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Function: Part II

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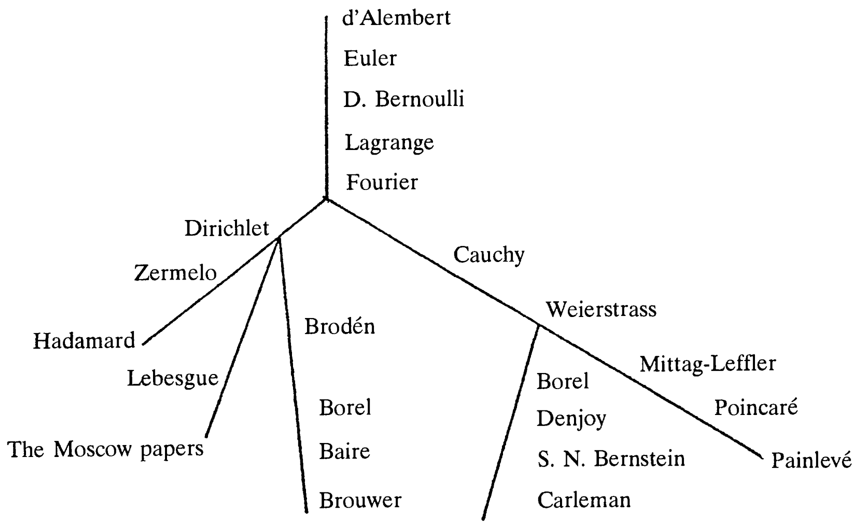
What follows is the second part of a translation by Abe Shenitzer (in two parts) of an article by N. Luzin in the first edition of **The Great Soviet Encyclopedia**, Vol. 59, pp. 314–334. The article describes the evolution of the function concept

The function concept after Fourier's discovery

The modern understanding of function and its definition, which seems correct to us, could arise only after Fourier's discovery. His discovery showed clearly that most of the misunderstandings that arose in the debate about the vibrating string were the result of confusing two seemingly identical but actually vastly different concepts, namely that of function and that of its *analytic representation.* Indeed, prior to Fourier's discovery no distinction was drawn between the concepts of "function" and of "analytic representation," and it was this discovery that brought about their disconnection. After this, the efforts of mathematicians were channelled in two different directions. On the one hand, the desire to maintain the mutual dependence of the parts of a curve gave rise to the modern *theory of functions of a complex variable*. The prospect on this road was the complete separation of the concepts of a function and of its analytic representation. This was done by Weierstrass in his concept of an "analytic" ("holomorphic") function. On the other hand, Fourier's discovery and the study of the values of analytic expressions destroyed all connections between different parts of a curve. It seemed that the only property of the values of an analytic expression was their *determinacy*, and that they were otherwise completely arbitrary, each independent of the others. This was the sense of the definition of the function concept given by Dirichlet. This definition turned out to be of fundamental importance for the contemporary *theory of functions of a real variable*. For a time, the definitions of function, given by Dirichlet and Weierstrass, respectively, brought great clarity and a certain serenity into the mathematical milieu. It seemed that this clarity was final and that all that remained to do was to develop the consequences of the solid definitions achieved after so many difficulties and efforts. But quite recently it became clear that not all mathematicians are of one mind concerning the value and the very sense of these definitions. Ever more frequent hints, supported by incontestable facts, suggested that the Weierstrass definition of function is overly restrictive. On the other hand, mathematicians concluded with utmost consternation that they were not all of one mind concerning the sense of Dirichlet's definition of function. Some found it perfect, others overly broad, and still others devoid of all meaning. It thus became clear that, in our own time, the controversy about the vibrating string has been renewed in another light and with a different content. The grouping of names below suggests the general pattern of the evolution of the function concept.



Functions of a real variable. Fourier's discovery showed that it is possible to view as a single function the ordinate of a continuous curve composed of arcs of curves that have nothing in common and thus of completely different nature. This discovery utterly destroyed the notion of an organic (logical) connection [presumably] existing between different parts of a curve described by means of a single analytic expression, especially an expression as simple as a trigonometric series. This being so, it seemed that the only available option was to ignore analytic expressions and declare that all there is to the meaning of the function concept is that it is a collection of numerical values corresponding to different values of x that are, in general, completely independent of one another. This was the idea behind the famous Dirichlet definition of a function, still in use today, which states that

y is a function of a variable x, defined on an interval $a < x < b$, *if to every value of the variable x in this interval there corresponds a definite value of the variable y. Also, it is irrelevant in what way this correspondence is established.* This definition immediately clarified a great many hitherto at best vaguely understood phenomena of mathematical analysis. At first this definition seemed so perfect that it was virtually unanimously accepted. For a long time it was viewed as a genuine discovery. Its formulation was thought to be so exactly suitable that no thought was given to the very possibility of its modification. The established view was that from then on mathematical analysis was to concern itself with the discovery of properties of various special classes of functions obtained by restricting Dirichlet's general definition of function. In this way there arose branches of analysis concerned with classes of functions such as *continuous* functions (in the sense of Cauchy); *monotonic* functions; functions *with a finite number of maxima and minima* [on an interval]; functions satisfying a *Lipschitz condition*; functions satisfying a *Dini condition*; *differentiable* functions, and so on. Only after these classes of functions had been singled out and investigated, voices arose that asked for clarification of the Dirichlet definition that was initially accepted without any reservations. The objections were directed against the clause: "*it is irrelevant in what way this* *correspondence is established.*" Later the arguments for and against this point linked up with the arguments for and against the so-called *axiom of choice*, first explicitly formulated by Zermelo. One of the first to clearly state his dissatisfaction with this "rider" to Dirichlet's definition of a function was Brodén (1897). Unfortunately, Brodén's argument was couched in rather general terms. As a result, not all contemporary mathematicians paid attention to his reservations. Brodén argued that the definition of a function must have a special property that would enable easy communication from mind to mind. To get an idea of what Brodén had in mind we subdivide the interval $[a, b]$ of definition of a function $y(x)$ into infinitely many subintervals $\delta_1, \delta_2, \delta_3, \dots$. Suppose that our function coincides on δ_1 with the ordinate of some straight line L_1, on δ_2 with the ordinate of some cycloid L_2, on δ_3 with the ordinate of some lemniscate L_3, and so on. Brodén asks: when may $y(x)$ be said to be defined? His answer is that $y(x)$ is defined if and only if we are given a definite law of choice of the curves $L_1, L_2, L_3, \dots,$ that is, when these curves have something in common and thus are in some sense "homogeneous" [as a class]. Brodén claims that we cannot study a function made up of infinitely many absolutely "heterogeneous" curves, since such a function can neither be prescribed nor given. The only time one can prescribe or give absolutely different curves is when their number is *finite*, in which case they can be given as absolutely independent of one another. According to Brodén, then, one cannot study infinitely many curves that are absolutely independent of one another. Somewhat later—independently of Brodén—Borel, Baire, and Lebesgue (1905) supported the requirement of a definite law, always *tacitly implied*, whenever one deals with the function concept. Baire pointed out that one should, once and for all, banish the analogy of a bag with balls passed from hand to hand from all discussions involving the infinite. While it is true that a function is, essentially, the totality of numerical values corresponding to the different values of the variable x, this totality cannot be passed from hand to hand like the previously mentioned bag with balls; here the description of the *law of correspondence* that associates a $y(x)$ to an x is absolutely indispensable, and that law must be communicable to anyone who wants to investigate the function $y(x)$. Baire notes that "for our minds all reduces to the finite". To describe accurately the difference between his own views and those of Zermelo and Hadamard, Borel performs the following thought experiment. He notes first that the decimal expansion of $\pi = 3.1415926535\dots$ must be viewed as *completely determined*, since every textbook of elementary geometry tells us how to compute as many of its decimal digits as we wish. This means that we may view each decimal digit, say the millionth one, as fully determined even if no one has as yet computed it. Then Borel makes a queue of a million people and makes each person name a decimal digit at random, thus obtaining a certain decimal expansion of a million digits. Borel regards this expansion as *fully determined.* Then he makes a queue of infinitely many, rather than a million, people and again makes each person name a decimal digit at random. Now Borel poses the question if one can continue to view the resulting infinite decimal as *fully determined*, that is, as fully determined as the infinite decimal expansion of π, say. Borel's reply is that mathematicians with the mental set of Zermelo and Hadamard will definitely regard this infinite decimal expansion as "fully determined", whereas he himself does not regard it as such. His reason is that the number obtained in this manner may not be ruled by any law, so that two mathematicians discussing this number will never be certain that they are talking *about the same number*; without a law of formation of its decimals they can never be certain of its identity. Lebesgue goes one step further and claims that a mathematician who does not have a law that realizes a function $y(x)$ he is considering cannot be certain that he is talking about the same function at different moments of his investigation; here we are no longer concerned about the common language of *two* mathematicians but about a mathematician agreeing *with himself*. In rebutting Borel's views Hadamard asserts that there is no difficulty in regarding a "lawless" decimal expansion as completely determined. Thus, for example, in the kinetic theory of gases one speaks of the velocities of the molecules in a given volume of a gas although no one will ever really know them. Hadamard points out that the requirement of a law that determines a function $y(x)$ under investigation strongly resembles the requirement of an *analytic expression* for that function, and that this is a throwback to the 18th century.

The mathematical papers of Baire and Lebesgue have shed a great deal of light on the question but have also made it extremely complex. Baire embarked on a systematic investigation of the representation of functions by means of analytic expressions. Since, by Weierstrass' theorem, every continuous function $f(x)$ is representable as the sum of a uniformly convergent series of polynomials $f(x) = \sum_{n=1}^{\infty} P_n(x)$, Baire calls all continuous functions *functions of class* 0. Baire defines the *functions of class* 1 as those discontinuous functions that are limits of continuous functions, that is, $f(x) = \lim_{n\to\infty} f_n(x)$. Baire calls functions not in the classes 0 and 1 that are limits of functions in class 1 *functions of class* 2, and so on. Baire's definition extends over all finite numbers and all countable transfinite numbers. Hence the famous *Baire classification of functions*:

$$K_0, K_1, K_2, \ldots, K_\omega, \ldots, K_\alpha, \ldots | \Omega.$$

Each function $f(x)$ in the Baire classification has a definite analytic representation by means of polynomials and finitely many or countably many symbols of passing to the limit. This is the kind of analytic expression considered by Baire. Lebesgue supplemented Baire's research in an essential way by showing that it is completely pointless to consider all other analytic operations, such as differentiation, expansion in series, integration, the use of transcendental functions such as $\sin x$, $\log x$, and so on; this because every function formed by using a finite or countable number of such operations is necessarily included in the Baire classification. Lebesgue also proved the fundamental fact that none of the Baire classes is empty and, finally, using a profound but extremely complex method, he found a specific function $f(x)$ outside the Baire classification. The impact of Lebesgue's discovery was just as stunning as that of Fourier's in his time. Lebesgue's result shows that a *logical* definition of a particular function is more extensive than a purely mathematical definition, since *a logical definition yielded a particular function $f(x)$ that cannot be obtained from polynomials by passing to the limit a finite or countable number of times*. The Lebesgue function that is outside the Baire classification is extremely complex and its nature has not yet been fully investigated. But the Moscow papers showed that the most delicate point of Lebesgue's considerations gives rise to objections. When Lebesgue showed that *every* analytic expression consisting of mathematical symbols, finite or countable in number, can be transformed into a Baire expression composed of simple (countable) passages to the limit he had no actual complete catalogue of all possible analytic expressions. This meant that he was exposing his enterprise to a great danger, for there could always turn up an analytic expression not transformable into a Baire expression. In fact, the Moscow papers showed that already the analytic expression

$$f(x) = \overline{\lim_{y\to\infty}}\ \overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} P_{m,n}(x, y),$$

where $P_{m,n}(x, y)$ is a polynomial in x and y, where the passing to the limit on m and n is *simple* (countable), and where the passing to $\overline{\lim}$ on y is *continuous* (uncountable), is not reducible to a Baire expression for the right choice of the polynomial $P_{m,n}(x, y)$. At the same time it became clear that, as Borel anticipated, very frequently analytic expressions are entirely useless in the sense that, *apparently*, even functions in class 1 of the Baire classification confront us with problems that are unsolvable in principle. The indicated problems concerning the nature of analytic expressions are far from being solved. But it should be pointed out that there are marked and important nuances of opinion among mathematicians who object to Dirichlet's definition. Thus while Lebesgue is willing to accept any law (logical or mathematical) as long as it yields an individual function, Borel insists on the restriction that the law be *countable* (that is, that it involve the natural numbers but not a continuum). Brouwer seems to go still further, for he refuses to consider even the infinity of natural numbers.

Functions of a complex variable. A very different fate befell the definition of function that aimed at a formulation of the function concept such that "knowledge of a small arc of the curve under consideration implies knowledge of the whole curve." It is a fact that just as Dirichlet, working with real variables, gave a definition of function that was viewed as final, so too Weierstrass, working with complex variables, gave a definition of function so perfect that to this day the majority of mathematicians regard it as unique and, at any rate, as meeting all requirements of practical applications. Whereas criticism directed against Dirichlet's definition calls for it to be *narrowed*, criticism directed against Weierstrass' definition calls for it to be *broadened*. The papers of Weierstrass were preceded by those of Cauchy (1789–1857). Cauchy was the first to realize that the property of a curve to be determined by a small arc called for the use of a *complex variable*, and that while it might play an auxiliary role, it was an indispensable role. Cauchy's ideas and basic theorems were ordered and systematized by Weierstrass (1815–1897). His fundamental idea was that of *analytic continuation*. Cauchy's investigations showed that every series $P(x - a)$ of positive powers of the difference $x - a$ converges in the interior of a definite circle C with center a and diverges outside C. The sum of the series inside C is infinitely differentiable. Weierstrass regarded this sum of the series $P(x - a)$ as an "analytic function" defined in C and relied on a special process for extending its existence domain. This process is based on the following fundamental theorem: *If the circles of convergence of two given series $P(x - a)$ and $P(x - b)$ intersect and if this intersection contains a point at which the values of the two sums and the values of all their corresponding derivatives are equal, then the two sums have the same values throughout the intersection of the two circles*. In this case, Weierstrass regards each of the two series as a *direct continuation* of the other and calls each of them an "element" of the analytic function that is being defined. Weierstrass' definition of an analytic function is the following: *An analytic function $f(x)$ is the totality of elements obtained from a given one by means of successive direct continuations*. Volterra and Poincaré contributed a final clarification to this definition by showing that the complete definition of an analytic function throughout its domain of existence required no more than a *countable* number of direct continuations. An analytic function $f(z)$ is called *single-valued* if there is no point z at which two different elements $P(x - a)$ and $P(x - b)$ have different values. The set of points z in the interior of the circles of the elements of the single-valued function $f(z)$ under consideration is called its *natural domain of existence*. A point on the boundary of the natural domain of existence of a single-valued function is called a *singular point* of that function. A basic theorem is that the *circle of convergence of each element of an analytic function $f(z)$ contains a singular point*. Weierstrass' definition immediately shed a bright light on a great many hitherto dark areas of mathematical analysis. It explained a great many paradoxes and gave rise to a flood of papers (continuing to this day) devoted to the study of properties of analytic functions. It seemed that one had found so perfect a definition of function that all that remained to do was to study its implications. Above all, one seemed to have finally unriddled the property of a function that "*the values of a small part of a curve determine all of it*": this property turned out to be just a consequence of the definition of function. In addition, many hitherto baffling properties of analytic expressions, primarily series and infinite products, became clear: it turned out that the sum of a uniformly convergent series of functions analytic in a domain D was an analytic function in D. The riddle of an analytic expression that converged to different functions in different domains was explained by noting that uniform convergence was disturbed between these domains. This explained why, for example, the series

$$\frac{1+z}{1-z}+\frac{2z}{z^2-1}+\frac{2z^2}{z^4-1}+\frac{2z^4}{z^8-1}+\cdots$$

converges to $+1$ inside the circle $|z|=1$ and to -1 in its exterior. Thus the concepts of an analytic function and an analytic expression became unlinked. Borel (1895) was the first to point out definite shortcomings of Weierstrass' definition and made a number of attempts to construct a theory more general than the Weierstrass theory. The first two of these attempts were found wanting by Poincaré and Painlevé and only the third one (1917) must be judged satisfactory. Borel devoted a significant part of his scientific work to the search for a class of functions wider than the class of Weierstrass' analytic functions. In this area he stated a number of profound ideas that became the foundation of virtually all papers of his followers in this direction. Borel's key objection to Weierstrass' definition was the complete artificiality of the boundary of the "natural domain of existence of a single-valued analytic function." This boundary is truly natural if it consists of a finite or countable number of points. But if it is a closed curve, then "this boundary—writes Borel—is frequently entirely artificial in the sense that the analytic expression that yields a function with this boundary turns out to also converge uniformly outside the boundary, and so yields an external function. From Weierstrass' viewpoint these two functions, the internal and external, are *completely different*, for neither one is a continuation of the other. But this is, in fact, a single function cut in two by a singular curve, for it is possible to find a class of analytic expressions such that if one part satisfies an algebraic or a differential relation then so does the other." The analytic expressions Borel has in mind are series of rational fractions

$$\sum A_N\big/(Z-a_n),$$

where the series $\Sigma|A_n|$ converges and the singular points a_n ("the poles of the analytic expression") are everywhere dense on the closed curve under consideration or accumulate in its vicinity.

Poincaré and Wolf were critical of Borel's first attempt [to modify the Weierstrass definition]. Poincaré pointed out that it is always possible to divide the curve under consideration into two parts A and B and to define two analytic (from

Weierstrass' viewpoint) functions $\varphi_1(z)$ and $\varphi_2(z)$ such that $\varphi_1(z)$ is analytic outside A, $\varphi_2(z)$ is analytic outside B, and yet $\varphi_1(z) + \varphi_2(z) = F_1(z)$ inside the curve and $\varphi_1(z) + \varphi_2(z) = F_2(z)$ outside the curve, where $F_1(z)$ and $F_2(z)$ are two *arbitrary* functions of which one is analytic inside the curve, the other is analytic outside the curve, and neither can be continued analytically anywhere across the curve. As for Wolf, he constructed a series $\Sigma A_n/(z - a_n)$ that converges to zero inside the curve, its poles a_n accumulate in a curve outside, and the series $\Sigma|A_n|$ converges. Following Poincaré's criticism Borel changed his theory and resorted to Mittag-Leffler star expansions. Mittag-Leffler star expansions are generalizations of Taylor series, for the n-th term of such an expansion is a linear combination of the first n coefficients $a_0, a_1, a_2, \dots, a_{n-1}$ of a Taylor series. Borel expressed the conviction that the Weierstrass concept of an analytic function is too strongly tied to a particular class of analytic expressions, namely Taylor series, and that if one took instead of a Taylor series $K(x - a)$ a Mittag-Leffler star expansion, constructed for an interior point of the curve, then one could slide past the poles of the analytic expression, located everywhere densely on the singular curve, into outside space along its rays. We note that the domain of convergence of a Mittag-Leffler star expansion for an analytic function $f(z)$ is obtained as follows. One lights a source of light at the initial point a of a Mittag-Leffler star expansion $M(x - a)$, and one drives in the plane opaque pegs into all singular points of the analytic function that is being expanded. The domain of convergence of the star expansion $M(x - a)$ to $f(z)$ is all the lit places (the "star"). Borel's computations seemed to confirm his idea, for it turned out that the star expansion $M(x - a)$ for an interior point a turned out to converge on the infinite set of rays of the star to the magnitude of the external function on these rays. But Painlevé wrote a brilliant, detailed, and extremely subtle paper in which he pointed out to Borel that all this could be accidental, for there are Mittag-Leffler star expansions that converge to zero on a segment of a ray without the whole expansion representing zero. Then Borel made a third attempt—this time a successful one—by assuming that the series $\Sigma|A_n|$ *converges extremely strongly* (at least to the order of $e^{-e^{n^4}}$). He linked this assumption that the "monogenicity on the set" (that is, to the existence of $f'(z)$ on the set). Borel's new theory stood the test; for a certain class of (complex) functions (in the sense of Dirichlet) the star expansions *invariably* converge to $f(z)$. This means that knowledge of the magnitude of the function and of its derivatives completely determines the function in its entirety. This is certainly the case when the function is known on a segment. A somewhat delayed confirmation of Borel's third theory came from the Moscow papers (Privalov, Luzin). Specifically, it was shown that if a function is analytic near a rectifiable curve and vanishes almost everywhere on the curve when its points are approached along tangent paths, the function must necessarily be identically equal to zero. And since the external Borel function takes on the same values almost everywhere on the (rectifiable) singular curve as the internal function, it follows that there can be *only one such* external function. This uniqueness confirms Borel's ideas concerning the *organic* connection between the internal and external noncontinuable functions.

In their search for the most natural generalization of the notion of an analytic function, Denjoy, Bernstein, and Carleman followed an entirely different path. The most original feature of their investigations was their determination to work with *real* rather than complex *variables*.

Bernstein begins with his results on best approximation of analytic functions. His starting point is the following theorem. If $f(x)$ is holomorphic at all points of an interval $[a, b]$ then the best approximation $E_n f$ of $f(x)$ by means of an n-th degree polynomial must satisfy the inequality $E_n f < M \cdot \rho^n$, where $\rho < 1$. Bernstein calls a function (P) *quasi-analytic* if there exists an infinite sequence of natural numbers $n_1 < n_2 < \cdots < n_k < \cdots$ such that $E_{n_k} f < M\rho^{n_k}$. These functions turn out to be remarkable for, as Bernstein's fundamental theorem asserts, every (P) quasi-analytic function is completely determined on the whole interval $[a, b]$ by its values on any of its subintervals $[a', b']$. This proposition enabled Bernstein to define (P) *quasi-analytic continuation as the preservation of the inequality* $E_{n_k} f < M \cdot \rho^{n_k}$ *in a larger interval* $[c, d]$ *containing the given interval* $[a, b]$. Bernstein likened the observed fact that changing the basis $n_1 < n_2 < \cdots < n_k < \cdots$ of a (P) quasi-analytic continuation produced entirely different continuations of the given function $f(x)$ outside the interval $[a, b]$ to the *multivaluedness* of ordinary analytic functions.

Carleman gave a different definition of quasi-analyticity. Whereas Bernstein's (P) quasi-analytic function may not have a derivative, Carleman insists that the functions $f(x)$ he considers have derivatives of *all* orders. He denotes by C_A the class of all functions $f(x)$ that satisfy on a given interval $[a, b]$ the inequality $|f^{(n)}(x)| < k^n \cdot A_n$, where $A_1, A_2, \ldots, A_n, \ldots$ is a sequence of natural numbers and k is a positive constant independent of n.

The fundamental Carleman-Denjoy theorem is the following important proposition. *A necessary and sufficient condition for the family of functions* C_A *to be quasi-analytic* (that is, that its members have the property that their values on a subinterval $[a', b']$ of $[a, b]$ determine them on all of $[a, b]$) *is that every monotonic majorant of the series* $\Sigma 1/\sqrt[n]{A_n}$ *diverges*. Denjoy proved only the sufficiency of this condition. Carleman's definition has already been applied to the theory of moments. Its connection with Bernstein's definition is indeterminate in the sense that we have here neither the relation of sameness nor the relation of the general to the particular.

BIBLIOGRAPHY

N. N. Luzin, *The integral and the trigonometric series*, Moscow, 1915. (Russian)

N. N. Luzin, *Leçons sur les ensembles analytiques et leurs applications*, Paris, 1930.

H. Burkhardt, *Entwicklungen nach oszillierenden Funktionen*, Leipzig, 1901.

E. W. Hobson, *The theory of functions of a real variable*, 2 vols., Cambridge, 1921–26.

É. Borel, *Leçons sur la théorie des fonctions*, Paris, 1905.

É. Borel, Méthodes et problémes de la théorie des fonctions, *Paris*, 1922.

H. Lebesgue, Sur les fonctions représentables analytiquement, *J. Math. Pures Appl.*, Paris, 1905.

S. Bernstein, Leçons sur les propriétés extrémales et al meilleure approximation des fonctions analytiques d'une variable réelle, *Paris*, 1925.

T. Carleman, *Fonctions quasi-analytiques*, Paris, 1926.