildings, how much to spend for public safety and defense. All of these decisions are based on information, and almost all of them take into account potentially conflicting information as well as multiple objectives, such as improving conditions and reducing costs. The fundamental problem then is that of understanding the effects of alternative decisions and identifying an appropriate balance of such effects. The results of possible decisions can be investigated by complementary statistical and nonstatistical models targeting causality, and this can all sometimes be put together by a formal decision process that is more effective, efficient, and objective than less structured individual or group decision making. After all, for the latter, just recall some recent market "flash crashes" or large-scale power outages, where retrospective analyses clearly indicate human decision failures (sometimes implemented through software) in stressful situations, such as fast moving or unanticipated circumstances. In other cases, psychological biases or political pressures can have significant negative effects. The problem of good decision making is so challenging in the public arena, where some politicians are likely to fan the flames of ignorance to their own advantage, that in more pessimistic moments one might be tempted to throw in the towel and set alternative goals on the belief that ignorance can be bliss if the majority of people are content with it. But little by little, I believe our responsibility as educators and scientists is to show people how to convert information to action in rational ways and to instill in the broader society a reasonable expectation and confidence that our leaders can do this. There is much room for improvement here.

This book begins with a wake-up call by showing in example after example how alternative conclusions have been drawn by supposed experts on the basis of statistical analyses of various datasets pertaining to the same decision problem. The emphasis here is on health impacts of environmental agents, such as particulate matter in the air or antibiotics used in the production of meat. It's the repetition of the story we tell our students from the first course in statistics: association is not causality. It's the story they repeat back to us on case studies and final examinations—but then they run off and do it anyway after the course is over. That's because the tempting combination of statistical associations and plausible logic can weaken the search for alternative explanations and become a self-fulfilling prophecy. It doesn't help objectivity that there may also be intense pressures for publication or paying clients who like what they see. Cox rigorously keeps the focus on the decision problem: what will be the likely impact on future outcomes from alternative decisions about proposed actions. For this we need to structure the decision process in a framework that can get beyond associations and on to causality.

Lots of progress on causality analysis has been made in the last twenty years, but it's still a complex field that requires cautious negotiation by professionals. Fortunately, some quite recent software packages can be used to facilitate this work. The author provides a nice overview of various approaches (including some older ones): interrupted time series analysis, change point analysis, quasi-experimental designs, panel data analysis, Granger causality, conditional independence tests, counterfactual regression models, and graph-theoretic analysis (chains and networks). This last category, closely related to the groundbreaking work of Judah Pearl, is developed through causal chains, path analysis, structural equations, directed acyclic graphs, and Bayesian networks. One would need to consult the references for details on the methods, but the overview and examples are excellent. Several very detailed case studies are presented in later chapters. The complementarity between statistical investigation and phenomenological models is shown with the use of ordinary differential equations to understand certain processes as well as game theoretic and optimization methods. These latter fit into the picture because the focus is consistently on making decisions that will affect the future, not just on interpreting data from the past.

The Sarbanes–Oxley Act of 2002 has led to a greater focus on formal risk assessment within public companies and many other institutions as a component of responsible management. This has given birth to a plethora of tools being marketed by consulting companies for the prioritization of risk reduction actions. My own experience suggests that these tools are hard for many clients to pick from, and those with the "slickest" human interface or ease of use may be selected more for that than for their technical merits. An important contribution of this book is to compare various approaches to making decisions about risk. The author shows, for example, the serious suboptimality of typical risking scoring systems, which are so common. The challenge I see is in bridging the gap between methods that an organization's leadership or constituencies will be willing to rely on and those that derive from a more sophisticated perspective. He discusses some ways to manage this challenge with more robust decision frameworks involving multiple models or adaptive decision stages.

In summary, this is an important contribution to the literature in that it bridges the gap from theory to practice, includes many fields of application, and is accessible to a wide audience.

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Nonlinear Inclusions and Hemivariational Inequalities: Models and Analysis of Contact Problems. *By S. Migórski, A. Ochal, and M. Sofonea.* Springer, New York, 2013. $169.00. xvi+285 pp., hardcover. ISBN 978-1-4614-4231-8.

Analysis of models from contact mechanics has a long history and has attracted much attention in the research community for over half a century. A contact problem is nonlinear, featured by its inequality form. When the problem has a convex structure, the inequality formulation is a variational inequality, and there has been a large amount of publications devoted to modeling, mathematical analysis, numerical algorithms, and simulations of variational inequalities arising in contact mechanics. Presence of a nonconvex term in the problem leads to a hemivariational inequality, which is in turn a nonlinear inclusion associated with a Clarke subdifferential operator. It is more challenging to study hemivariational inequalities, and the size of the literature on hemivariational inequalities is much smaller than that on variational inequalities. The book under review presents a rigorous introduction to the theory of nonlinear inclusions and hemivariational inequalities with applications to contact mechanics.

The book consists of three parts. Part I reviews concisely basic notions and results from functional analysis that is needed later in the book. This part is intended to make the book as self-contained as possible. It covers materials on normed spaces, duality, measure theory, function spaces, various differentiation notions for nonsmooth functions, and single-valued and set-valued operators of monotone type. Most of the results are stated without proof, but extensive references are provided for the proofs and for further discussions. The other two parts are based on the authors' research work. Part II provides existence and uniqueness results for abstract nonlinear inclusions, both stationary and evolutionary, and applies the results in the study of stationary and evolutionary hemivariational inequalities. Detailed proofs of the statements are given, based on techniques of monotonicity, compactness, and fixed-point arguments. The theory developed in Part II is employed in Part III to analyze some hemivariational inequalities modeling static and dynamic frictional contact problems. The constitutive law is elastic, or piezoelectric, or viscoelastic with short-term or long-term memory. The contact condition is modeled with normal compliance or normal damped response, while the friction condition is modeled with variants or regularizations of Coulomb's law. For each contact problem studied, a hemivariational inequality is derived, and solution existence or both existence and uniqueness are shown.

The book is very well written. In addition to a rather comprehensive list of relevant references, the authors include several pages of bibliographical notes at the end of each part to aid the reader for references with additional information and further study. The book is highly recommended to researchers, graduate students included, who are interested in the mathematical theory of hemivariational inequalities and/or in the study of contact problems involving nonmonotone and set-valued constitutive laws or boundary conditions.

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Invisible in the Storm: The Role of Mathematics in Understanding Weather. *By Ian Roulstone and John Norbury.* Princeton University Press, Princeton, NJ, 2013. $35.00. x+325 pp., hardcover. ISBN 978-0-691-15272-1.

This is a book about the history of meteorology and the essential role mathematics and computing have played in it. It emphasizes the people who developed the subject, illustrating the ideas with good graphics and the personalities with stories and descriptions of the work, carried out through international cooperation and accumulated understanding and capability throughout the last century.

The authors convince the reader that the subject of climate is much more complicated than astronomy and more important than suggesting what clothes we should wear. The heroes include the Norwegian Vilhelm Bjerknes, the British Quaker Lewis Fry Richardson, the Swede Carl-Gustav Rossby, and the Americans Jule Charney and Edward Lorenz. The emphasis is numerical weather prediction, based on simplifying the physics, the global grid, and data assimilation. The basic hydrostatic and geostrophic balances are described, as is the chaotic behavior of the solutions to the three-dimensional nonlinear deterministic model of Lorenz and the resulting need for ensemble forecasting. They accomplish all this with a minimum of technicality.

Readers will realize that much better forecasting has resulted because of combined progress regarding computers, mathematics, and the atmospheric sciences. We should expect accelerated continuing success, based on adequate funding and attention by all.

Roulstone and Norbury have done an outstanding job and provide readers a fine bibliography to continue their education on this fascinating topic.

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A Primer on Mathematical Models in Biology. *By Lee A. Segel and Leah Edelstein-Keshet.* SIAM, Philadelphia, 2013. $69.00. xxiv+424 pp., softcover. ISBN 978-1-611972-49-8.

Lee Segel (1932–2005) was a superb applied mathematician and gifted teacher and expositor who modeled problems in biology and other fields, and his former student Leah Edelstein-Keshet of the University of British Columbia has earned similar recognition. Both are authors of well-known and often-used textbooks in the SIAM Classics in Applied Mathematics series. We are very fortunate that Leah was able to retrieve Lee's most recent lecture notes from the Weizmann Institute and modify and extend them to provide us this extraordinary primer, aimed primarily at biology students. The presentation benefits tremendously by its large number of illustrations, primarily generated using Bard Ermentrout's XPP software (available online and described in his SIAM publication).

The result, despite its low level of mathematical prerequisites, is very sophisticated. Readers really learn and appreciate phase plane analysis, as well as scaling, limit cycles, bifurcations, etc. Students will especially be convinced that understanding solution behavior involves lots of simplifications and geometric interpretations of the dynamics. They'll be provided exposure to analytical detail by working on the many biologically meaningful exercises and projects and consulting the many references, both biological and mathematical. Applications include the spread of infections, the extensive treatment of the quasi-steady state hypothesis in biochemical kinetics, the FitzHugh–

Nagumo equations, and networks of genes and cells.

The authors don't preach but convincingly demonstrate what successful modeling is and how essential a role mathematics (mostly systems of ordinary differential equations) plays in the process. They've provided us a new and unique classic!

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Differential-Algebraic Equations: A Projector Based Analysis. *By Rene Lamour, Roswitha Marz, and Caren Tischendorf.* Springer, Berlin, 2013. $149.00. xxviii+649 pp., softcover. ISBN 978-3-642-27554-8.

How does one handle a system

$$F(t, x, x') = 0$$

of differential equations when one cannot solve for the derivative? What initial conditions should be provided? One could proceed geometrically or algebraically.

Special cases abound, such as a differential equation

$$y' = f(t, y, z)$$

subject to the constraint

$$0 = g(t, y, z).$$

They are worthy of study because they are essential to important applications, such as circuit theory. Their algebraic analysis goes back to the work of Weierstrass and Kronecker on matrix pencils, so it's fitting that a foremost center for the study of differential-algebraic equations (DAEs) developed at Humboldt University in Berlin, under the leadership of Roswitha Marz. Indeed, the 1986 book of Griepentrop and Marz on DAEs and their numerical treatment (Tuebner, Leipzig) has been very influential. This big new book brings scholars up to date on this school's progress and initiates a Springer series Differential-Algebraic Equations Forum, with an international editorial board.

For linear equations, it's natural to develop projector-based analyses and related numerical methods based on the underlying notion of index (related to differentiating the constraints a number of times, hoping to specify the solution). The authors present their ideas for a zoo of specific problems in detailed step-by-step fashion, with simple illustrative examples. Variable coefficients and nonlinearity are considered, under various restrictive hypotheses. Canonical forms, singular points, regularity, the pseudoinverse, and much new terminology come into play. A variety of numerical methods are presented, especially for index-1 problems. The final chapter introduces abstract DAEs.

Altogether, the book is well written and well motivated. Nonetheless, the presentation is necessarily very technical and far from an easy read.

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The Divergence Theorem and Sets of Finite Perimeter. *By Washek F. Pfeffer.* CRC Press, Boca Raton, FL, 2012. $99.95. xvi+243 pp., hardcover. ISBN 978-1-4665-0719-7.

This monograph is devoted to the detailed and comprehensive development of the most general form of the divergence theorem, a result of fundamental importance in mathematical analysis. In its simplest form, which is presented in most calculus books, it states the following: If $f : E \to \mathbb{R}^n$ is a smooth vector field defined on a suitably simple set $E \subset \mathbb{R}^n$, then

$$\int_E \operatorname{div} f(x)\, dx = \int_{\partial E} f(y) \cdot \nu(y) d\,\sigma, \tag{1}$$

where σ denotes surface measure. Research on the problem of finding the most natural and general form of the divergence theorem has been one of the successes of geometric measure theory, by describing the most general form of the sets E for which (1) remains valid. Thus, in order to give the exposition the greatest clarity, the book is divided into 3 parts:

1. Extending the family of vector fields for which the divergence theorem holds on simple sets.

2. Extending the family of sets for which the divergence theorem holds for Lipschitz vector fields.

3. Proving the divergence theorem when the vector fields and sets are extended simultaneously.

Part 1 begins a review of some of the fundamental facts of topology, including linear spaces and their duals as well as the weak* and strong topologies. All this is followed by a review of measure theory which includes a discussion of Hausdorff measure and the theorems of Egoroff and Lusin and the Saks–Henstock lemma. The most important portion of Part 1 deals with a proof of (1) in which the set $E \subset \mathbb{R}^n$ is replaced by a finite union of dyadic cubes and where f is assumed to be differentiable on the interior of E with $\operatorname{div} f \in L^1(E)$. This result is important because of the simplicity of the proof, the fact that it will be used in establishing the general form of (1), and the fact that it can be used in the study of removable singularities of some classical differential equations. All the material in Part 1 should be accessible to a first-year graduate student in analysis.

The material of Part 2 is directed to the development of material which is necessary to understand the main results of the book. For this purpose the class of BV functions $f : U \to \mathbb{R}^1$ such that the partial derivatives of f are totally finite measures in the sense of distributions are introduced, as well as their associated *sets of finite perimeter*. A measurable set is said to have finite perimeter, $P(E) < \infty$, if the indicator function χ_E is of bounded variation. This is the analytic formulation. There is also the measure-theoretic version, which is as follows: The measure-theoretic boundary of a measurable set E is the set of all $x \in \mathbb{R}^n$ such that x is not a point of density of E and is not a point of density of the complement of E. The measure-theoretic boundary of E is denoted by $\partial_* E$. The fact that a set of finite perimeter is characterized by $\mathcal{H}^{n-1}(\partial_* E) < \infty$ is one of the main results of this section. Related concepts to measure-theoretic boundary are the analogues of the topological boundary, namely, the sets $\text{ext}_* E$, $\text{int}_* E$, and cl_*E, which are, respectively, points of dispersion of E, $\text{int}_* E := \text{ext}_* \mathbb{R}^n \setminus E$, and cl_*E:$=\mathbb{R}^n \setminus \text{ext}_* E$. In order to exploit the connection between sets of finite perimeter and BV functions, the author proves the Fleming–Rishel co-area formula for BV functions [5]: If $f \in BV$, then the set $E_t := \{f > t\}$ is of finite perimeter for almost all $t \in \mathbb{R}$ and $\|Df\| = \int_{\mathbb{R}} (E_t) dt$, where $\|Df\|$ denotes the total variation of the gradient measure Df.

The next big step in the development is the introduction of the *reduced boundary of a set* E, which is the set {x: $x \in spt(\|D\chi_E\|)$ } such that $|\nu_E(x)| = 1$ and

(2)
$$\lim_{r\to 0} \frac{1}{\mu_E(B(x,r))} \int_{B(x,r)} \nu_E(y)\, d\mu_E(y) = \nu_E(x),$$

where $\nu_E(x)$ is the Radon–Nikodým derivative of $D\chi_E$ with respect to $\|D\chi_E\|$. The notion of reduced boundary is due to [2] and is denoted by $\partial_* E$. Note that $\partial^* E \subset \partial_* E$. Next, it is shown that $\nu(E,x)$ has a geometric meaning. Indeed, it is shown that if $H_\pm(E,x)$ are the two half-spaces that are determined by the hyperplane $H(E,x) := \{y \in \mathbb{R}^n : \nu(E,x) \cdot (y - x) = 0\}$, then $H(E,x)$ is tangent to E at x in the measure-theoretic sense:

1. $\Theta(H_+(E,x) \cap E, x) = \Theta(H_-(E,x) \setminus E, x) = 0$.
2. $\Theta\nu(E,x) = 1/2$.

Here, $\Theta(A,x)$ denotes the density of a set $A \subset \mathbb{R}^n$ at x.
These results are very satisfying intuitively, but their proofs are not easy. They are critical in establishing the following result, due to Federer [3, 4].

THEOREM 1. *If E is a set of finite perimeter and $v : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz, then*

$$\int_E \operatorname{div}(x)\, dx = \int_{\partial_* E} v(y) \cdot \nu(E,y)\, d\mathcal{H}^{n-1}(y).$$

Part 3 is concerned with generalizing Theorem 1 for vector fields that are more general than being Lipschitz. The author considers vector fields in which each component of $v \in BV$, and therefore the divergence of v is a signed measure. For such a vector field and for $\Omega \subset \mathbb{R}^n$, a Lipschitz domain, he proves that the trace of v,

$$Tv(x) = \lim_{r\to 0} \frac{1}{r^{n-1}} \|Dv\| (\Omega \cap B(x,r)),$$

exists for $\mathcal{H}^{n-1}$a.e. $x \in \partial\Omega$. Furthermore, it is shown that whenever $v \in BV(\Omega;\mathbb{R}^n)$, and Ω is Lipschitz and compact, then there exists $C > 0$ such that

$$\|Tv\|_{L^1(\partial\Omega,\mathcal{H}^{n-1})} \le C\,\|v\|_{BV(\Omega;\mathbb{R}^n)}\,.$$

With this notion of trace, the following is proved.

THEOREM 2. *If Ω is Lipschitz with compact boundary, then*

$$\operatorname{div} W(\Omega) = \int_{\partial\Omega} (Tw)\cdot \nu_\Omega \, dH^{n-1},$$

whenever $w \in BV(\Omega;\mathbb{R}^n)$.[1]

Since, up to this point, the divergence theorem is known only for vector fields, v, for which $\operatorname{div} v = \mu$ and for which $v \in L^\infty$, the author investigates the situation where v is allowed to be unbounded near certain singular sets, K. The growth of v near K is controlled by a suitable power of the distance from K. The power is determined by $M^{*t}(K)$, the upper Minkowski content of K of dimension t. A mapping $v : E \subset \mathbb{R}^n \to \mathbb{R}^n$ is called *bounded* at $x \in \mathrm{cl}E$, provided there is a neighborhood U of x such that

$$\sup\{|v(y)| : y \in E \cap U\} < \infty.$$

The set of $x \in E$ such that v is not bounded at x is denoted by E_∞.

Let $A \subset \mathbb{R}^n$ be a bounded set of finite perimeter. It is shown that if $v : \mathrm{cl}E \to \mathbb{R}^n$, with compact sets K_j such that $E_\infty = \bigcup K_j$, for each j, there are numbers $0 \le t_j < n-1$ such that $\mathcal{M}^{*t_j}(K_j) < \infty$ and $M^{*t_j}(K_j, \partial_* A) < \infty$, and if there are decreasing, integrable functions $\beta_j : \mathbb{R}^1 \to \mathbb{R}^1$ such that for $w := v|\partial_* A$

$$|w(x)| = O(1)\beta_j[\boldsymbol{d}(x,K_j)]\boldsymbol{d}(x,K_j)^{t+2-n} \quad \text{as } x \to K_j, \tag{3}$$

then under these conditions, it is shown that when $\operatorname{div} v$ is calculated relative to $\mathrm{cl}_* A$ it exists at almost all points of A. Furthermore, if $\operatorname{div}_{\mathrm{cl}_* A} \in L^1(A)$, then

$$\int_A \operatorname{div}_{\mathrm{cl}_* A} v(x)\,dx = \int_{\partial_* A} v(y)\cdot\nu_A(y)\,d\mathcal{H}^{n-1}(y).$$

The author pursues another version of the divergence which may exist at x while the classical divergence may not. For this he considers the flux, F, of $v \in L^\infty_{\mathrm{loc}}(\Omega,\mathcal{H}^{n-1})$ which is defined for all dyadic figures contained in the open set Ω, that is, for the finite union of dyadic cubes,

$$F(A) := \int_{\partial A} v\cdot\nu_A \, d\mathcal{H}^{n-1}.$$

It turns out that F is an additive function. For a given set $E \subset \Omega$, and $\delta > 0$, let

$$V_\delta F(E) := \sup_P \sum_{i=1}^p |F(C_i)|\,,$$

where P is a δ-fine partition of Ω. The *critical variation* of F is defined as

$$VF(E) := \inf_{\delta>0} V_\delta F(E).$$

Interestingly, the author shows that VF is a Borel measure on Ω. If the measure VF is absolutely continuous, it is then shown that the derivative F' exists a.e. in Ω, which is called the *mean divergence* of v at x. Furthermore, if $F' \in L^1_{\mathrm{loc}}$, then

$$F(A) = \int_A F'(x)\,dx.$$

Thus, we have the following theorem.

THEOREM 3. *Suppose the flux of $v \in L^\infty_{\mathrm{loc}}(\Omega,\mathcal{H}^{n-1})$ is absolutely continuous on Ω. Then the mean divergence,* div_m, *exists*

[1] The author notes here that this result has been extended to $v \in L^\infty(E)$, independently by [6] and [1]. The result states that for such a v with $\operatorname{div}(v) = \mu$, a signed measure, there is a trace operator, $Tv : \partial_* E \to \mathbb{R}^n, Tv \in L^\infty(\partial_* E,\mathcal{H}^{n-1})$, which satisfies the equation

$$\int_E \operatorname{div} v := \mu(E) = \int_{\partial_* E} (Tv)\cdot\nu_E\,d\mathcal{H}^{n-1}.$$

a.e. in Ω, *and if* $\operatorname{div}_m \in L^1_{\mathrm{lcc}}(\Omega)$, *then*

$$\int_A \operatorname{div}_m(x) = \int_{\partial A} v \cdot \nu_A \, dH^{n-1}.$$

The author brings the book to a conclusion by considering the following question: How can one characterize those distributions L for which the divergence equation $\operatorname{div} v = L$?

Depending on L, the answer is sought for finding a v that has some regularity such as boundedness or continuity, or both. He begins the discussion by considering the relatively easy case of when $L \in \mathbb{D}'_p(\Omega)$, where $\mathbb{D}'_p(\Omega) := \{L \in D'(\Omega) : \|L\|_p < \infty\}$ and where

$$\|L_p\| := \sup\Big\{L(\varphi) := \varphi \in \mathbb{D}(\Omega) \text{ and } \|D\varphi\|_{L^p_{(\Omega;\mathbb{R}^n)}} \leq 1\Big\}.$$

Using the Hahn–Banach theorem, we obtain the following.

THEOREM 4. *Let* $\infty = q > 1$. *The equation* $\operatorname{div} v = L$ *has a weak solution* $v \in L^q(\Omega;\mathbb{R}^n)$ *if and only if* $L \in \mathbb{D}'_P(\Omega)$. *Moreover the solution* v *can be chosen so that* $\|v\|_{L^q(\Omega;\mathbb{R}^n)} = \|L\|_p$. *Here,* $\frac{1}{q} + \frac{1}{p} = 1$.

An application of the Gagliardo–Nirenberg–Sobolev inequality yields the following.

THEOREM 5. *For each* $f \in L^p(\Omega)$ *with* $p \in (1, n]$, *there exists a weak solution* $v \in L^{p^*}(\Omega)$ *to the equation* $\operatorname{div} v = f$ *such that*

$$\|v\|_{L^{p^*}(\Omega;\mathbb{R}^n)} \leq \kappa \|f\|_{L^p(\Omega)}.$$

Here, $p^* := \frac{np}{n-p}$ *is the Sobolev conjugate of* p.

In order to discuss the harder case of finding a continuous weak solution, the notion of a *charge* is introduced. A distribution $F \in \mathbb{D}'(\Omega)$ is called a charge if for each open set U compactly contained in Ω and $\varepsilon > 0$ there is $\theta > 0$ such that

$$F(\varphi) \leq \theta \|\varphi\|_{L^1(U)} + \varepsilon \|D\varphi\|_{L^1(U;\mathbb{R}^n)}$$

for all test functions $\varphi \in \mathbb{D}(\Omega)$.

It turns out that the notion of charge provides an elegant answer to the question of whether a continuous solution exists.

THEOREM 6. *Let* $L \in \mathbb{D}'(\Omega)$. *Then the equation* $\operatorname{div} v = L$ *has a weak solution* $v \in C(\Omega;\mathbb{R}^n)$ *if and only if* L *is a charge in* Ω.

If $f \in L^n_{\mathrm{loc}}(\Omega)$, then it's an easy consequence of both Hölder's and Sobolev's inequalities that f is a charge, and therefore the equation has continuous solutions. In view of Theorem 5, it has bounded solutions as well. A result due to T. Depauw shows that it has a solution that is both bounded and continuous simultaneously.

THEOREM 7. *If* $f \in L^n(\Omega)$, *then there exists a* κ *such that* $\operatorname{div} v = f$ *has weak solution* $v \in C(\Omega;\mathbb{R}^n)$ *with* $\|v\|_{L^\infty_{\mathrm{loc}}(\Omega)} \leq \kappa \|f\|_{L^n(\Omega)}$.

In conclusion, this book has admirably accomplished its goal of the detailed and comprehensive development of the most general form of the divergence theorem. Though this development is necessarily technical and complex, the author has succeeded in presenting the material in such a way as to make it accessible to a first-year graduate student in analysis. This he has done by providing many examples and in presenting some of the material in small print, which may be considered marginal, but also as an enhancement of the text. Some of the small-print material would be of greater interest to the expert in that it represents mathematics at the frontiers. In general, because of the importance of the divergence theorem in mathematics, this book will appeal to a wide audience of mathematicians, even those whose primary interest is not in analysis. I believe this book will find a secure place in virtually any mathematician's personal library as an important reference book.

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