Theodore Lyman.
June 1902.

AN ESSAY

ON THE

APPLICATION

OF

MATHEMATICAL ANALYSIS TO THE THEORIES OF ELECTRICITY AND MAGNETISM.

BY

GEORGE GREEN.

Nottingham:

PRINTED FOR THE AUTHOR, BY T. WHEELHOUSE.

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1828.

FACSIMILE-DRUCK IN 100 EXEMPLAREN. BERLIN. MAYER & MÜLLER. 1889.

TO

HIS GRACE THE DUKE OF NEWCASTLE, K. G.

Lord Lieutenant of the County of Nottingham;

VICE PRESIDENT OF THE ROYAL SOCIETY OF LITERATURE,

&c. &c. &c.

MY LORD DUKE,

I AVAIL myself of your GRACE's kind permission to introduce the following Essay to the notice of the Public, under your high auspices; and I deem myself singularly fortunate, that my first attempt to illustrate some of the most interesting phenomena of nature, should make its appearance under the patronage of a NOBLEMAN, who has always evinced a most lively interest in the promotion of Science and Literature, and particularly in the County over which he so eminently presides.

I have the honor,

MY LORD DUKE,

To be your Grace's most obedient

and grateful Servant,

GEORGE GREEN.

Sneinton, near Nottingham, March 29th. 1828.

PREFACE.

AFTER I had composed the following Essay, I naturally felt anxious to become acquainted with what had been effected by former writers on the same subject, and, had it been practicable, I should have been glad to have given, in this place, an historical sketch of its progress; my limited sources of information, however, will by no means permit me to do so; but probably I may here be allowed to make one or two observations on the few works which have fallen in my way, more particularly as an opportunity will thus offer itself, of noticing an excellent paper, presented to the Royal Society by one of the most illustrious members of that learned body, which appears to have attracted little attention, but which, on examination, will be found not unworthy the man who was able to lay the foundations of pneumatic chymistry, and to discover that water, far from being according to the opinions then received, an elementary substance, was a compound of two of the most important gasses in nature.

It is almost needless to say the author just alluded to is the celebrated CAVENDISH, who, having confined himself to such simple methods, as may readily be understood by any one possessed of an elementary knowledge of geometry and fluxions, has rendered his paper accessible to a great number of readers; and although, from subsequent remarks, he appears dissatisfied with an hypothesis which enabled him to draw some important conclusions, it will readily be perceived, on an attentive perusal of his paper, that a trifling alteration will suffice to render the whole perfectly legitimate.*

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* In order to make this quite clear, let us select one of CAVENDISH's propositions, the twentieth for instance, and examine with some attention the method there employed. The object of this proposition is to show, that when two similar conducting bodies communicate by means of a long slender canal, and are charged with electricity, the respective quantities of redundant fluid contained in them, will be proportional to the $n-1$ power of their corresponding diameters: supposing the electric repulsion to vary inversely as the n power of the distance. This is proved by considering the canal as cylindrical, and filled with incompressible fluid of uniform density: then the quantities of electricity in the interior of the two bodies are determined by a very simple geometrical construction, so that the total action exerted on the whole canal by one of them, shall exactly balance that arising from the other; and from some remarks in the 27th proposition, it appears the results thus obtained, agree very well with experiments in which real canals are employed, whether they are straight or crooked, provided, as has since been shown by COULOMB, n is equal to two. The author however confesses he is by no means able to demonstrate this, although, as we shall see immediately, it may very easily be deduced from the propositions contained in this paper.

For this purpose, let us conceive an incompressible fluid of uniform density, whose particles do not act on each other, but which are subject to the same actions from all the electricity in their vicinity, as real electric fluid of like density would be; then supposing an infinitely thin canal of this hypothetical fluid, whose perpendicular sections are all equal and similar, to pass from a point a on the surface of one of the bodies, through a portion of its mass, along the interior of the real canal, and through a part of the other body, so as to reach a point A on its surface, and then proceed from A to a in a right line, forming thus a closed circuit, it is evident from the principles of hydrostatics, and may be proved from our author's 23d proposition, that the whole of the hypothetical canal will be in equilibrium, and as every particle of the portion contained within the system is necessarily so, the rectilinear portion aA must therefore be in equilibrium. This simple consideration serves to complete CAVENDISH's demonstration, whatever may be the form or thickness of the real canal, provided the quantity of electricity in it is very small compared with that contained in the bodies. An analagous application of it will render the demonstration of the 22d proposition complete, when the two coatings of the glass plate communicate with their respective conducting bodies, by fine metallic wires of any form.

Little appears to have been effected in the mathematical theory of electricity, except immediate deductions from known formulæ, that first presented themselves in researches on the figure of the earth, of which the pricipal are,—the determination of the law of the electric density on the surfaces of conducting bodies differing little from a sphere, and on those of ellipsoids, from 1771, the date of CAVENDISH's paper, until about 1812, when M. POISSON presented to the French Institute two memoirs of singular elegance, relative to the distribution of electricity on the surfaces of conducting spheres, previously electrified and put in presence of each other. It would be quite impossible to give any idea of them here: to be duly appretiated they must be read. It will therefore only be remarked, that they are in fact founded upon the consideration of what have, in this Essay, been termed potential functions, and by means of an equation in variable differences, which may immediately be obtained from the one given in our tenth article, serving to express the relation between the two potential functions arising from any spherical surface, the author deduces the values of these functions belonging to each of the two spheres under consideration, and thence the general expression of the electric density on the surface of either, together with their actions on any exterior point.

I am not aware of any material accessions to the theory of electricity, strictly so called, except those before noticed; but since the electric and magnetic fluids are subject to one common law of action, and their theory, considered in a mathematical point of view, consists merely in developing the consequences which flow from this law, modified only by considerations arising from the peculiar constitution of natural bodies with respect to these two kinds of fluid, it is evident the mathematical theory of the latter, must be very intimately connected with that of the former; nevertheless, because it is here necessary to consider bodies as formed of an immense number of insulated particles, all acting upon each other mutually, it is easy to conceive that superior difficulties must, on this account, present themselves, and indeed, until within the last four or five years, no successful attempt to overcome them had been published. For this farther extension of the domain of analysis, we are again indebted to M. POISSON, who has already furnished us with three memoirs on magnetism: the two first contain the general equations on which the magnetic state of a body depends, whatever may be its form, together with their complete solution in case the body under consideration is a hollow spherical shell, of uniform thickness, acted upon by any exterior forces, and also when it is a solid ellipsoid subject to the influence of the earth's action. By supposing magnetic changes to require time, although an exceedingly short one, to complete them, it had been suggested that M. ARAGO's discovery relative to the magnetic effects developed in copper, wood, glass, *etc.*, by rotation, might be explained. On this hypothesis M. POISSON has founded his third memoir, and thence deduced formulæ applicable to magnetism in a state of motion. Whether the preceding hypothesis will serve to explain the singular phenomena observed by M. ARAGO or not, it would ill become me to decide; but it is probably quite adequate to account for those produced by the rapid rotation of iron bodies.

We have just taken a cursory view of what has hitherto been written, to the best of my knowledge, on subjects connected with the mathematical theory of electricity; and although many of the artifices employed in the works before mentioned are remarkable for their elegance, it is easy to see they are adapted only to particular objects, and that some general method, capable of being employed in every case, is still wanting. Indeed M.

Poisson, in the commencement of his first memoir (Mem. de l'Institut 1811), has incidentally given a method for determining the distribution of electricity on the surface of a spheroid of any form, which would naturally present itself to a person occupied in these researches, being in fact nothing more than the ordinary one noticed in our introductory observations, as requiring the resolution of the equation *(a)*. Instead however of supposing, as we have done, that the point p must be upon the surface, in order that the equation may subsist, M. Poisson availing himself of a general fact, which was then supported by experiment only, has conceived the equation to hold good wherever this point may be situated, provided it is within the spheroid, but even with this extension the method is liable to the same objection as before.

Considering how desirable it was that a power of universal agency, like electricity, should, as far as possible, be submitted to calculation, and reflecting on the advantages that arise in the solution of many difficult problems, from dispensing altogether with a particular examination of each of the forces which actuate the various bodies in any system, by confining the attention solely to that peculiar function on whose differentials they all depend, I was induced to try whether it would be possible to discover any general relations, existing between this function and the quantities of electricity in the bodies producing it. The advantages Laplace had derived in the third book of the Mecanique Celeste, from the use of a partial differential equation of the second order, there given, were too marked to escape the notice of any one engaged with the present subject, and naturally served to suggest that this equation might be made subservient to the object I had in view. Recollecting, after some attempts to accomplish it, that previous researches on partial differential equations, had shown me the necessity of attending to what have, in this Essay, been denominated the singular values of functions, I found, by combining this consideration with the preceding, that the resulting method was capable of being applied with great advantage to the electrical theory, and was thus, in a short time, enabled to demonstrate the general formulæ contained in the preliminary part of the Essay. The remaining part ought to be regarded principally as furnishing particular examples of the use of these general formulæ; their number might with great ease have been increased, but those which are given, it is hoped, will suffice to point out to mathematicians, the mode of applying the preliminary results to any case they may wish to investigate. The hypotheses on which the received theory of magnetism is founded, are by no means so certain as the facts on which the electrical theory rests; it is however not the less necessary to have the means of submitting them to calculation, for the only way that appears open to us in the investigation of these subjects, which seem as it were desirous to conceal themselves from our view, is to form the most probable hypotheses we can, to deduce rigorously the consequences which flow from them, and to examine whether such consequences agree numerically with accurate experiments.

The applications of analysis to the physical Sciences, have the double advantage of manifesting the extraordinary powers of this wonderful instrument of thought, and at the same time of serving to increase them; numberless are the instances of the truth of this assertion. To select one we may remark, that M. Fourier, by his investigations relative to heat, has not only discovered the general equations on which its motion depends, but has likewise been led to new analytical formulæ, by whose aid M. M. Cauchy & Poisson have been enabled to give the complete theory of the motion of the waves in an indefinitely extended fluid. The same formulæ have also put us in possession of the solutions of many other interesting problems too numerous to be detailed here.—It must certainly be regarded as a pleasing prospect to analists, that at a time when astronomy, from the state of perfection to which it has attained, leaves little room for farther applications of their art, the rest of the physical sciences should show themselves daily more and more willing to submit to it; and, amongst other things, probably the theory that supposes light to depend on the undulations of a luminiferous fluid, and to which the celebrated Dr. T. Young has given such plausibility, may furnish a useful subject of research, by affording new opportunities of applying the general theory of the motion of fluids. The number of these opportunities can scarcely be too great, as it must be evident to those who have examined the subject, that, although we have long been in possession of the general equations on which this kind of motion depends, we are not yet well acquainted with the various limitations it will be necessary to introduce, in order to adapt them to the different physical circumstances which may occur.

Should the present Essay tend in any way to facilitate the application of analysis to one of the most interesting of the physical sciences, the author will deem himself amply repaid for any labour he may have bestowed upon it; and it is hoped the difficulty of the subject will incline mathematicians to read this work with indulgence, more particularly when they are informed that it was written by a young man, who has been obliged to obtain the little knowledge he possesses, at such intervals and by such means, as other indispensable avocations which offer but few opportunities of mental improvement, afforded.

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AN ESSAY

ON THE APPLICATION OF

MATHEMATICAL ANALYSIS TO THE THEORIES OF ELECTRICITY AND MAGNETISM.

INTRODUCTORY OBSERVATIONS.

THE object of this Essay is to submit to Mathematical Analysis the phenomena of the equilibrium of the Electric and Magnetic Fluids, and to lay down some general principles equally applicable to perfect and imperfect conductors; but, before entering upon the calculus, it may not be amiss to give a general idea of the method that has enabled us to arrive at results, remarkable for their simplicity and generality, which it would be very difficult if not impossible to demonstrate in the ordinary way.

It is well known, that nearly all the attractive and repulsive forces existing in nature are such, that if we consider any material point p, the effect, in a given direction, of all the forces acting upon that point, arising from any system of bodies S under consideration, will be expressed by a partial differential of a certain function of the co-ordinates which serve to define the point's position in space. The consideration of this function is of great importance in many inquiries, and probably there are none in which its utility is more marked than in those about to engage our attention. In the sequel we shall often have occasion to speak of this function, and will therefore, for abridgment, call it the potential function arising from the system S. If p be a particle of positive electricity under the influence of forces arising from any electrified body, the function in question, as is well known, will be obtained by dividing the quantity of electricity in each element of the body, by its distance from the particle p, and taking the total sum of these quotients for the whole body, the quantities of electricity in those elements which are negatively electrified, being regarded as negative.

It is by considering the relations existing between the density of the electricity in any system, and the potential functions thence arising, that we have been enabled to submit many electrical phenomena to calculation, which had hitherto resisted the attempts of analysts; and the generality of the consideration here employed, ought necessarily, and does, in fact, introduce a great generality into the results obtained from it. There is one consideration peculiar to the analysis itself, the nature and utility of which will be best illustrated by the following sketch:

Suppose it were required to determine the law of the distribution of the electricity on a closed conducting surface A without thickness, when placed under the influence of any electrical forces whatever: these forces, for greater simplicity, being reduced to three, X, Y, and Z, in the direction of the rectangular co-ordinates, and tending to increase them. Then ρ representing the density of the electricity on an element $d\sigma$ of the surface, and r the distance between $d\sigma$ and p, any other point of the surface, the equation for determining ρ which would be employed in the ordinary method, when the problem is reduced to its simplest form, is known to be

$$\text{Cons} = a = \int \frac{\rho d\sigma}{r} - \int (Xdx + Ydy + Zdz) ; \ldots . (a)$$

the first integral relative to $d\sigma$ extending over the whole surface A, and the second representing the function whose complete differential is $Xdx + Ydy + Zdz$, x, y and z being the co-ordinates of p.

This equation is supposed to subsist, whatever may be the position of p, provided it is situate upon A. But we have no general theory of equations of this description, and whenever we are enabled to resolve one of them, it is because some consideration peculiar to the problem renders, in that particular case, the solution comparatively simple, and must be looked upon as the effect of chance, rather than of any regular and scientific procedure.

We will now take a cursory view of the method it is proposed to substitute in the place of the one just mentioned.

Let us make $B = \int (Xdx + Ydy + Zdz)$ whatever may be the position of the point p, $V = \int \frac{\rho d\sigma}{r}$ when p is situate any where within the surface A, and $V' = \int \frac{\rho d\sigma}{r}$ when p is exterior to it: the two quantities V and V', although expressed by the same definite integral, are essentially distinct functions of x, y, and z, the rectangular co-ordinates of p; these functions, as is well known, having the property of satisfying the partial differential equations

$$0 = \frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2},$$
$$0 = \frac{d^2 V'}{dx^2} + \frac{d^2 V'}{dy^2} + \frac{d^2 V'}{dz^2}.$$

If now we could obtain the values of V and V' from these equations, we should have immediately, by differentiation, the required value of ρ, as will be shown in the sequel.

In the first place, let us consider the function V, whose value at the surface A is given by the equation (a), since this may be written

$$a = \overline{V} - \overline{B},$$

the horizontal line over a quantity indicating that it belongs to the surface A. But, as the general integral of the partial differential equation ought to contain two arbitrary functions, some other condition is requisite for the complete determination of V. Now since $V = \int \frac{\rho d\sigma}{r}$, it is evident that none of its differential co-efficients can become infinite when p is situate any where within the surface A, and it is worthy of remark, that this is precisely the condition required: for, as will be afterwards shown, when it is satisfied we shall have generally

$$V = -\int (\rho) d\sigma \overline{V};$$

the integral extending over the whole surface, and (ρ) being a quantity dependant upon the respective positions of p and $d\sigma$.

All the difficulty therefore reduces itself to finding a function V, which satisfies the partial differential equation, becomes equal to the known value of $\overline{V}$ at the surface, and is moreover such that none of its differential coefficients shall be infinite when p is within A.

In like manner, in order to find V', we shall obtain $\overline{V}'$, its value at A, by means of the equation (a), since this evidently becomes

$$a = \overline{V}' - \overline{B}, \quad \text{i. e.} \quad \overline{V}' = \overline{V}.$$

Moreover it is clear, that none of the differential co-efficients of $V' = \int \frac{\rho d\sigma}{r}$ can be infinite when p is exterior to the surface A, and when p is at an infinite distance from A, V' is equal to *zero*. These two conditions combined with the partial differential equation in V', are sufficient in conjunction with its known value $\overline{V}'$ at the surface A for the complete determination of V', since it will be proved hereafter, that when they are satisfied we shall have

$$V' = -\int (\rho) d\sigma \overline{V}';$$

the integral, as before, extending over the whole surface A, and (ρ) being a quantity dependant upon the respective position of p and $d\sigma$.

It only remains therefore to find a function V' which satisfies the partial differential equation, becomes equal to $\overline{V}'$ when p is upon the surface A, vanishes when p is at an infinite distance from A, and is besides such, that none of its differential co-efficients shall be infinite, when the point p is exterior to A.

All those to whom the practice of analysis is familiar, will readily perceive that the problem just mentioned, is far less difficult than the direct resolution of the equation (a), and therefore the solution of the question originally proposed has been rendered much easier by what has preceded. The peculiar consideration relative to the differential co efficients of V and V', by restricting the generality of the integral of the partial differential equation, so that it can in fact contain only one arbitrary function, in the place of two which it ought otherwise to have contained, and, which has thus enabled us to effect the simplification in question, seems worthy of the attention of analysts, and may be of use in other researches where equations of this nature are employed.

We will now give a brief account of what is contained in the following Essay. The first seven articles are employed in demonstrating some very general relations existing between the density of the electricity on surfaces and in solids, and the corresponding potential functions. These serve as a foundation to the more particular applications which follow them. As it would be difficult to give any idea of this part without employing analytical symbols, we shall content ourselves with remarking, that it contains a number of singular equations of great generality and simplicity, which seem capable of being applied to many departments of the electrical theory besides those considered in the following pages.

In the eight article we have determined the general values of the densities of the electricity on the inner and outer surfaces of an insulated electrical jar, when, for greater generality, these surfaces are supposed to be connected with separate conductors charged in any way whatever; and have proved, that for the same jar, they depend solely on the difference existing between the two constant quantities, which express the values of the potential functions within the respective conductors. Afterwards, from these general values the following consequences have been deduced:—

When in an insulated electrical jar we consider only the electricity accumulated on the two surfaces of the glass itself, the total quantity on the inner surface is precisely equal to that on the outer surface, and of a contrary sign, notwithstanding the great accumulation of electricity on each of them: so that if a communication were established between the two sides of the jar, the sum of the quantities of electricity which would manifest themselves on the two metallic coatings, after the discharge, is exactly equal to that which, before it had taken place, would have been observed to have existed on the surfaces of the coatings farthest from the glass, the only portions then sensible to the electrometer.

If an electrical jar communicates by means of a long slender wire with a spherical conductor, and is charged in the ordinary way, the density of the electricity at any point of the interior surface of the jar, is to the density on the conductor itself, as the radius of the spherical conductor to the thickness of the glass in that point.

The total quantity of electricity contained in the interior of any number of equal and similar jars, when one of them communicates with the prime conductor and the others are charged *by cascade*, is precisely equal to that, which one only would receive, if placed in communication with the same conductor, its exterior surface being connected with the common reservoir. This method of charging batteries, therefore, must not be employed when any great accumulation of electricity is required.

It has been shown by M. Poisson, in his first Memoir on Magnetism (Mem. de l'Acad. de Sciences, 1821 et 1822), that when an electrified body is placed in the interior of a hollow spherical conducting shell of uniform thickness, it will not be acted upon in the slightest degree by any bodies exterior to the shell, however intensely they may be electrified. In the ninth article of the present Essay this is proved to be generally true, whatever may be the form or thickness of the conducting shell.

In the tenth article there will be found some simple equations, by means of which the density of the electricity induced on a spherical conducting surface, placed under the influence of any electrical forces whatever, is immediately given; and thence the general value of the potential function for any point either within or without this surface is determined from the arbitrary value at the surface itself, by the aid of a definite integral. The proportion in which the electricity will divide itself between two insulated conducting spheres of different diameters, connected by a very fine wire, is afterwards considered; and it is proved, that when the radius of one of them is small compared with the distance between their surfaces, the product of the mean density of the electricity on either sphere, by the radius of that sphere, and again by the shortest distance of its surface from the centre of the other sphere, will be the same for both. Hence when their distance is very great, the densities are in the inverse ratio of the radii of the spheres.

When any hollow conducting shell is charged with electricity, the whole of the fluid is carried to the exterior surface, without leaving any portion on the interior one, as may be immediately shown from the fourth and fifth articles. In the experimental verification of this, it is necessary to leave a small orifice in the shell: it became therefore a problem of some interest to determine the modification which this alteration would produce. We have, on this account, terminated the present article, by investigating the law of the distribution of electricity on a thin spherical conducting shell, having a small circular orifice, and have found that its density is very nearly constant on the exterior surface, except in the immediate vicinity of the orifice; and the density at any point p of the inner surface, is to the constant density on the outer one, as the product of the diameter of a circle into the cube of the radius of the orifice, is to the product of three times the circumference of that circle into the cube of the distance of p from the centre of the orifice; excepting as before those points in its immediate vicinity. Hence, if the diameter of the sphere were twelve inches, and that of the orifice one inch, the density at the point on the inner surface opposite the centre of the orifice, would be less than the hundred and thirty thousandth part of the constant density on the exterior surface.

In the eleventh article some of the effects due to atmospherical electricity are considered; the subject is not however insisted upon, as the great variability of the cause which produces them, and the impossibility of measuring it, gives a degree of vagueness to these determinations.

The form of a conducting body being given, it is in general a problem of great difficulty, to determine the law of the distribution of the electric fluid on its surface: but it is possible to give different forms, of almost every imaginable variety of shape, to conducting bodies; such, that the values of the density of the electricity on their surfaces may be rigorously assignable by the most simple calculations: the manner of doing this is explained in the twelfth article, and two examples of its use are given. In the last, the resulting form of the conducting body is an oblong spheriod, and the density of the electricity on its surface, here found, agrees with the one long since deduced from other methods.

Thus far perfect conductors only have been considered. In order to give an example of the application of theory to bodies which are not so, we have, in the thirteenth article, supposed the matter of which they are formed to be endowed with a constant coercive force equal to β, and analagous to friction in its operation, so that when the resultant of the electric forces acting upon any one of their elements is less than β, the electrical state of this element shall remain unchanged; but, so soon as it begins to exceed β, a change shall ensue. Then imagining a solid of revolution to turn continually about its axis, and to be subject to a constant electrical force f acting in parallel right lines, we determine the permanent electrical state at which the body will ultimately arrive. The result of the analysis is, that in consequence of the coercive force β, the solid will receive a new polarity, equal to that which would be induced in it if it were a perfect conductor and acted upon by the constant force β, directed in lines parallel to one in the body's equator, making the angle $90°+\gamma$, with a plane passing through its axis and parallel to the direction of f: f being supposed resolved into two forces, one in the direction of the body's axis, the other

b directed along the intersection of its equator with the plane just mentioned, and γ being determined by the equation

$$\operatorname{Sin} \gamma = \frac{\beta}{b}.$$

In the latter part of the present article the same problem is considered under a more general point of view, and treated by a different analysis: the body's progress from the initial, towards that permanent state it was the object of the former part to determine is exhibited, and the great rapidity of this progress made evident by an example.

The phenomena which present themselves during the rotation of iron bodies, subject to the influence of the earth's magnetism, having lately engaged the attention of experimental philosophers, we have been induced to dwell a little on the solution of the preceding problem, since it may serve in some measure to illustrate what takes place in these cases. Indeed, if there were any substances in nature whose magnetic powers, like those of iron and nickel, admit of considerable developement, and in which moreover the coercive force was, as we have here supposed it, the same for all their elements, the results of the preceding theory ought scarcely to differ from what would be observed in bodies formed of such substances, provided no one of their dimensions was very small, compared with the others. The hypothesis of a constant coercive force was adopted in this article, in order to simplify the calculations: probably, however, this is not exactly the case of nature, for a bar of the hardest steel has been shown (I think by Mr. Barlow) to have a very considerable degree of magnetism induced in it by the earth's action, which appears to indicate, that although the coercive force of some of its particles is very great, there are others in which it is so small as not to be able to resist the feeble action of the earth. Nevertheless, when iron bodies are turned slowly round their axes, it would seem that our theory ought not to differ greatly from observation; and in particular, it is very probable the angle γ might be rendered sensible to experiment, by sufficiently reducing b the component of the force f.

The remaining articles treat of the theory of magnetism. This theory is here founded on an hypothesis relative to the constitution of magnetic bodies, first proposed by Coulomb, and afterwards generally received by philosophers, in which they are considered as formed of an infinite number of conducting elements, separated by intervals absolutely impervious to the magnetic fluid, and by means of the general results contained in the former part of the Essay, we readily obtain the necessary equations for determining the magnetic state induced in a body of any form, by the action of exterior magnetic forces. These equations accord with those M. Poisson has found by a very different method. (Mem. de l'Acad. des Sciences, 1821 et 1822.)

If the body in question be a hollow spherical shell of constant thickness, the analysis used by Laplace (Mec. Cel. Liv. 3) is applicable, and the problem capable of a complete solution, whatever may be the situation of the centres of the magnetic forces acting upon it. After having given the general solution, we have supposed the radius of the shell to become infinite, its thickness remaining unchanged, and have thence deduced formula belonging to an indefinitely extended plate of uniform thickness. From these it follows, that when the point p, and the centres of the magnetic forces are situate on opposite sides of a soft iron plate of great extent, the total action on p will have the same direction as the resultant of all the forces, which would be exerted on the points p, p', p'', p''' *etc. in infinitum* if no plate were interposed, and will be equal to this resultant multiplied by a very small constant quantity: the points p, p', p'', p''' *etc.* being all on a right line perpendicular to the flat surfaces of the plate, and receding from it so, that the distance between any two consecutive points may be equal to twice the plate's thickness.

What has just been advanced will be sensibly correct, on the supposition of the distances between the point p and the magnetic centres not being very great, compared with the plate's thickness, for, when these distances are exceedingly great, the interposition of the plate will make no sensible alteration in the force with which p is solicited.

When an elongated body, as a steel wire for instance, has, under the influence of powerful magnets, received a greater degree of magnetism than it can retain alone, and is afterwards left to itself, it is said to be magnetized to saturation. Now if in this state we consider any one of its conducting elements, the force with which a particle p of magnetism situate within the element tends to move, will evidently be precisely equal to its coercive force f, and in equilibrium with it. Supposing therefore this force to be the same for every element, it is clear that the degree of magnetism retained by the wire in a state of saturation, is, on account of its elongated form, exactly the same as would be induced by the action of a constant force, equal to f, directed along lines parallel to its axis, if all the elements were perfect conductors; and consequently, may readily be determined by the general theory. The number and accuracy of COULOMB's experiments on cylindric wires magnetized to saturation, rendered an application of theory to this particular case very desirable, in order to compare it with experience. We have therefore effected this in the last article, and the result of the comparison is of the most satisfactory kind.

GENERAL PRELIMINARY RESULTS.

(1.) THE function which represents the sum of all the electric particles acting on a given point divided by their respective distances from this point, has the property of giving, in a very simple form, the forces by which it is solicited, arising from the whole electrified mass.—We shall, in what follows, endeavour to discover some relations between this function, and the density of the electricity in the mass or masses producing it, and apply the relations thus obtained to the theory of electricity.

Firstly, let us consider a body of any form whatever, through which the electricity is distributed according to any given law, and fixed there, and let x', y', z', be the rectangular co-ordinates of a particle of this body, ρ' the density of the electricity in this particle, so that $dx'dy'dz'$ being the volume of the particle, $\rho'dx'dy'dz'$ shall be the quantity of electricity it contains: moreover, let r' be the distance between this particle and a point p

exterior to the body, and V represent the sum of all the particles of electricity divided by their respective distances from this point, whose co-ordinates are supposed to be x, y, z, then shall we have

$$r' = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2},$$

and

$$V = \int \frac{\rho' dx' dy' dz'}{r'};$$

the integral comprehending every particle in the electrified mass under consideration.

LAPLACE has shown, in his Mec. Celeste, that the function V has the property of satisfying the equation

$$0 = \frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2},$$

and as this equation will be incessantly recurring in what follows, we shall write it in the abridged form $0 = \delta V$; the symbol δ being used in no other sense throughout the whole of this Essay.

In order to prove that $0 = \delta V$, we have only to remark, that by differentiation we immediately obtain $0 = \delta \frac{1}{r'}$, and consequently each element of V substituted for V in the above equation satisfies it; hence the whole integral (being considered as the sum of all these elements) will also satisfy it. This reasoning ceases to hold good when the point p is within the body, for then, the co-efficients of some of the elements which enter into V becoming infinite, it does not therefore necessarily follow that V satisfies the equation

$$0 = \delta V,$$

although each of its elements, considered separately, may do so.

In order to determine what δV becomes for any point within the body, conceive an exceedingly small sphere whose radius is a inclosing the point p at the distance b from its centre, a and b being exceedingly small quantities. Then, the value of V may be considered as composed of two parts, one due to the sphere itself, the other due to the whole mass exterior to it: but the last part evidently becomes equal to *zero* when substituted for V in δV, we have therefore only to determine the value of δV for the small sphere itself, which value is known to be

$$\delta\left(2\pi a^2 \rho - \frac{2\pi}{3} b^2 \rho\right);$$

ρ being equal to the density within the sphere and consequently to the value of ρ' at p. If now $x_{\prime}, y_{\prime}, z_{\prime}$ be the co-ordinates of the centre of the sphere, we have

$$b^2 = (x_{\prime} - x)^2 + (y_{\prime} - y)^2 + (z_{\prime} - z)^2,$$

and consequently $\delta\left(2\pi a^2 \rho - \frac{2\pi}{3} b^2 \rho\right) = -4\pi\rho.$

Hence, throughout the interior of the mass

$$0 = \delta V + 4\pi\rho;$$

of which, the equation $0 = \delta V$ for any point exterior to the body is a particular case, seeing that, here $\rho = 0$.

Let now q be any line terminating in the point p, supposed without the body, then $-\left(\frac{dV}{dq}\right)=$ the force tending to impel a particle of positive electricity in the direction of q, and tending to increase it. This is evident, because each of the elements of V substituted for V in $-\left(\frac{dV}{dq}\right)$, will give the force arising from this element in the direction tending to increase q, and consequently, $-\left(\frac{dV}{dq}\right)$ will give the sum of all the forces due to every element of V, or the total force acting on p in the same direction. In order to show that this will still hold good, although the point p be within the body; conceive the value of V to be divided into two parts as before, and moreover let p be at the surface of the small sphere, or $b=a$, then the force exerted by this small sphere will be expressed by

$$\frac{4\pi a\rho}{3}\left(\frac{da}{dq}\right);$$

da being the increment of the radius a, corresponding to the increment dq of q, which force evidently vanishes when $a=0$: we need therefore have regard only to the part due to the mass exterior to the sphere, and this is evidently equal to

$$V-\frac{4\pi}{3}a^2\rho.$$

But as the first differentials of this quantity are the same as those of V when a is made to vanish, it is clear, that whether the point p be within or without the mass, the force acting upon it in the direction of q increasing, is always given by $-\left(\frac{dV}{dq}\right)$.

Although in what precedes we have spoken of one body only, the reasoning there employed is general, and will apply equally to a system of any number of bodies whatever, in those cases even, where there is a finite quantity of electricity spread over their surfaces, and it is evident that we shall have for a point p in the interior of any one of these bodies

$$0=\delta V+4\pi\rho. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Moreover, the force tending to increase a line q ending in any point p within or without the bodies, will be likewise given by $-\left(\frac{dV}{dq}\right)$; the function V representing the sum of all the electric particles in the system divided by their respective distances from p. As this function, which gives in so simple a form the values of the forces by which a particle p of electricity, any how situated, is impelled, will recur very frequently in what follows, we have ventured to call it the potential function belonging to the system, and it will evidently be a function of the co-ordinates of the particle p under consideration.

(2.) It has been long known from experience, that whenever the electric fluid is in a state of equilibrium in any system whatever of perfectly conducting bodies, the whole of the electric fluid will be carried to the surface of those bodies, without the smallest portion of electricity remaining in their interior: but I do not know that this has ever been shown to be a necessary consequence of the law of electric repulsion, which is found to take place in nature. This however may be shown to be the case for every imaginable system of conducting bodies, and is an immediate consequence of what has preceded. For let x, y, z, be the rectangular co-ordinates of any particle p in the interior of one of the bodies; then will $-\left(\frac{dV}{dx}\right)$ be the force with which p is impelled in the direction of the co-ordinate x, and tending to increase it. In the same way $-\frac{dV}{dy}$ and $-\frac{dV}{dz}$ will be the forces in y and z, and since the fluid is in equilibrium all these forces are equal to *zero:* hence

$$0=\frac{dV}{dx}dx+\frac{dV}{dy}dy+\frac{dV}{dz}dz=dV,$$

which equation being integrated gives

$$V=\text{const.}$$

This value of V being substituted in the equation (1) of the preceding number gives

$$\rho=0,$$

and consequently shows, that the density of the electricity at any point in the interior of any body in the system is equal to *zero*.

The same equation (1) will give the value of ρ the density of the electricity in the interior of any of the bodies, when there are not perfect conductors, provided we can ascertain the value of the potential function V in their interior.

(3.) Before proceeding to make known some relations which exist between the density of the electric fluid at the surfaces of bodies, and the corresponding values of the potential functions within and without those surfaces, the electric fluid being confined to them alone, we shall in the first place, lay down a general theorem which will afterwards be very useful to us. This theorem may be thus ennunciated:

Let U and V be two continuous functions of the rectangular co-ordinates x, y, z, whose differential co-efficients do not become infinite at any point within a solid body of any form whatever; then will

$$\int dxdydz\, U\delta V+\int d\sigma\, U\left(\frac{dV}{dw}\right)=\int dxdydz\, V\delta U+\int d\sigma\, V\left(\frac{dU}{dw}\right);$$

the triple integrals extending over the whole interior of the body, and those relative to $d\sigma$, over its surface, of which $d\sigma$ represents an element: dw being an infinitely small line perpendicular to the surface, and measured from this surface towards the interior of the body.

To prove this let us consider the triple integral

$$\int dxdydz\left\{\left(\frac{dV}{dx}\right)\left(\frac{dU}{dx}\right)+\left(\frac{dV}{dy}\right)\left(\frac{dU}{dy}\right)+\left(\frac{dV}{dz}\right)\left(\frac{dU}{dz}\right)\right\}.$$

The method of integration by parts, reduces this to

$$\int dydz V'' \frac{dU''}{dx}-\int dydz V' \frac{dU'}{dx}+\int dxdz V'' \frac{dU''}{dy}-\int dxdz V' \frac{dU'}{dy}$$

$$+\int dxdy V'' \frac{dU''}{dz}-\int dxdy V' \frac{dU'}{dz}-\int dxdydz V\left\{\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}+\frac{d^2U}{dz^2}\right\};$$

the accents over the quantities indicating, as usual, the values of those quantities at the limits of the integral, which in the present case are on the surface of the body, over whose interior the triple integrals are supposed to extend.

Let us now consider the part $\int dydzV'' \frac{dU''}{dx}$ due to the greater values of x. It is easy to see since dw is every where perpendicular to the surface of the solid, that if $d\sigma''$ be the element of this surface corresponding to $dydz$, we shall have

$$dydz = -\frac{dx}{dw} d\sigma'',$$

and hence by substitution

$$\int dydzV'' \frac{dU''}{dx} = -\int d\sigma'' \frac{dx}{dw} V'' \frac{dU''}{dx}.$$

In like manner it is seen, that in the part $-\int dydzV' \frac{dU'}{dx}$ due to the smaller values of x, we shall have $dydz = +\frac{dx}{dw} d\sigma'$, and consequently

$$-\int dydzV' \frac{dU'}{dx} = -\int d\sigma' \frac{dx}{dw} V' \frac{dU'}{dx}.$$

Then, since the sum of the elements represented by $d\sigma'$, together with those represented by $d\sigma''$, constitute the whole surface of the body, we have by adding these two parts

$$\int dydz\left(V'' \frac{dU''}{dx} - V' \frac{dU'}{dx}\right) = -\int d\sigma \frac{dx}{dw} V \frac{dU}{dx}:$$

where the integral relative to $d\sigma$ is supposed to extend over the whole surface, and dx to be the increment of x corresponding to the increment dw.

In precisely the same way we have

$$\int dxdz\left(V'' \frac{dU''}{dy} - V' \frac{dU'}{dy}\right) = -\int d\sigma \frac{dy}{dw} V \frac{dU}{dy},$$

$$\text{and} \int dxdy\left(V'' \frac{dU''}{dz} - V' \frac{dU'}{dz}\right) = -\int d\sigma \frac{dz}{dw} V \frac{dU}{dz};$$

therefore, the sum of all the double integrals in the expression before given will be ob tained by adding together the three parts just found; we shall thus have

$$-\int d\sigma V \left\{ \frac{dU}{dx}\frac{dx}{dw} + \frac{dU}{dy}\frac{dy}{dw} + \frac{dU}{dz}\frac{dz}{dw} \right\} = -\int d\sigma V \frac{dU}{dw};$$

where V and $\frac{dU}{dw}$ represent the values at the surface of the body. Hence, the integral

$$\int dxdydz \left\{ \frac{dV}{dx}\frac{dU}{dx} + \frac{dV}{dy}\frac{dU}{dy} + \frac{dV}{dz}\frac{dU}{dz} \right\},$$

by using the characteristic δ in order to abridge the expression, becomes

$$-\int d\sigma V \frac{dU}{dw} - \int dxdydz V \delta U.$$

Since the value of the integral just given remains unchanged when we substitute V in the place of U and reciprocally, it is clear, that it will also be expressed by

$$-\int d\sigma U \frac{dV}{dw} - \int dxdydz\, U\delta V.$$

Hence, if we equate these two expressions of the same quantity, after having changed their signs, we shall have

$$\int d\sigma V \frac{dU}{dw} + \int dxdydz\, V\delta U = \int d\sigma U \frac{dV}{dw} + \int dxdydz\, U\delta V. \quad . \quad . \quad (2)$$

Thus the theorem appears to be completely established, whatever may be the form of the functions U and V.

In our ennunciation of the theorem, we have supposed the differentials of U and V to be finite within the body under consideration, a condition, the necessity of which does not appear explicitly in the demonstation, but, which is understood in the method of integration by parts there employed.

In order to show more clearly the necessity of this condition, we will now determine the modification which the formula must undergo, when one of the functions, U for example, becomes infinite within the body; and let us suppose it to do so in one point p' only: moreover, infinitely near this point let U be sensibly equal to $\frac{1}{r}$; r being the distance between the point p' and the element $dxdydz$. Then if we suppose an infinitely small sphere whose radius is a to be described round p', it is clear that our theorem is applicable to the whole of the body exterior to this sphere, and since, $\delta U = \delta \frac{1}{r} = 0$ within the sphere, it is evident, the triple integrals may still be supposed to extend over the whole body, as the greatest error that this supposition can induce, is a quantity of the order a^2. Moreover, the part of $\int d\sigma U \frac{dV}{dw}$, due to the surface of the small sphere is only an infinitely small quantity of the order a; there only remains therefore to consider, the part of $\int d\sigma V \frac{dU}{dw}$ due to this same surface, which, since we have here $\frac{dU}{dw} = \frac{dU}{dr} = \frac{d\frac{1}{r}}{dr} = \frac{-1}{r^2} = \frac{-1}{a^2}$, becomes

$$-4\pi V'$$

when the radius a is supposed to vanish. Thus, the equation (2) becomes

$$\int dxdydz\, U\delta V + \int d\sigma U \frac{dV}{dw} = \int dxdydz\, V\delta U + \int d\sigma V \frac{dU}{dw} - 4\pi V'; \quad . \quad . \quad (3)$$

where, as in the former equation, the triple integrals extend over the whole volume of the body, and those relative to $d\sigma$, over its exterior surface: V' being the value of V at the point p'.

In like manner, if the function V be such, that it becomes infinite for any point p'' within the body, and is moreover, sensibly equal to $\frac{1}{r'}$, infinitely near this point, as U is infinitely near to the point p', it is evident from what has preceded that we shall have

$$\int dxdydzU\delta V+\int d\sigma U\frac{dV}{dw}-4\pi U''=\int dxdydzV\delta U+\int d\sigma V\frac{dU}{dw}-4\pi V'; \quad . \quad . \quad (3')$$

the integrals being taken as before, and U'' representing the value of U, at the point p'' where V becomes infinite. The same proeess will evidently apply, however great may be the number of similar points belonging to the functions U and V.

For abridgment, we shall in what follows, call those singular values of a given function, where its differential co-efficients become infinite, and the condition originally imposed upon U and V will be expressed by saying, that neither of them has any singular values within the solid body under consideration.

(4.) We will now proceed to determine some relations existing between the density of the electric fluid at the surface of a body, and the potential functions thence arising, within and without this surface. For this, let $\rho d\sigma$ be the quantity of electricity on an element $d\sigma$ of the surface, and V, the value of the potential function for any point p within it, of which the co-ordinates are x, y, z. Then, if V' be the value of this function for any other point p' exterior to this surface, we shall have

$$V=\int\frac{\rho d\sigma}{\sqrt{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2}};$$

ξ, η, ζ, being the co-ordinates of $d\sigma$, and

$$V'=\int\frac{\rho d\sigma}{\sqrt{(\xi-x')^2+(\eta-y')^2+(\zeta-z')^2}}:$$

the integrals relative to $d\sigma$ extending over the whole surface of the body.

It might appear at first view, that to obtain the value of V' from that of V, we should merely have to change x, y, z, into x', y', z': but, this is by no means the case; for, the form of the potential function changes suddenly, in passing from the space within to that without the surface. Of this, we may give a very simple example, by supposing the surface to be a sphere whose radius is a and centre at the origin of the co-ordinates; then, if the density ρ be constant, we shall have

$$V=4\pi\rho a \text{ and } V'=\frac{4\pi a^2}{\sqrt{x'^2+y'^2+z'^2}};$$

which are essentially distinct functions.

With respect to the functions V and V' in the general case, it is clear that each of them will satisfy LAPLACE's equation, and consequently

$$0=\delta V \text{ and } 0=\delta' V':$$

moreover, neither of them will have singular values; for any point of the spaces to which they respectively belong, and at the surface itself, we shall have

$$\overline{V}=\overline{V'}$$

the horizontal lines over the quantities indicating that they belong to the surface. At an infinite distance from this surface, we shall likewise have

$$V'=0.$$

We will now show, that if any two functions whatever are taken, satisfying these conditions, it will always be in our power to assign one, and only one value of ρ, which will produce them for corresponding potential functions. For this we may remark, that the equation (3) art. 3 being applied to the space within the body, becomes, by making $U=\frac{1}{r}$,

$$\int \frac{d\sigma}{r}\left(\frac{d\overline{V}}{dw}\right)=\int d\sigma \overline{V}\left(\frac{d\frac{1}{r}}{dw}\right)-4\pi V;$$

since $U=\frac{1}{r}$, has but one singular point, viz. p; and, we have also $\delta V=0$ and $\delta \frac{1}{r}=0$: r being the distance between the point p to which V belongs, and the element $d\sigma$.

If now, we conceive a surface inclosing the body at an infinite distance from it, we shall have, by applying the formula (2) of the same article to the space between the surface of the body and this imaginary exterior surface (seeing that here $\frac{1}{r}=U$ has no singular value)

$$\int \frac{d\sigma}{r}\left(\frac{d\overline{V'}}{dw'}\right)=\int d\sigma \overline{V'}\left(\frac{d\frac{1}{r}}{dw'}\right):$$

since the part due to the infinite surface may be neglected, because V' is there equal to *zero*. In this last equation, it is evident that dw' is measured from the surface, into the exterior space, and hence

$$\left(\frac{d\frac{1}{r}}{dw}\right)=-\left(\frac{d\frac{1}{r}}{dw'}\right) \quad i.\ e. \quad 0=\left(\frac{d\frac{1}{r}}{dw}\right)+\left(\frac{d\frac{1}{r}}{dw'}\right);$$

which equation reduces the sum of the two just given to

$$\int \frac{d\sigma}{r}\left\{\left(\frac{d\overline{V}}{dw}\right)+\left(\frac{d\overline{V'}}{dw'}\right)\right\}=-4\pi V.$$

In exactly the same way, for the point p' exterior to the surface, we shall obtain

$$\int \frac{d\sigma}{r'}\left\{\left(\frac{d\overline{V}}{dw}\right)+\left(\frac{d\overline{V'}}{dw'}\right)\right\}=-4\pi V'.$$

Hence it appears, that there exists a value of ρ, viz. $\rho=\frac{-1}{4\pi}\left\{\left(\frac{d\overline{V}}{dw}\right)+\left(\frac{d\overline{V'}}{dw'}\right)\right\}$, which will give V and V', for the two potential functions, within and without the surface.

Again, $-\left(\frac{d\overline{V}}{dw}\right)=$ force with which a particle of positive electricity p, placed within the surface and infinitely near it, is impelled in the direction dw perpendicular to this surface, and directed inwards; and $-\left(\frac{d\overline{V'}}{dw'}\right)$ expresses the force with which a similar particle p' placed without this surface, on the same normal with p, and also infinitely near it, is impelled outwards in the direction of this normal: but the sum of these two forces is equal to double the force that an infinite plane would exert upon p, supposing it uniformly covered with electricity of the same density as at the foot of the normal on which p is; and this last force is easily shown to be expressed by $2\pi\rho$, hence by equating

$$4\pi\rho = -\left\{\frac{\overline{dV}}{dw} + \frac{\overline{dV'}}{dw'}\right\}, \quad . \quad . \quad . \quad . \quad (4)$$

and consequently there is only one value of ρ, which can produce V and V' as corresponding potential functions.

Although in what precedes, we have considered the surface of one body only, the same arguments apply, how great soever may be their number; for the potential functions V and V' would still be given by the formulæ

$$V = \int \frac{\rho d\sigma}{r} \text{ and } V' = \int \frac{\rho d\sigma}{r'};$$

the only difference would be, that the integrations must now extend over the surface of all the bodies, and, that the number of functions represented by V, would be equal to the number of the bodies, one for each. In this case, if there were given a value of V for each body, together with V' belonging to the exterior space; and moreover, if these functions satisfied to the above mentioned conditions, it would always be possible to determine the density on the surface of each body, so as to produce these values as potential functions, and there would be but one density, viz. that given by

$$0 = 4\pi\rho + \frac{\overline{dV}}{dw} + \frac{\overline{dV'}}{dw'} \quad . \quad . \quad . \quad . \quad (4')$$

which could do so: ρ, $\frac{\overline{dV}}{dw}$ and $\frac{\overline{dV'}}{dw'}$ belonging to a point on the surface of any of these bodies.

(5.) From what has been before established (art. 3), it is easy to prove, that when the value of the potential function $\overline{V}$ is given on any closed surface, there is but one function which can satisfy at the same time the equation

$$0 = \delta V,$$

and the condition, that V shall have no singular values within this surface. For the equation (3) art. 3, becomes by supposing $\delta U = 0$,

$$\int d\sigma \overline{U} \frac{\overline{dV}}{dw} = \int d\sigma \overline{V} \frac{\overline{dU}}{dw} - 4\pi V'.$$

In this equation, U is supposed to have only one singular value within the surface, viz. at the point p', and, infinitely near to this point, to be sensibly equal to $\frac{1}{r}$; r being the distance from p'. If now we had a value of U, which, besides satisfying the above written conditions, was equal to *zero* at the surface itself, we should have $\overline{U} = 0$, and this equation would become

$$0 = \int d\sigma \overline{V} \frac{\overline{dU}}{dw} - 4\pi V', \quad . \quad . \quad . \quad . \quad (5)$$

which shows, that V' the value of V at the point p' is given, when $\overline{V}$ its value at the surface is known.

To convince ourselves that there does exist such a function as we have supposed U to be; conceive the surface to be a perfect conductor put in communication with the earth, and a unit of positive electricity to be concentrated in the point p', then the total potential function arising from p', and from the electricity it will induce upon the surface, will be the required value of U. For, in consequence of the communication established between the conducting surface and the earth, the total potential function at this surface must be constant, and equal to that of the earth itself, i. e. to *zero* (seeing that in this state they form but one conducting body). Taking, therefore, this total potential function for U, we have evidently $0=\overline{U}$, $0=\delta U$, and $U=\frac{1}{r}$ for those parts infinitely near to p'. As moreover, this function has no other singular points within the surface, it evidently possesses all the properties assigned to U in the preceding proof.

Again, since we have evidently $U'=0$, for all the space exterior to the surface, the equation (4) art. 4 gives

$$0=4\pi(\rho)+\frac{\overline{dU}}{dw};$$

where (ρ) is the density of the electricity induced on the surface, by the action of a unit of electricity concentrated in the point p'. Thus, the equation (5) of this article becomes

$$V'=-\int d\sigma(\rho)\overline{V}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This equation is remarkable on account of its simplicity and singularity, seeing that it gives the value of the potential for any point p', within the surface, when $\overline{V}$, its value at the surface itself is known, together with (ρ), the density that a unit of electricity concentrated in p' would induce on this surface, if it conducted electricity perfectly, and were put in communication with the earth.

Having thus proved, that V' the value of the potential function V, at any point p' within the surface is given, provided its value $\overline{V}$ is known at this surface, we will now show, that whatever the value of $\overline{V}$ may be, the general value of V deduced from it by the formula just given shall satisfy the equation

$$0=\delta V.$$

For, the value of V at any point p whose co-ordinates are x, y, z, deduced from the assumed value of $\overline{V}$, by the above written formula, is

$$4\pi V=\int d\sigma\overline{V}\left(\frac{\overline{dU}}{dw}\right),$$

U being the total potential function within the surface, arising from a unit of electricity concentrated in the point p, and the electricity induced on the surface itself by its action. Then, since $\overline{V}$ is evidently independent of x, y, z, we immediately deduce

$$4\pi\delta V=\int d\sigma\overline{V}\delta\left(\frac{\overline{dU}}{dw}\right).$$

Now the general value of U will depend upon the position of the point p producing it, and upon that of any other point p' whose co-ordinates are x', y', z', to which it is referred, and will consequently be a function of the six quantities x, y, z, x', y', z'. But we may conceive U to be divided into two parts, one $=\frac{1}{r}$ (r being the distance pp') arising from the electricity in p, the other, due to the electricity induced on the surface by the action of p, and which we shall call $U_{\prime}$. Then since $U_{\prime}$ has no singular values within the surface, we may deduce its general value from that at the surface, by a formula similar to the one just given. Thus

$$4\pi U_{\prime}=\int d\sigma \overline{U}_{\prime}\left(\frac{\overline{dU'}}{dw}\right);$$

where U' is the total potential function, which would be produced by a unit of electricity in p', and therefore, $\left(\frac{\overline{dU'}}{dw}\right)$ is independent of the co-ordinates x, y, z, of p, to which δ refers. Hence

$$4\pi\delta U_{\prime}=\int d\sigma\left(\frac{\overline{dU'}}{dw}\right)\delta\overline{U}_{\prime}.$$

We have before supposed

$$U=\frac{1}{r}+U_{\prime},$$

and as $\delta\frac{1}{r}=0$, we immediately obtain

$$\delta U=\delta U_{\prime}.$$

Again, since we have at the surface itself $0=\overline{U}=\frac{1}{\overline{r}}+\overline{U}_{\prime}$; $\overline{r}$ being the distance between p and the element $d\sigma$, we hence deduce

$$0=\delta\overline{U}_{\prime};$$

this substituted in the general value of $\delta U_{\prime}$ before given, there arises $\delta U_{\prime}=0$, and consequently $0=\delta U$. The result just obtained being general, and applicable to any point p' within the surface, gives immediately

$$0=\delta\left(\frac{\overline{dU}}{dw}\right),$$

and we have by substituting in the equation determining δV

$$0=\delta V.$$

In a preceding part of this article, we have obtained the equation

$$0=4\pi(\rho)+\left(\frac{\overline{dU}}{dw}\right),$$

which combined with $0=\delta\left(\frac{\overline{dU}}{dw}\right)$, gives

$$0=\delta(\rho),$$

and therefore the density (ρ) induced on any element $d\sigma$, which is evidently a function of the co-ordinates x, y, z, of p, is also such a function as will satisfy the equation $0=\delta(\rho)$: it is moreover evident, that (ρ) can never become infinite when p is within the surface.

F

It now remains to prove, that the formula

$$V = \frac{1}{4\pi}\int d\sigma \overline{V}\left(\frac{\overline{dU}}{dw}\right) = -\int d\sigma (\rho) \overline{V}$$

shall always give $V = \overline{V}$, for any point within the surface and infinitely near it, whatever may be the assumed value of $\overline{V}$.

For this, suppose the point p to approach infinitely near the surface, then it is clear that the value of (ρ), the density of the electricity induced by p, will be insensible, except for those parts infinitely near to p, and in these parts it is easy to see, that the value of (ρ) will be independant of the form of the surface, and depend only on the distance $p, d\sigma$. But, we shall afterwards show (art. 10), that when this surface is a sphere of any radius whatever, the value of (ρ) is

$$(\rho) = \frac{-a}{2\pi \cdot f^3};$$

a being the shortest distance between p and the surface, and f representing the distance $p, d\sigma$. This expression will give an idea of the rapidity with which (ρ) decreases, in passing from the infinitely small portion of the surface in the immediate vicinity of p, to any other part situate at a finite distance from it, and when substituted in the above written value of V, gives, by supposing a to vanish,

$$V = \overline{V}.$$

It is also evident, that the function V, determined by the above written formula, will have no singular values within the surface under consideration.

What was before proved, for the space within any closed surface, may likewise be shown to hold good, for that exterior to a number of closed surfaces, of any forms whatever, provided we introduce the condition, that V' shall be equal to *zero* at an infinite distance from these surfaces. For, conceive a surface at an infinite distance from those under consideration; then, what we have before said, may be applied to the whole space within the infinite surface and exterior to the others; consequently

$$4\pi V' = \int d\sigma \overline{V}'\left(\frac{\overline{dU}}{dw}\right); \quad . \quad . \quad . \quad (5')$$

where the sign of integration must extend over all the surfaces, (seeing that the part due to the infinite surface is destroyed by the condition, that V' is there equal to *zero*) and dw must evidently be measured from the surfaces, into the exterior space to which V' now belongs.

The form of the equation (6) remains also unaltered, and

$$V' = -\int (\rho) d\sigma \overline{V}'; \quad . \quad . \quad . \quad . \quad . \quad (6')$$

the sign of integration extending over all the surfaces, and (ρ) being the density of the electricity which would be induced on each of the bodies, in presence of each other, supposing they all communicated with the earth by means of infinitely thin conducting wires.

(6.) Let now A be any closed surface, conducting electricity perfectly, and p a point within it, in which a given quantity of electricity Q is concentrated, and suppose this to induce an electrical state in A; then will V, the value of the potential function arising from the surface only, at any other point p', also within it, be such a function of the co-ordinates p and p', that we may change the co-ordinates of p, into those of p', and reciprocally, without altering its value. Or, in other words, the value of the potential function at p', due to the surface alone, when the inducing electricity Q is concentrated in p, is equal to that which would have place at p, if the same electricity Q were concentrated in p'.

For, in consequence of the equilibrium at the surface, we have evidently, in the first case, when the inducing electricity is concentrated in p,

$$\frac{Q}{\overline{r}}+\overline{V}=\beta;$$

$\overline{r}$ being the distance between p and $d\sigma'$ an element of the surface A, and β a constant quantity dependant upon the quantity of electricity originally placed on A. Now the value of V at p' is

$$V=-\int(\rho')d\sigma'\overline{V},$$

by what has been shown (art. 5); (ρ') being, as in that article, the density of the electricity which would be induced on the element $d\sigma'$ by a unit of electricity in p', if the surface A were put in communication with the earth. This equation gives

$$\delta V=-\int(\rho')\,d\sigma'\delta\overline{V}=0;$$

since $\delta\overline{V}=-\delta\frac{Q}{\overline{r}}=0$: the symbol δ referring to the co-ordinates x, y, z, of p. But we know that $0=\delta' V$; where δ' refers in a similar way to the co-ordinates x', y', z', of p' only. Hence we have simultaneously

$$0=\delta V \quad \text{and} \quad 0=\delta' V;$$

where it must be remarked, that the function V has no singular values, provided the points p and p' are both situate within the surface A. This being the case the first equation evidently gives (art. 5)

$$V=-\int(\rho)d\sigma\overline{V};$$

$\overline{V}$ being what V would become, if the inducing point p were carried to $d\sigma$, p' remaining fixed. Where $\overline{V}$ is a function of x', y', z', and ξ, η, ζ, the co-ordinates of $d\sigma$, whereas (ρ) is a function of $x, y, z, \xi, \eta, \zeta$, independent of x', y', z'; hence by the second equation

$$0=\delta' V=-\int(\rho)d\sigma\delta'\overline{V},$$

which could not hold generally whatever might be the situation of p, unless we had

$$0=\delta'\overline{V};$$

where we must be cautious, not to confound the present value of $\overline{V}$, with that employed at the beginning of this article in proving the equation $0=\delta V$, which last, having performed its office, will be no longer employed.

The equation $0=\delta' \overline{V}'$ gives in the same way

$$\overline{V}=-\int(\rho')d\sigma'\overline{V}';$$

$\overline{\overline{V}}$ being what V becomes by bringing the point p' to any other element $d\sigma'$ of the surface A. This substituted for V, in the expression before given, there arises

$$V = +\iint (\rho)(\rho')\, d\sigma\, d\sigma' \overline{\overline{V}}:$$

in which double integral, the signs of integration, relative to each of the independent elements $d\sigma$ and $d\sigma'$, must extend over the whole surface.

If now, we represent by V', the value of the potential function at p arising from the surface A, when the electricity Q is concentrated in p', we shall evidently have

$$V_{\prime} = +\int (\rho')(\rho)\, d\sigma'\, d\sigma\, \overline{\overset{\prime}{V}}_{\prime}.$$

where the order of integrations alone is changed, the limits remaining unaltered: $\overline{\overset{\prime}{V}}_{\prime}$ being what $V_{\prime}$ would become, by first bringing the electrical point p' to the surface, and afterward the point p to which $V_{\prime}$ belongs. This being done, it is clear that $\overline{\overline{V}}$ and $\overline{\overset{\prime}{V}}_{\prime}$ represent but one and the same quantity, seeing that each of them serves to express the value of the potential function, at any point of the surface A, arising from the surface itself, when the electricity is induced upon it by the action of an electrified point, situate in any other point of the same surface, and hence we have evidently

$$V = V_{\prime},$$

as was asserted at the commencement of this article.

It is evident from art. 5, that our preceding arguments will be equally applicable to the space exterior to the surfaces of any number of conducting bodies, provided we introduce the condition, that the potential function V, belonging to this space, shall be equal to *zero*, when either p or p' shall remove to an infinite distance from these bodies, which condition will evidently be satisfied, provided all the bodies are originally in a natural state. Supposing this therefore to be the case, we see that the potential function belonging to any point p' of the exterior space, arising from the electricity induced on the surfaces of any number of conducting bodies, by an electrified point in p, is equal to that which would have place at p, if the electrified point were removed to p'.

What has been just advanced, being perfectly independant of the number and magnitude of the conducting bodies, may be applied to the case of an infinite number of particles, in each of which the fluid may move freely, but which are so constituted that it cannot pass from one to another. This is what is always supposed to take place in the theory of magnetism, and the present article will be found of great use to us when in the sequel we come to treat of that theory.

(7.) These things being established with respect to electrified surfaces; the general theory of the relations between the density of the electric fluid and the corresponding potential functions, when the electricity is disseminated through the interior of solid bodies as well as over their surfaces, will very readily flow from what has been proved (art. 1).

For this let V' represent the value of the potential function at a point p', within a solid body of any form, arising from the whole of the electric fluid contained in it, and ρ' be the density of the electricity in its interior; ρ' being a function of the three rectangular co-ordinates x, y, z: then if ρ be the density at the surface of the body, we shall have

$$V' = \int \frac{dxdydz\rho'}{r'} + \int \frac{d\sigma\rho}{r};$$

r' being the distance between the point p' whose co-ordinates are x', y', z', and that whose co-ordinates are x, y, z, to which ρ' belongs, also r the distance between p' and $d\sigma$, an element of the surface of the body: V' being evidently a function of x', y', z'. If now V be what V' becomes by changing x', y', z', into x, y, z, it is clear from art. 1, that ρ' will be given by

$$0 = 4\pi\rho' + \delta V.$$

Substituting for ρ', the value which results from this equation, in that immediately preceding we obtain

$$V' = -\int \frac{dxdydz\delta V}{4\pi r'} + \int \frac{\rho d\sigma}{r},$$

which, by means of the equation (3) art. 3, becomes

$$\int \frac{\rho d\sigma}{r} = \frac{1}{4\pi} \left\{ \int d\sigma \overline{V} \left(\frac{d\frac{1}{r}}{dw} \right) - \int \frac{d\sigma}{r} \left(\overline{\frac{dV}{dw}} \right) \right\};$$

the horizontal lines over the quantities, indicating that they belong to the surface itself.

Suppose $V_{\prime}$ to be the value of the potential function in the space exterior to the body, which, by art. 5, will depend on the value of V at the surface only; and the equation (2) art. 3, applied to this exterior space, will give since $\delta V_{\prime} = 0$ and $\delta \frac{1}{r} = 0$,

$$\int d\sigma \overline{V} \left(\frac{d\frac{1}{r}}{dw'} \right) = \int d\sigma \overline{V}_{\prime} \left(\frac{d\frac{1}{r}}{dw'} \right) = \int \frac{d\sigma}{r} \left(\overline{\frac{dV_{\prime}}{dw'}} \right);$$

where dw' is measured from the surface into the exterior space to which $V_{\prime}$ belongs, as dw is, into the interior space. Consequently $dw = -dw'$, and therefore

$$\int d\sigma \overline{V} \left(\frac{d\frac{1}{r}}{dw} \right) = -\int d\sigma \overline{V} \left(\frac{d\frac{1}{r}}{dw'} \right) = -\int \frac{d\sigma}{r} \left(\overline{\frac{dV_{\prime}}{dw'}} \right).$$

Hence the equation determining ρ becomes, by substituting for $\int d\sigma \overline{V} \left(\frac{d\frac{1}{r}}{dw} \right)$ its value just given,

$$\int \frac{\rho d\sigma}{r} = \frac{-1}{4\pi} \int \frac{d\sigma}{r} \left\{ \left(\overline{\frac{dV}{dw}} \right) + \left(\overline{\frac{dV_{\prime}}{dw'}} \right) \right\},$$

an equation which could not subsist generally, unless

$$\rho = \frac{-1}{4\pi} \left\{ \overline{\frac{dV}{dw}} + \overline{\frac{dV_{\prime}}{dw'}} \right\}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (7)$$

Thus the whole difficulty is reduced to finding the value $V_{\prime}$ of the potential function exterior to the body.

G

Although we have considered only one body, it is clear that the same theory is applicable to any number of bodies, and that the values of ρ and ρ' will be given by precisely the same formulæ, however great that number may be: $V_{\prime}$ being the exterior potential function common to all the bodies.

In case the bodies under consideration are all perfect conductors, we have seen (art. 1), that the whole of the electricity will be carried to their surfaces, and therefore there is here no place for the application of the theory contained in this article; but as there are probably no perfectly conducting bodies in nature, this theory becomes indispensably necessary, if we would investigate the electrical phenomena in all their generality.

Having in this, and the preceding articles, laid down the most general principles of the electrical theory, we shall in what follows apply these principles to more special cases; and the necessity of confining this Essay within a moderate extent, will compel us to limit ourselves to a brief examination of the more interesting phenomena.

APPLICATION OF THE PRECEDING RESULTS TO THE THEORY OF ELECTRICITY.

(8.) THE first application we shall make of the foregoing principles, will be to the theory of the Leyden phial. For this, we will call the inner surface of the phial A, and suppose it to be of any form whatever, plane or curved, then, B being its outer surface, and θ the thickness of the glass measured along a normal to A; θ will be a very small quantity, which, for greater generality, we will suppose to vary in any way, in passing from one point of the surface A to another. If now the inner coating of the phial be put in communication with a conductor C, charged with any quantity of electricity, and the outer one be also made to communicate with another conducting body C', containing any other quantity of electricity, it is evident, in consequence of the communications here established, that the total potential function, arising from the whole system, will be constant throughout the interior of the inner metallic coating, and of the body C. We shall here represent this constant quantity by

$$\beta.$$

Moreover, the same potential function within the substance of the outer coating, and in the interior of the conductor C', will be equal to another constant quantity

$$\beta'.$$

Then designating by $V_{\prime}$ the value of this function, for the whole of the space exterior to the conducting bodies of the system, and consequently for that within the substance of the glass itself; we shall have (art. 4)

$$\overline{V}=\beta \quad \text{and} \quad \overline{\overline{V}}=\beta' :$$

One horizontal line over any quantity, indicating that it belongs to the inner surface A; and two showing that it belongs to the outer one B.

At any point of the surface A, suppose a normal to it to be drawn, and let this be the axes of $\overline{\varpi}$: then $\overline{\varpi}'$, $\overline{\varpi}''$, being two other rectangular axes, which are necessarily in the plane tangent to A at this point; V may be considered as a function of $\overline{\varpi}$, $\overline{\varpi}'$ and $\overline{\varpi}''$, and we shall have by TAYLOR's theorem, since $\overline{\varpi}'=0$ and $\overline{\varpi}''=0$ at the axis of $\overline{\varpi}$ along which θ is measured,

$$\overline{\overline{V}}=\overline{V}+\frac{d\overline{V}}{d\overline{\varpi}}\cdot\frac{\theta}{1}+\frac{d^2\overline{V}}{d\overline{\varpi}^2}\cdot\frac{\theta^2}{1\cdot 2}+etc.;$$

where, on account of the smallness of θ, the series converges very rapidly. By writing in the above, for $\overline{V}$ and $\overline{\overline{V}}$ their values just given, we obtain

$$\beta'-\beta=\frac{d\overline{V}}{d\overline{\varpi}}\cdot\frac{\theta}{1}+\frac{d^2\overline{V}}{d\overline{\varpi}^2}\cdot\frac{\theta^2}{1\cdot 2}+etc.$$

In the same way, if $\overline{\overline{\varpi}}$ be a normal to B, directed towards A, and $\theta_{\prime}$ be the thickness of the glass measured along this normal, we shall have

$$\beta-\beta'=\frac{d\overline{\overline{V}}}{d\overline{\overline{\varpi}}}\cdot\frac{\theta_{\prime}}{1}+\frac{d^2\overline{\overline{V}}}{d\overline{\overline{\varpi}}^2}\cdot\frac{\theta_{\prime}^2}{1\cdot 2}+etc.$$

But, if we neglect quantities of the order θ, compared with those retained, the following equation will evidently hold good,

$$\frac{d^n\overline{\overline{V}}}{d\overline{\overline{\varpi}}^n}=(-1)^n\frac{d^n\overline{V}}{d\overline{\varpi}^n};$$

n being any whole positive number, the factor $(-1)^n$ being introduced because $\overline{\varpi}$ and $\overline{\overline{\varpi}}$ are measured in opposite directions. Now by article 4

$$-4\pi\overline{\rho}=\frac{d\overline{V}}{d\overline{\varpi}} \text{ and } -4\pi\overline{\overline{\rho}}=\frac{d\overline{\overline{V}}}{d\overline{\overline{\varpi}}};$$

$\overline{\rho}$ and $\overline{\overline{\rho}}$ being the densities of the electric fluid at the surfaces A and B respectively. Permitting ourselves, in what follows, to neglect quantities of the order θ^2 compared with those retained, it is clear that we may write θ for $\theta_{\prime}$, and hence by substitution

$$\beta'-\beta=-4\pi\overline{\rho}\theta+\left(\frac{d^2\overline{V}}{d\overline{\varpi}^2}\right)\frac{\theta^2}{1\cdot 2}$$

$$\beta-\beta'=-4\pi\overline{\overline{\rho}}\theta+\left(\frac{d^2\overline{V}}{d\overline{\varpi}^2}\right)\frac{\theta^2}{1\cdot 2};$$

where V and ρ are quantities of the order $\frac{1}{\theta}$; β' and β being the order θ^0 or unity. The only thing which now remains to be determined, is the value of $\frac{d^2\overline{V}}{d\overline{w}^2}$ for any point on the surface A.

Throughout the substance of the glass, the potential function V will satisfy the equation $0=\delta V$, and therefore at a point on the surface of A, where of necessity, w, w', and w'', are each equal to *zero*, we have

$$0=\frac{d^2\overline{V}}{dw^2}+\frac{d^2\overline{V}}{dw'^2}+\frac{d^2\overline{V}}{dw''^2}=\delta\overline{V};$$

the horizontal mark over w, w' and w'' being, for simplicity, omitted. Then since $w'=0$,

$$\frac{d^2\overline{V}}{dw'^2}=(V_0-2V_{dw}'+V_{2dw}')\colon dw'^2,$$

and as V is constant and equal to β at the surface A, there hence arises

$$V_0=\beta;\ V_{dw}'=\beta+\frac{d\overline{V}}{dw}\,\frac{dw'^2}{2R},\ V_{2dw}'=\beta+\frac{d\overline{V}}{dw}\,\frac{4dw'^2}{2R};$$

R being the radius of curvature of the surface A, in the plane (w, w'). Substituting these values in the expression immediately preceding, we get

$$\frac{d^2\overline{V}}{dw'^2}=\frac{1}{R}\,\frac{d\overline{V}}{dw}=\frac{-4\pi\bar{\rho}}{R}.$$

In precisely the same way we obtain, by writing R' for the radius of curvature in the plane (w, w''),

$$\frac{d^2\overline{V}}{dw''^2}=\frac{-4\pi\bar{\rho}}{R'}:$$

both rays being accounted positive on the side where w, i. e. $\overline{w}$ is negative. These values substituted in $0=\delta\overline{V}$, there results

$$\frac{d^2\overline{V}}{dw^2}=4\pi\bar{\rho}\left(\frac{1}{R}+\frac{1}{R'}\right)$$

for the required value of $\frac{d^2\overline{V}}{dw^2}$, and thus the sum of the two equations into which it enters, yields

$$\bar{\rho}\left\{1+\left(\frac{1}{R}+\frac{1}{R'}\right)\theta\right\}=-\bar{\bar{\rho}},$$

and the difference of the same equations, gives

$$\beta-\beta'=2\pi(\bar{\rho}-\bar{\bar{\rho}})\theta,$$

therefore the required values of the densities $\bar{\rho}$ and $\bar{\bar{\rho}}$ are

$$\left.\begin{array}{l}\bar{\rho}=\dfrac{\beta-\beta'}{4\pi\theta}\left\{1+\dfrac{\theta}{2}\left(\dfrac{1}{R}+\dfrac{1}{R'}\right)\right\}\\[2ex] \bar{\bar{\rho}}=\dfrac{\beta'-\beta}{4\pi\theta}\left\{1-\dfrac{\theta}{2}\left(\dfrac{1}{R}+\dfrac{1}{R'}\right)\right\}\end{array}\right\}; \quad . \quad . \quad . \quad . \quad (8)$$

which values are correct to quantities of the order $\theta^2\bar{\rho}$, or, which is the same thing, to quantities of the order θ; these having been neglected in the latter part of the preceding analysis, as unworthy of notice.

Suppose $d\sigma$ is an element of the surface A, the corresponding element of B, cut off by normals to A, will be $d\sigma\left\{1+\theta\left(\frac{1}{R}+\frac{1}{R'}\right)\right\}$, and therefore the quantity of fluid on this last element will be $\bar{\bar{\rho}}d\sigma\left\{1+\theta\left(\frac{1}{R}+\frac{1}{R'}\right)\right\}$; substituting for $\bar{\bar{\rho}}$ its value before found, $\bar{\bar{\rho}}=-\bar{\rho}\left\{1-\theta\left(\frac{1}{R}+\frac{1}{R'}\right)\right\}$, and neglecting $\theta^2\bar{\rho}$, we obtain

$$-\bar{\rho}\,d\sigma,$$

the same quantity as on the element $d\sigma$ of the first surface. If therefore, we conceive any portion of the surface A, bounded by a closed curve, and a corresponding portion of the surface B, which would be cut off by a normal to A, passing completely round this curve; the sum of the two quantities of electric fluid, on these corresponding portions, will be equal to *zero;* and consequently, in an electrical jar any how charged, the total quantity of electricity in the jar may be found, by calculating the quantity, on the two exterior surfaces of the metallic coatings farthest from the glass, as the portions of electricity, on the two surfaces adjacent to the glass, exactly neutralise each other. This result will appear singular, when we consider the immense quantity of fluid collected on these last surfaces, and moreover, it would not be difficult to verify it by experiment.

As a particular example of the use of this general theory: suppose a spherical conductor whose radius a, to communicate with the inside of an electrical jar, by means of a long slender wire, the outside being in communication with the common reservoir; and let the whole be charged: then P representing the density of the electricity on the surface of the conductor, which will be very nearly constant, the value of the potential function within the sphere, and, in consequence of the communication established, at the inner coating A also, will be $4\pi aP$ very nearly, since we may, without sensible error, neglect the action of the wire and jar itſelf in calculating it. Hence

$$\beta=4\pi aP \text{ and } \beta'=0,$$

and the equations (8), by neglecting quantities of the order θ, give

$$\bar{\rho}=\frac{\beta}{4\pi\theta}=\frac{a}{\theta}P \text{ and } \bar{\bar{\rho}}=\frac{-\beta}{4\pi\theta}=-\frac{a}{\theta}P.$$

We thus obtain, by the most simple calculation, the values of the densities, at any point on either of the surfaces A and B, next the glass, when that on the spherical conductor is known.

The theory of the condenser, electrophorous, &c. depends upon what has been proved in this article; but these are details into which the limits of this Essay will not permit me to enter; there is, however, one result, relative to charging a number of jars *by cascade*, that appears worthy of notice, and which flows so readily from the equations (8), that I cannot refrain from introducing it here.

Conceive any number of equal and similar insulated Leyden phials, of uniform thickness, so disposed, that the exterior coating of the first, may communicate with the interior one of the second; the exterior one of the second, with the interior one of the third; and so on throughout the whole series, to the exterior surface of the last, which we will suppose in communication with the earth. Then, if the interior of the first phial, be made to communicate with the prime conductor of an electrical machine, in a state of action, all the phials will receive a certain charge, and this mode of operating is called charging *by cascade*. Permitting ourselves to neglect the small quantities of free fluid on the exterior surfaces of the metallic coatings, and other quantities of the same order, we may readily determine the electrical state of each phial in the series: for thus, the equations (8) become

$$\bar{\rho}=\frac{\beta-\beta'}{4\pi\theta},\quad \bar{\bar{\rho}}=\frac{\beta'-\beta}{4\pi\theta}.$$

Designating now, by an index at the foot of any letter, the number of the phial to which it belongs, so that, $\bar{\rho}_1$ may belong to the first, $\bar{\rho}_2$ to the second phial, and so on; we shall have, by supposing their whole number to be n, since θ is the same for every one,

$$\bar{\rho}_1=\frac{\beta_1-\beta'_1}{4\pi\theta} \qquad \bar{\bar{\rho}}_1=\frac{\beta'_1-\beta_1}{4\pi\theta}$$

$$\bar{\rho}_2=\frac{\beta_2-\beta'_2}{4\pi\theta} \qquad \bar{\bar{\rho}}_2=\frac{\beta'_2-\beta_2}{4\pi\theta}$$

etc. *etc.*

$$\bar{\rho}_n=\frac{\beta_n-\beta'_n}{4\pi\theta} \qquad \bar{\bar{\rho}}_n=\frac{\beta'_n-\beta_n}{4\pi\theta}$$

Now β represents the value of the total potential function, within the prime conductor and interior coating of the first phial, and in consequence of the communications established in this system, we have in regular succession, beginning with the prime conductor, and ending with the exterior surface of the last phial, which communicates with the earth,

$$\beta=\beta_1;\ \beta'_1=\beta_2;\ \beta'_2=\beta_3\ etc.\ \ldots\ \beta'_{n-1}=\beta_n;\ \beta'_n=0$$
$$0=\bar{\bar{\rho}}_1+\bar{\rho}_2;\ 0=\bar{\bar{\rho}}_2+\bar{\rho}_3;\ etc.\ \ldots\ 0=\bar{\bar{\rho}}_{n-1}+\bar{\rho}_n.$$

But the first system of equations gives $0=\bar{\rho}_s+\bar{\bar{\rho}}_s$, whatever whole number s may be, and the second line of that just exhibited is expressed by $0=\bar{\rho}_{s-1}+\bar{\bar{\rho}}_s$, hence by comparing these two last equations

$$\bar{\rho}_s=\bar{\rho}_{s-1},$$

which shows that every phial of the system is equally charged. Moreover, if we sum up vertically, each of the columns of the first system, there will arise in virtue of the second

$$\overline{\rho}_1+\overline{\rho}_2+\overline{\rho}_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad +\overline{\rho}_n=\frac{\beta}{4\pi\theta}$$

$$\overline{\overline{\rho}}_1+\overline{\overline{\rho}}_2+\overline{\overline{\rho}}_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad +\overline{\overline{\rho}}_n=\frac{-\beta}{4\pi\theta}.$$

We therefore see, that the total charge of all the phials is precisely the same, as that which one only would receive, if placed in communication with the same conductor, provided its exterior coating were connected with the earth. Hence this mode of charging, although it may save time, will never produce a greater accumulation of fluid, than would take place, if one phial only were employed.

(9.) Conceive now, a hollow shell of perfectly conducting matter, of any form and thickness whatever, to be acted upon by any electrified bodies, situate without it; and suppose them to induce an electrical state in the shell; then will this induced state be such, that the total action on an electrified particle, placed any where within it, will be absolutely null.

For let V represent the value of the total potential function, at any point p within the shell, then we shall have at its inner surface, which is a closed one,

$$\overline{V}=\beta;$$

β being the constant quantity, which expresses the value of the potential function, within the substance of the shell, where the electricity is, by the supposition, in equilibrium, in virtue of the actions of the exterior bodies, combined with that arising from the electricity induced in the shell itself. Moreover, V evidently satisfies the equation $0=\delta V$, and has no singular value within the closed surface to which it belongs: it follows therefore, from art. 5, that its general value is

$$V=\beta,$$

and as the forces acting upon p, are given by the differentials of V, these forces are evidently all equal to *zero*.

If, on the contrary, the electrified bodies are all within the shell, and its exterior surface is put in communication with the earth, it is equally easy to prove, that there will not be the slightest action on any electrified point exterior to it; but, the action of the electricity induced on its inner surface, by the electrified bodies within it, will exactly balance the direct action of the bodies themselves. Or more generally:

Suppose we have a hollow, and perfectly conducting shell, bounded by any two closed surfaces, and a number of electrical bodies are placed, some within and some without it, at will; then, if the inner surface and interior bodies be called the interior system; also, the outer surface and exterior bodies the exterior system; all the electrical phenomena of the interior system, relative to attractions, repulsions, and densities, will be the same as would take place if there were no exterior system, and the inner surface were a perfect conductor, put in communication with the earth; and all those of the exterior system will be the same, as if the interior one did not exist, and the outer surface were a perfect conductor, containing a quantity of electricity, equal to the whole of that originally contained in the shell itself, and in all the interior bodies.

This is so direct a consequence of what has been shown in articles 4 and 5, that a formal demonstration would be quite superfluous, as it is easy to see, the only difference which could exist, relative to the interior system, between the case where there is an exterior system, and where there is not one, would be in the addition of a constant quantity, to the total potential function within the exterior surface, which constant quantity must necessarily disappear in the differentials of this function, and consequently, in the values of the attractions, repulsions, and densities, which all depend on these differentials alone. In the exterior system there is not even this difference, but the total potential function exterior to the inner surface is precisely the same, whether we suppose the interior system to exist or not.

(10.) The consideration of the electrical phenomena, which arise from spheres variously arranged, is rather interesting, on account of the ease with which all the results obtained from theory, may be put to the test of experiment; but, the complete solution of the simple case of two spheres only, previously electrified, and put in presence of each other, requires the aid of a profound analysis, and has been most ably treated by M. Poisson (Mem. de l'Institut. 1811). Our object, in the present article, is merely to give one or two examples of determinations, relative to the distribution of electricity on spheres, which may be expressed by very simple formulæ.

Suppose a spherical surface whose radius is a, to be covered with electric matter, and let its variable density be represented by ρ; then if, as in the Mec. Celeste, we expand the potential function V, belonging to a point p within the sphere, in the form

$$V = U^{(0)} + U^{(1)}\frac{r}{a} + U^{(2)}\frac{r^2}{a^2} + U^{(3)}\frac{r^3}{a^3} + \text{etc.};$$

r being the distance between p and the centre of the sphere, and $U^{(0)}$, $U^{(1)}$, *etc.* functions of the two other polar co-ordinates of p, it is clear, by what has been shown in the admirable work just mentioned, that the potential function V', arising from the same spherical surface, and belonging to a point p', exterior to this surface, at the distance r' from its centre, and on the radius r produced, will be

$$V' = U^{(0)}\frac{a}{r'} + U^{(1)}\frac{a^2}{r'^2} + U^{(2)}\frac{a^3}{r'^3} + \text{etc.}$$

If, therefore, we make $V=\phi(r)$, and $V'=\psi(r')$, the two functions ϕ and ψ will satisfy the equation

$$\psi(r) = \frac{a}{r}\phi\left(\frac{a^2}{r}\right) \quad \text{or} \quad \phi(r) = \frac{a}{r}\psi\left(\frac{a^2}{r}\right).$$

But (art. 4)

$$4\pi\rho = -\frac{d\overline{V}}{dw} - \frac{d\overline{V'}}{dw'} = +\frac{d\overline{V}}{dr} - \frac{d\overline{V'}}{dr'} = \phi'(a) - \psi'(a),$$

and the equation between ϕ and ψ, in its first form, gives, by differentiation,

$$\psi'(r) = -\frac{a}{r^2}\phi\left(\frac{a^2}{r}\right) - \frac{a^3}{r^3}\phi'\left(\frac{a^2}{r}\right).$$

Making now $r=a$ there arises

$$\psi'(a)=-\frac{\phi(a)}{a}-\phi'(a);$$

ϕ' and ψ' being the characteristics of the differential co-efficients of ϕ and ψ, according to LAGRANGE's notation.

In the same way the equation in its second form yields

$$\phi'(a)=-\frac{\psi(a)}{a}-\psi'(a).$$

These substituted successively, in the equation by which ρ is determined, we have the following

$$\left.\begin{aligned} 4\pi\rho&=2\phi'(a)+\frac{\phi(a)}{a}=2\frac{d\overline{V}}{dr}+\frac{\overline{V}}{a} \\ 4\pi\rho&=-2\psi'(a)-\frac{\psi(a)}{a}=-2\frac{d\overline{V'}}{dr'}-\frac{\overline{V'}}{a} \end{aligned}\right\} \quad \ldots\ldots \quad (9)$$

If, therefore, the value of the potential function be known, either for the space within the surface, or, for that without it, the value of the density ρ will be immediately given, by one or other of these equations.

From what has preceded, we may readily determine how the electric fluid will distribute itself, in a conducting sphere whose radius is a, when acted upon by any bodies situate without it; the electrical state of these bodies being given. In this case, we have immediately the value of the potential function arising from them. Let this value, for any point p within the sphere, be represented by A; A being a function of the radius r, and two other polar co-ordinates. Then the whole of the electricity will be carried to the surface (art. 1), and if V be the potential function arising from this electrified surface, for the same point p, we shall have, in virtue of the equilibrium within the sphere,

$$V+A=\beta \quad \text{or} \quad V=\beta-A;$$

β being a constant quantity. This value of V being substituted in the first of the equations (9), there results

$$4\pi\rho=-2\frac{d\overline{A}}{dr}-\frac{\overline{A}}{a}+\frac{\beta}{a}:$$

the horizontal lines indicating, as before, that the quantities under them belong to the surface itself.

In case the sphere communicates with the earth, β is evidently equal to *zero*, and ρ is completely determined by the above: but if the sphere is insulated, and contains any quantity Q of electricity, the value of β may be ascertained as follows: Let V' be the value of the potential function without the surface, corresponding to the value $V=\beta-A$ within it; then, by what precedes

$$V'=\frac{\beta}{r'}-A';$$

A' being determined from A by the following equations:

$$A=\phi_{\prime}(r),\ \psi_{\prime}(r)=\frac{a}{r}\phi_{\prime}\left(\frac{a^2}{r}\right),\ A'=\psi_{\prime}(r'),$$

and r', being the radius corresponding to the point p', exterior to the sphere, to which A' belongs. When r' is infinite, we have evidently $V'=\frac{Q}{r'}$. Therefore by equating

$$\frac{Q}{r'}=\frac{\beta}{r'}-A' \quad \text{or} \quad \beta=Q+r'A';$$

r' being made infinite. Having thus the value of β, the value of ρ becomes known.

To give an example of the application of the second equation in ρ; let us suppose a spherical conducting surface, whose radius is a, in communication with the earth, to be acted upon by any bodies situate within it, and B' to be the value of the potential function arising from them, for a point p' exterior to it. The total potential function, arising from the interior bodies and surface itself, will evidently be equal to *zero* at this surface, and consequently (art. 5), at any point exterior to it. Hence $V'+B'=0$; V' being due to the surface. Thus the second of the equations (9) becomes

$$4\pi\rho=2\frac{d\overline{B'}}{dr'}+\frac{\overline{B'}}{a}.$$

We are therefore able, by means of this very simple equation, to determine the density of the electricity induced on the surface in question.

Suppose now, all the interior bodies to reduce themselves to a single point P, in which a unit of electricity is concentrated, and f to be the distance Pp': the potential function arising from P will be $\frac{1}{f}$, and hence

$$B'=\frac{1}{f};$$

r' being, as before, the distance between p' and the centre O of the shell. Let now b represent the distance OP, and θ the angle POp', then will $f^2=b^2-2br'.\cos\theta+r'^2$. From which equation we deduce successively,

$$\left(\frac{df}{dr'}\right)=\frac{r'-b\cos\theta}{f}, \text{ and } 2\frac{dB'}{dr'}=-\frac{2}{f^2}\left(\frac{df}{dr'}\right)=\frac{-2r'+2b.\cos.\theta}{f^3}.$$

Making $r'=a$ in this, and in the value of B' before given, in order to obtain those which belong to the surface, there results

$$2\frac{d\overline{B'}}{dr'}+\frac{\overline{B'}}{a}=\frac{-2a^2+2ab.\cos.\theta+f^2}{af^3}=\frac{b^2-a^2}{af^3}.$$

This substituted in the general equation written above, there arises

$$\rho=\frac{b^2-a^2}{4\pi af^3}.$$

If P is supposed to approach infinitely near to the surface, so that $b=a-\alpha$; α being an infinitely small quantity, this would become

$$\rho=\frac{-\alpha}{2\pi f^2}.$$

In the same way, by the aid of the equation between A and ρ, the density of the electric fluid, induced on the surface of a sphere whose radius is a, when the electrified point P is exterior to it, is found to be

$$\rho=\frac{a^2-b^2}{4\pi a f^3};$$

supposing the sphere to communicate, by means of an infinitely fine wire, with the earth, at so great a distance, that we might neglect the influence of the electricity induced upon it by the action of P. If the distance of P from the surface, be equal to an infinitely small quantity α, we shall have in this case, as in the foregoing,

$$\rho=\frac{-\alpha}{2\pi f^2}$$

From what has preceded, we may readily deduce the general value of V, belonging to any point P, within the sphere, when $\overline{V}$ its value at the surface is known. For (ρ), the density induced upon an element $d\sigma$ of the surface, by a unit of electricity concentrated in P, has just been shown to be

$$\frac{b^2-a^2}{4\pi a f^3};$$

f being the distance $P, d\sigma$. This substituted in the general equation (6), art. 5, gives

$$V=-\int d\sigma(\rho)\overline{V}=\frac{a^2-b^2}{4\pi a}\int\frac{d\sigma}{f^3}\overline{V}. \quad . \quad . \quad . \quad . \quad (10)$$

In the same way we shall have, when the point P is exterior to the sphere,

$$V=\frac{b^2-a^2}{4\pi a}\int\frac{d\sigma}{f^3}\overline{V}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The use of these two equations will appear almost immediately, when we come to determine the distribution of the electric fluid, on a thin spherical shell, perforated with a small circular orifice.

The results just given, may be readily obtained by means of LAPLACE's much admired analysis (Mec. Cel. Liv. 3, Ch. 2), and indeed, our general equations (9), flow very easily from the equation (2) art. 10 of that chapter. Want of room compels me to omit these confirmations of our analysis, and this I do the more freely, as the manner of deducing them must immediately occur, to any one who has read this part of the Mecanique Celeste.

Conceive now, two spheres S and S', whose radii are a and a', to communicate with each other by means of an infinitely fine wire: it is required to determine the ratio of the quantities of electric fluid on these spheres, when in a state of equilibrium; supposing the distance of their centres to be represented by b.

The value of the potential function, arising from the electricity on the surface of S, at a point p, placed in its centre, is

$$\int \frac{\rho d\sigma}{a} = \frac{1}{a} \int \rho d\sigma = \frac{Q}{a};$$

$d\sigma$ being an element of the surface of the sphere, ρ the density of the fluid on this element, and Q the total quantity on the sphere. If now, we represent by F', the value of the potential function for the same point p, arising from S', we shall have, by adding together both parts,

$$F' + \frac{Q}{a},$$

the value of the total potential function belonging to p, the centre of S. In like manner, the value of this function at p', the centre of S', will be

$$F + \frac{Q'}{a'};$$

F being the part arising from S, and Q' the total quantity of electricity on S'. But in consequence of the equilibrium of the system, the total potential function throughout its whole interior is a constant quantity. Hence

$$F' + \frac{Q}{a} = F + \frac{Q'}{a'}.$$

Although it is difficult to assign the rigorous values of F and F'; yet, when the distance between the surfaces of the two spheres is considerable, compared with the radius of one of them, it is easy to see, that F and F' will be very nearly the same, as if the electricity on each of the spheres producing them, was concentrated in their respective centres, and therefore, we have very nearly

$$F = \frac{Q}{b} \quad \text{and} \quad F' = \frac{Q'}{b}.$$

These substituted in the above, there arises

$$\frac{Q}{b} + \frac{Q'}{a'} = \frac{Q'}{b} + \frac{Q}{a} \quad i.e. \quad Q\left(\frac{1}{a} - \frac{1}{b}\right) = Q'\left(\frac{1}{a'} - \frac{1}{b}\right).$$

Thus the ratio of Q to Q' is given by a very simple equation, whatever may be the form of the connecting wire, provided it be a very fine one.

If we wished to put this result of calculation to the test of experiment, it would be more simple to write P and P' for the mean densities of the fluid on the spheres, or those which would be observed when, after being connected as above, they were separated to such a distance, as not to influence each other sensibly. Then since

$$Q = 4\pi a^2 P \quad \text{and} \quad Q' = 4\pi a'^2 P',$$

we have by substitution, *etc.*

$$\frac{P}{P'} = \frac{a(b-a)}{a'(b-a')}.$$

We therefore see, that when the distance b between the centres of the spheres is very great, the mean densities will be inversely as the radii; and these last remaining unchanged, the density on the smaller sphere will decrease, and that on the larger increase in a very simple way, by making them approach each other.

Lastly, let us endeavour to determine the law of the distribution of the electric fluid, when in equilibrium on a very thin spherical shell, in which there is a small circular orifice. Then, if we neglect quantities of the order of the thickness of the shell, compared with its radius, we may consider it as an infinitely thin spherical surface, of which the greater segment S is a perfect conductor, and the smaller one s constitutes the circular orifice. In virtue of the equilibrium, the value of the potential function, on the conducting segment, will be equal to a constant quantity, as F, and if there were no orifice, the corresponding value of the density would be

$$\frac{F}{4\pi a};$$

a being the radius of the spherical surface. Moreover on this supposition, the value of the potential function for any point P, within the surface, would be

$$F.$$

Let therefore, $\frac{F}{4\pi a}+\rho$ represent the general value of the density, at any point on the surface of either segment of the sphere, and $F+V$, that of the corresponding potential function for the point P. The value of the potential function for any point on the surface of the sphere, will be $F+\overline{V}$, which equated to F, its value on S, gives for the whole of this segment

$$0=\overline{V}.$$

Thus the equation (10) of this article becomes

$$V=\frac{a^2-b^2}{4\pi a}\int\frac{d\sigma}{f^3}\overline{V};$$

the integral extending over the surface of the smaller segment s only, which, without sensible error, may be considered as a plane.

But, since it is evident, that ρ is the density corresponding to the potential function V, we shall have for any point on the segment s, treated as a plane,

$$\rho=\frac{-1}{2\pi}\frac{d\overline{V}}{dw},$$

as it is easy to see, from what has been before shown (art. 4); dw being perpendicular to the surface, and directed towards the centre of the sphere; the horizontal line always serving to indicate quantities belonging to the surface. When the point P is very near the plane s, and z is a perpendicular from P upon s, z will be a very small quantity, of which the square and higher powers may be neglected. Thus $b=a-z$, and by substitution

$$V=\frac{z}{2\pi}\int\frac{d\sigma}{f^3}\overline{V};$$

the integral extending over the surface of the small plane s, and f being, as before, the distance $P, d\sigma$. Now $\frac{d\overline{V}}{dw}=\frac{d\overline{V}}{dz}$ at the surface of s, and $\frac{z}{f^3}=-\frac{d}{dz}\frac{1}{f}$; hence

$$\rho = \frac{-1}{2\pi}\frac{d\overline{V}}{dw} = \frac{-1}{2\pi}\frac{d\overline{V}}{dz} = \frac{-1}{4\pi^2}\frac{d}{dz}\int\frac{z\,d\sigma}{f^3}\overline{V} = \frac{1}{4\pi^2}\frac{d^2}{dz^2}\int\frac{d\sigma}{f}\overline{V};$$

provided we suppose $z=0$ at the end of the calculus. Now the density $\frac{F}{4\pi a}+\rho$, upon the surface of the orifice s, is equal to *zero*, and therefore, we have for the whole of this surface $\rho = -\frac{F}{4\pi a}$. Hence by substitution

$$\frac{-F\pi}{a} = \frac{d^2}{dz^2}\int\frac{d\sigma}{f}\overline{V}; \quad . \quad . \quad . \quad . \quad . \quad (12)$$

the integral extending over the whole of the plane s, of which $d\sigma$ is an element, and z being supposed equal to *zero*, after all the operations have been effected.

It now only remains to determine the value of $\overline{V}$ from this equation. For this, let β now represent the linear radius of s, and y, the distance between its centre C and the foot of the perpendicular z: then, if we conceive an infinitely thin oblate spheroid, of uniform density, of which the circular plane s constitutes the equator, the value of the potential function at the point P, arising from this spheroid, will be

$$\phi = k\int\frac{d\sigma}{f}\sqrt{\beta^2-\eta^2};$$

η being the distance $d\sigma$, C, and k a constant quantity. The attraction exerted by this spheroid, in the direction of the perpendicular z, will be $-\frac{d\phi}{dz}$, and by the known formulæ relative to the attractions of homogeneous spheroids, we have

$$-\frac{d\phi}{dz} = \frac{3Mz}{\beta^3}(\tan\theta - \theta);$$

M representing the mass of the spheroid, and θ being determined by the equations

$$\alpha^2 = \tfrac{1}{2}(z^2+y^2-\beta^2) + \tfrac{1}{2}\sqrt{\{(z^2+y^2-\beta^2)^2+4\beta^2z^2\}}$$

$$\tan\theta = \frac{\beta}{\alpha}.$$

Supposing now z very small, since it is to vanish at the end of the calculus, and $y \angle \beta$, in order that the point P may fall within the limits of s, we shall have by neglecting quantities of the order z^2 compared with those retained

$$\theta = \frac{\pi}{2} - \frac{z}{\sqrt{\beta^2-y^2}},$$

and consequently

$$-\frac{d\phi}{dz} = \frac{-d}{dz}k\int\frac{d\sigma}{f}\sqrt{\beta^2-\eta^2} = \frac{3M\sqrt{\beta^2-y^2}}{\beta^3} - \frac{3M\pi}{2\beta^3}z.$$

This expression, being differentiated again relative to z, gives

$$\frac{d^2}{dz^2}k\int\frac{d\sigma}{f}\sqrt{\beta^2-\eta^2} = \frac{3M\pi}{2\beta^3}.$$

But the mass M is given by

$$M = k\int d\sigma\,\sqrt{\beta^2-\eta^2} = 2\pi k\int \eta d\eta\,\sqrt{\beta^2-\eta^2} = \frac{2\pi k\beta^3}{3}$$

Hence by substitution

$$\frac{d^2}{dz^2}k\int\frac{d\sigma}{f}\sqrt{\beta^2-\eta^2} = \pi^2 k:$$

which expression is rigorously exact when $z=0$. Comparing this result with the equation (12) of the present article, we see that if $\overline{V}=k\sqrt{\beta^2-\eta^2}$, the constant quantity k may be always determined, so as to satisfy (12). In fact, we have only to make

$$\pi^2 k = \frac{-F\pi}{a} \qquad i.\,e. \qquad k=\frac{-F}{a\pi}.$$

Having thus the value of $\overline{V}$, the general value of V is known, since

$$V=\frac{a^2-b^2}{4\pi a}\int\frac{d\sigma}{f^3}\overline{V} = -\frac{a^2-b^2}{4\pi az}\frac{d}{dz}\int\frac{d\sigma}{f}\left(\overline{V}=k\sqrt{\beta^2-\eta^2}\right)=$$

$$=\frac{a^2-b^2}{4\pi az}\times-\frac{d\phi}{dz}=\frac{a^2-b^2}{4\pi az}\times\frac{3Mz}{\beta^3}\,(\tan\theta-\theta)=-\frac{a^2-b^2}{2\pi a^2}F\,(\tan\theta-\theta).$$

The value of the potential function, for any point P within the shell, being $F+V$, and that in the interior of the conducting matter of the shell being constant, in virtue of the equilibrium, the value ρ' of the density, at any point on the inner surface of the shell, will be given immediately by the general formula (4) art. 4. Thus

$$\rho'=\frac{-1}{4\pi}\frac{d\overline{V}}{dw}=\frac{1}{4\pi}\frac{d\overline{V}}{db}=\frac{+F}{4\pi^2 a}\,(\tan\theta-\theta):$$

in which equation, the point P is supposed to be upon the element $d\sigma'$ of the interior surface, to which ρ' belongs. If now, R be the distance between C, the centre of the orifice, and $d\sigma'$, we shall have $R^2=y^2+z^2$, and by neglecting quantities of the order $\frac{\beta^2}{R^2}$ compared with those retained, we have successively

$$\alpha=R,\ \theta=\frac{\beta}{R} \ \text{ and } \ \tan\theta-\theta=\frac{\theta^3}{3}=\frac{\beta^3}{3R^3}.$$

Thus the value of ρ' becomes

$$\rho'=\frac{F}{12\pi^2 a}\,\frac{\beta^3}{R^3}.$$

In the same way, it is easy to show from the equation (11) of this article, that ρ'', the value of the density on an element $d\sigma''$ of the exterior surface of the shell, corresponding to the element $d\sigma'$ of the interior surface, will be

$$\rho''=\frac{F}{4\pi a}+\rho',$$

which, on account of the smallness of ρ' for every part of the surface, except very near the orifice s, is sensibly constant and equal to $\frac{F}{4\pi a}$, therefore

$$\frac{\rho'}{\rho''}=\frac{\beta^3}{3\pi\cdot R^3}:$$

which equation shows, how very small the density within the shell is, even when the orifice is considerable.

(11.) The determination of the electrical phenomena, which result from long metallic wires, insulated and suspended in the atmosphere, depends upon the most simple calculations. As an example, let us conceive two spheres A and B, connected by a long slender conducting wire; then $\rho dxdydz$ representing the quantity of electricity in an element $dxdydz$ of the exterior space, (whether it results from the ground in the vicinity of the wire having become slightly electrical, or from a mist, or even a passing cloud,) and r being the distance of this element from A's centre; also r' its distance from B's, the value of the potential function at A's centre, arising from the whole exterior space, will be

$$\int\frac{\rho dxdydz}{r},$$

and the value of the same function at B's centre, will be

$$\int\frac{\rho dxdydz}{r'},$$

the integrals extending over all the space exterior to the conducting system under consideration.

If now, Q be the total quantity of electricity on A's surface, and Q' that on B's, their radii being a and a'; it is clear, the value of the potential function at A's centre, arising from the system itself, will be

$$\frac{Q}{a};$$

seeing that, we may neglect the part due to the wire, on account of its fineness, and that due to the other sphere, on account of its distance. In a similar way, the value of the same function at B's centre, will be found to be

$$\frac{Q'}{a'}.$$

But (art. 1), the value of the total potential function must be constant throughout the whole interior of the conducting system, and therefore, its value at the two centres must be equal; hence

$$\frac{Q}{a}+\int\frac{\rho dxdydz}{r}=\frac{Q'}{a'}+\int\frac{\rho dxdydz}{r'}.$$

Although ρ, in the present case, is exceedingly small, the integrals contained in this equation, may not only be considerable, but very great, since they are of the second dimension relative to space. The spheres, when at a great distance from each other, may therefore become highly electrical, according to the observations of experimental philosophers, and the charge they will receive in any proposed case may readily be calculated; the value of ρ being supposed given. When one of the spheres, *B* for instance, is connected with the ground, Q' will be equal to *zero*, and consequently Q immediately given. If, on the contrary, the whole system were insulated and retained its natural quantity of electricity, we should have, neglecting that on the wire,

$$0=Q+Q',$$

and hence Q and Q' would be known.

If it were required, to determine the electrical state of the sphere *A*, when in communication with a wire, of which one extremity is elevated into the atmosphere, and terminates in a fine point p, we should only have to make the radius of *B*, and consequently, Q', vanish in the expression before given. Hence in this case

$$\frac{Q}{a}=\int\frac{\rho dxdydz}{r'}-\int\frac{\rho dxdydz}{r},$$

r' being the distance between p and the element $dxdydz$. Since the object of the present article, is merely to indicate the cause of some phenomena of atmospherical electricity, it is useless to extend it to a greater length, more particularly, as the extreme difficulty of determining correctly the electrical state of the atmosphere at any given time, precludes the possibility of putting this part of the theory to the test of accurate experiment.

(12.) Supposing the form of a conducting body to be given, it is in general impossible to assign, rigorously, the law of the density of the electric fluid on it ssurface in a state of equilibrium, when not acted upon by any exterior bodies, and; at present, there has not even been found any convenient mode of approximation applicable to this problem. It is, however, extremely easy to give such forms to conducting bodies, that this law shall be rigorously assignable by the most simple means. The following method, depending upon art. 4 and 5, seems to give to these forms the greatest degree of generality of which they are susceptible, as, by a tentative process, any form whatever might be approximated indefinitely.

Take any continuous function V', of the rectangular co-ordinates x', y', z', of a point p', which satisfies the partial differential equation $0=\delta V'$, and vanishes when p' is removed to an infinite distance from the origin of the co-ordinates.

Choose a constant quantity b, such that $V'=b$ may be the equation of a closed surface *A*, and that V' may have no singular values, so long as p' is exterior to this surface: then if we form a conducting body, whose outer surface is *A*, the density of the electric fluid in equilibrium upon it, will be represented by

$$\rho=\frac{-h}{4\pi}\,\frac{d\overline{V'}}{dw'},$$

and the potential function due to this fluid, for any point p', exterior to the body, will be

$$hV';$$

h being a constant quantity dependant upon the total quantity of electricity Q, communicated to the body. This is evident from what has been proved in the articles cited.

Let R represent the distance between p', and any point within A; then the potential function arising from the electricity upon it, will be expressed by $\frac{Q}{R}$, when R is infinite. Hence the condition

$$\frac{Q}{R} = hV' \quad (R \text{ being infinite}).$$

which will serve to determine h, when Q is given.

In the application of this general method, we may assume for V', either some analytical expression containing the co-ordinates of p', which is known to satisfy the equation $0=\delta V'$, and to vanish when p' is removed to an infinite distance from the origin of the co-ordinates; as, for instance, some of those given by LAPLACE (Mec. Celeste, Liv. 3, Ch. 2), or, the value a potential function, which would arise from a quantity of electricity any how distributed within a finite space, at a point p' without that space; since this last will always satisfy the conditions to which V' is subject.

It may be proper to give an example of each of these cases. In the first place, let us take the general expression given by LAPLACE,

$$V = \frac{U^{(0)}}{r} + \frac{U^{(1)}}{r^2} + \frac{U^{(2)}}{r^3} + etc.,$$

then, by confining ourselves to the two first terms, the assumed value of V' will be

$$V' = \frac{U^{(0)}}{r} + \frac{U^{(1)}}{r^2};$$

r being the distance of p' from the origin of the co-ordinates, and $U^{(0)}$, $U^{(1)}$, *etc.* functions of the two other polar co-ordinates θ and ϖ. This expression by changing the direction of the axes, may always be reduced to the form

$$V' = \frac{2a}{r} + \frac{k^2 \cos\theta}{r^2};$$

a and k being two constant quantities, which we will suppose positive. Then if b be a very small positive quantity, the form of the surface given by the equation $V'=b$, will differ but little from a sphere, whose radius is $\frac{2a}{b}$: by gradually increasing b, the difference becomes greater, until $b=\frac{a^2}{k^2}$, and afterwards, the form assigned by $V'=b$, becomes improper for our purpose. Making therefore $b=\frac{a^2}{k^2}$, in order to have a surface differing as much from a sphere, as the assumed value of V' admits, the equation of the surface A becomes

$$V' = \frac{2a}{r} + \frac{k^2 \cos\theta}{r^2} = \frac{a^2}{k^2}.$$

From which we obtain

$$r = \frac{k^2}{a}\left(1 + \sqrt{2}\cos\frac{\theta}{2}\right).$$

If now ϕ represents the angle formed by dr and dw', we have

$$\frac{-dr}{r\,d\theta} = \frac{\sqrt{2}\sin\frac{\theta}{2}}{2+2\sqrt{2}\cos\frac{\theta}{2}} = \tan\phi,$$

and as the electricity is in equilibrium upon A, the force with which a particle p, infinitely near to it, would be repelled, must be directed along dw': but the value of this force is $-\frac{d\overline{V'}}{dw'}$, and consequently its effect in the direction of the radius r, and tending to increase it, will be $-\frac{d\overline{V'}}{dw'}\cos\phi$. This last quantity is equally represented by $-\frac{d\overline{V'}}{dr}$, and therefore

$$-\frac{d\overline{V'}}{dr} = -\frac{d\overline{V'}}{dw'}\cos\phi;$$

the horizontal lines over quantities, indicating, as before, that they belong to the surface itself. The value of $-\left(\frac{d\overline{V'}}{dw'}\right)$, deduced from this equation, is

$$-\frac{d\overline{V'}}{dw'} = \frac{-1}{\cos\phi}\frac{d\overline{V'}}{dr} = \frac{1}{\cos\phi}\left\{\frac{2a}{r^2}+\frac{2k^2\cos\theta}{r^3}\right\} = \frac{2a\sqrt{2}\cos\frac{\theta}{2}}{r^2\cos\phi},$$

this substituted in the general value of ρ, before given, there arises

$$\rho = \frac{-h}{4\pi}\frac{d\overline{V'}}{dw'} = \frac{ha\sqrt{2}\cos\frac{\theta}{2}}{2\pi r^2\cos\phi}.$$

Supposing Q is the quantity of electricity communicated to the surface, the condition

$$\frac{Q}{R} = hV \quad \textit{(where R is infinite)}$$

before given, becomes, since r may here be subst tuted for R, seeing that it is measured from a point within the surface,

$$\frac{Q}{r} = \frac{2ah}{r} \qquad i.\,e. \qquad h = \frac{Q}{2a}.$$

We have thus the rigorous value of ρ for the surface A whose equation is $r=\frac{k^2}{a}\left(1+\sqrt{2}\cos\frac{\theta}{2}\right)$ when the quantity Q of electricity upon it is known, and by substituting for r and h their values just given, there results

$$\rho = \frac{Qa^2\sqrt{2}\cos\frac{\theta}{2}}{4\pi k^4\cos\phi\left(1+\sqrt{2}\cos\frac{\theta}{2}\right)^2}.$$

Moreover the value of the potential function for the point p' whose polar co-ordinates are r, θ, and ϖ, is

$$hV' = \frac{Q}{r} + \frac{Qk^2 \cos\theta}{2ar^2}.$$

From which we may immediately deduce the forces acting on any point p' exterior to A.

In tracing the surface A, θ is supposed to extend from $\theta=0$ to $\theta=\pi$, and ϖ, from $\varpi=0$ to $\varpi=2\pi$: it is therefore evident, by constructing the curve whose equation is

$$r = \frac{k^2}{a}\left(1+\sqrt{2}\cos.\frac{\theta}{2}\right),$$

that the parts about P, where $\theta=\pi$, approximate continually in form towards a cone whose apex is P, and as the density of the electricity at P is null, in the example before us, we may make this general inference: when any body whatever, has a part of its surface in the form of a cone, directed inwards; the density of the electricity in equilibrium upon it, will be null at its apex, precisely the reverse of what would take place, if it were directed outwards, for then, the density at the apex would become infinite.*

As a second example, we will assume for V', the value of the potential function arising from the action of a line uniformly covered with electricity. Let $2a$ be the length of the line, y the perpendicular falling from any point p' upon it, x the distance of the foot of this perpendicular form the middle of the line, and x' that of the element dx' from the same point: then taking the element dx', as the measure of the quantity of electricity it contains, the assumed value of V' will be

$$V' = \int \frac{dx'}{\sqrt{y^2+(x-x')^2}} = \log \frac{a-x+\sqrt{y^2+(a-x)^2}}{-a-x+\sqrt{y^2+(a+x)^2}};$$

the integral being taken from $x'=-a$ to $x'=+a$. Making this equal to a constant quantity $\log b$, we shall have, for the equation of the surface A,

$$\frac{a-x+\sqrt{y^2+(a-x)^2}}{-a-x+\sqrt{y^2+(a+x)^2}} = b,$$

which by reduction becomes

$$0 = y^2(1-b^2)^2 + x^2 \cdot 4b(1-b)^2 - a^2 . 4b(1+b)^2.$$

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* Since this was written, I have obtained formulæ serving to express, generally, the law of the distribution of the electric fluid near the apex O of a cone, which forms part of a conducting surface of revolution having the same axis. From these formulæ it results that, when the apex of the cone is directed inwards, the density of the electric fluid at any point p, near to it, is proportional to r^{n-1}; r being the distance Op, and the exponent n very nearly such as would satisfy the simple equation $(4n+2)\,\beta = 3\pi$: where 2β is the angle at the summit of the cone. If 2β exceeds π, this summit is directed outwards, and when the excess is not very considerable, n will be given as above: but 2β still increasing, until it becomes $2\pi-2\gamma$; the angle 2γ at the summit of the cone, which is now directed outwards, being very small, n will be given by $2n \log \frac{2}{\gamma} = 1$, and in case the conducting body is a sphere whose radius is b, on which P represents the mean density of the electric fluid, ρ, the value of the density near the apex O, will be determined by the formula

$$\rho = \frac{2Pbn}{(a+b)\gamma}\left(\frac{r}{a}\right)^{n-1};$$

a being the length of the cone.

We thus see that this surface is a spheroid produced by the revolution of an ellipsis about its greatest diameter; the semi-transverse axis being $a\frac{1+b}{1-b}=\varsigma$, and semi-conjugate $a\frac{2\sqrt{b}}{1-b}=\gamma$.

By differentiating the general value of V', just given, and substituting for y its value at the surface A, we obtain

$$\frac{d\overline{V}'}{dx}=\frac{-2x\frac{1-b}{1+b}}{\left(\frac{1+b}{1-b}\right)^2a^2-\left(\frac{1-b}{1+b}\right)^2x^2}=\frac{-2a\varsigma x}{\varsigma^4-a^2x^2}.$$

Now writing ϕ for the angle formed by dx and dw', we have

$$\frac{1}{\cos\phi}=\frac{ds}{-dy}=\frac{1-b}{2x\sqrt{b}}\sqrt{\left(\frac{1+b}{1-b}\right)^4a^2-x^2}=\frac{\sqrt{\varsigma^4-a^2x^2}}{\gamma x};$$

ds being an element of the generating ellipsis. Hence, as in the preceding example, we shall have,

$$\frac{d\overline{V}'}{dw'}=\frac{1}{\cos\phi}\cdot\frac{d\overline{V}'}{dx}=\frac{-2a\varsigma}{\gamma\sqrt{\varsigma^4-a^2x^2}}.$$

On the surface A therefore, in this example, the general value of ρ is

$$\rho=\frac{-h}{4\pi}\frac{d\overline{V}'}{dw'}=\frac{ah\varsigma}{2\pi\gamma\sqrt{\varsigma^4-a^2x^2}},$$

and the potential function for any point p', exterior to A, is

$$hV'=h\log\frac{a-x+\sqrt{y^2+(a-x)^2}}{-a-x+\sqrt{y^2+(a+x)^2}}.$$

Making now x and y both infinite, in order that p' may be at an infinite distance, there results

$$hV'=\frac{2ah}{\sqrt{x^2+y^2}},$$

and thus the condition determining h, in Q, the quantity of electricity upon the surface, is, since R may be supposed equal to $\sqrt{x^2+y^2}$,

$$\frac{Q}{R}=hV'=\frac{2ah}{\sqrt{x^2+y^2}} \quad i.\ e. \quad h=\frac{Q}{2a}.$$

These results of our analysis, agree with what has been long known concerning the law of the distribution of electric fluid on the surface of a spheroid, when in a state of equilibrium.

(13.) In what has preceded, we have confined ourselves to the consideration of perfect conductors. We will now give an example of the application of our general method, to a body that is supposed to conduct electricity imperfectly, and which will, moreover, be interesting, as it serves to illustrate the magnetic phenomena, produced by the rotation of bodies under the influence of the earth's magnetism.

If any solid body whatever of revolution, turn about its axis, it is required to determine what will take place, when the matter of this solid is not perfectly conducting, supposing it under the influence of a constant electrical force, acting parallel to any given right line fixed in space, the body being originally in a natural state.

Let β designate the coercive force of the body, which we will suppose analogous to friction in its operation, so that as long as the total force acting upon any particle within the body is less than β, its electrical state shall remain unchanged, but when it begins to exceed β, a change shall ensue.

In the first place, suppose the constant electrical force, which we will designate by b, to act in a direction parallel to a line passing through the centre of the body, and perpendicular to its axis of revolution; and let us consider this line as the axis of x, that of revolution being the axis of z, and y the other rectangular co-ordinate of a point p, within the body and fixed in space. Thus, if V be the value of the total potential function for the same point p, at any instant of time, arising from the electricity of the body and the exterior force,

$$bx + V$$

will be the part due to the body itself at the same instant; since $-bx$ is that due to the constant force b, acting in the direction of x, and tending to increase it. If now we make

$$z = r\cos\theta, \quad x = r\sin\theta\cos\varpi, \quad y = r\sin\theta\sin\varpi;$$

the angle ϖ being supposed to increase in the direction of the body's revolution, the part due to the body itself becomes

$$br\sin\theta\cos\varpi + V.$$

Were we to suppose the value of the potential function V given at any instant, we might find its value at the next instant, by conceiving, that whilst the body moves forward through the infinitely small angle $d\varpi$, the electricity within it shall remain fixed, and then be permitted to move, until it is in equilibrium with the coercive force.

Now the value of the potential function at p, arising from the body itself, after having moved through the angle $d\varpi$ (the electricity being fixed), will evidently be obtained by changing ϖ into $\varpi - d\varpi$ in the expression just given, and is therefore

$$br\sin\theta\cos\varpi + V + br\sin\theta\sin\varpi\, d\varpi - \frac{dV}{d\varpi}\, d\varpi,$$

adding now the part $-bx = -br\sin\theta\cos\varpi$, due to the exterior bodies, and restoring x, y, *etc.* we have since $\frac{dV}{d\varpi} = -y\frac{dV}{dx} + x\frac{dV}{dy}$,

$$V + d\varpi\left\{by + y\frac{dV}{dx} - x\frac{dV}{dy}\right\}$$

for the value of the total potential function at the end of the next instant, the electricity being still supposed fixed. We have now only to determine what this will become, by allowing the electricity to move forward until the total forces acting on points within the body, which may now exceed the coercive force by an infinitely small quantity, are again reduced to an equilibrium with it. If this were done, we should, when the initial state of the body was given, be able to determine, successively, its state for every one of the following instants. But since it is evident from the nature of the problem, that the body, by revolving, will quickly arrive at a permanent state, in which the value of V will afterwards remain unchanged and be independent of its initial value, we will here confine ourselves to the determination of this permanent state. It is easy to see, by considering the forces arising from the new total potential function, whose value has just been given, that in this case the electricity will be in motion over the whole interior of the body, and consequently

$$\beta^2 = \left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2,$$

which equation expresses that the total force to move any particle p, within the body, is just equal to β, the coercive force. Now if we can assume any value for V, satisfying the above, and such, that it shall reproduce itself after the electricity belonging to the new total potential function (art. 7), is allowed to find its equilibrium with the coercive force, it is evident this will be the required value, since the rest of the electricity is exactly in equilibrium with the exterior force b, and may therefore be here neglected. To be able to do this the more easily, conceive two new axes X', Y', in advance of the old ones X, Y, and making the angle γ with them; then the value of the new potential function, before given, becomes

$$V + d\omega . \left\{ by' \cos\gamma + bx' \sin\gamma + y' \frac{dV}{dx'} - x' \frac{dV}{dy'} \right\},$$

which, by assuming $V = \beta y'$, and determining γ by the equation

$$0 = b \sin\gamma - \beta,$$

reduces itself to

$$y' (\beta + b \cos\gamma\, d\omega),$$

Considering now the symmetrical distribution of the electricity belonging to this potential function, with regard to the plane whose equation is $0 = y'$, it will be evident that, after the electricity has found its equilibrium, the value of V at this plane must be equal to *zero:* a condition which, combined with the partial differential equation before given, will serve to determine, completely, the value of V at the next instant, and this value of V will be

$$V = \beta y'.$$

We thus see that the assumed value of V reproduces itself at the end of the following instant, and is therefore the one required belonging to the permanent state.

If the body had been a perfect conductor, the value of V would evidently have been equal to *zero*, seeing that it was supposed originally in a natural state: that just found is therefore due to the rotation combined with the coercive force, and we thus see that their effect is to polarise the body in the direction of y' positive, making the angle $\frac{\pi}{2} + \gamma$ with the direction of the constant force b; and the degree of polarity will be the same as would be produced by a force equal to β, acting in this direction on a perfectly conducting body of the same dimensions.

We have hitherto supposed the constant force to act in a direction parallel to the equatorial plane of the body, but whatever may be its direction, we may conceive it decomposed into two: one equal to b as before, and parallel to this plane, the other perpendicular to it, which last will evidently produce no effect on the value of V, as this is due to the coercive force, and would still be equal to *zero* under the influence of the new force, if the body conducted electricity perfectly.

Knowing the value of the potential function at the surface of the body, due to the rotation, its value for all the exterior space may be considered as determined (art. 5), and if the body be a solid sphere, may easily be expressed analytically; for it is evident (art. 7), from the value of V just given, that even in the present case all the electricity will be confined to the surface of the solid; and it has been shown (art. 10), that when the value of the potential function for the point p within a spherical surface, whose radius is a, is represented by

$$\phi(r),$$

the value of the same function for a point p', situate without this sphere, on the prolongation of r, and at the distance r' from its centre, will be

$$\frac{a}{r'}\phi\left(\frac{a^2}{r'}\right).$$

But we have seen that the value of V due to the rotation, for the point p, is

$$V = \beta y' = \beta r \cos\theta';$$

θ' being the angle formed by the ray r and the axis of y'; the corresponding value for the point p' will therefore be

$$V' = \frac{\beta a^3 \cos\theta'}{r'^2}.$$

And hence, by differentiation, we immediately obtain the value of the forces acting on any particle situate without the sphere, which arise from its rotation; but, if we would determine the total forces arising from the sphere, we must, to the value of the potential function just found, add that part which would be produced by the action of the constant force upon this sphere, when it is supposed to conduct electricity perfectly, which will be given in precisely the same way as the former. In fact, f designating the constant force, and θ'' the angle formed by r and a line parallel to the direction of f, the potential function arising from it, for the point p, will be

$$-fr\cos\theta'',$$

and consequently the part arising from the electricity, induced by its action, must be

$$+fr\cos\theta'',$$

seeing that their sum ought to be equal to *zero*. The corresponding value for the point p', exterior to the sphere, is therefore

$$\frac{fa^3\cos\theta''}{r'^2},$$

this added to the value of V', before found, will give the value of the total potential function for the point p', arising from the sphere itself.

It will be seen when we come to treat of the theory of magnetism, that the results of this theory, in general, agree very nearly with those which would arise from supposing the magnetic fluid at liberty to move from one part of a magnetized body to another; at least, for bodies whose magnetic powers admit of considerable developement, as iron and nickel for example; the errors of the latter supposition being of the order $1-g$ only; g being a constant quantity dependant on the nature of the body, which in those just mentioned, differs very little from unity. It is therefore evident that when a solid of revolution, formed of iron, is caused to revolve slowly round its axis, and placed under the influence of the earth's magnetic force f, the act of revolving, combined with the coercive force β of the body, will produce a new polarity, whose direction and quantity will be very nearly the same as those before determined. Now f having been supposed resolved into two forces, one equal to b in the plane of the body's equator, and another perpendicular to this plane; if β be very small compared with b, the angle γ will be very small, and the direction of the new polarity will be very nearly at right angles to the direction of b, a result which has been confirmed by many experiments: but by our analysis we moreover see that when b is sufficiently reduced, the angle γ may be rendered sensible, and the direction of the new polarity will then form with that of b the angle $\frac{\pi}{2}+\gamma$; γ being determined by the equation

$$\sin\gamma = \frac{\beta}{b}.$$

This would be very easily put to the test of experiment by employing a solid sphere of iron.

The values of the forces induced by the rotation of the body, which would be observed in the space exterior to it, may be obtained by differentiating that of V' before given, and will be found to agree with the observations of Mr. BARLOW (Phil. Tran. 1825), on the supposition of β being very small.

As the experimental investigation of the magnetic phenomena developed by the rotation of bodies, has lately engaged the attention of several distinguished philosophers, it may not be amiss to consider the subject in a more general way, as we shall thus not only confirm the preceding analysis, but be able to show with what rapidity the body approaches that permanent state, which it has been the object of the preceding part of this article to determine.

Let us now, therefore, consider a body A fixed in space, under the influence of electric forces which vary according to any given law; then we might propose to determine the electrical state of the body, after a certain interval of time, from the knowledge of its initial state; supposing a constant coercive force to exist within it. To resolve this in its most general form, it would be necessary to distinguish between those parts of the body where the fluid was at rest, from the forces acting there being less than the coercive force, and those where it would be in motion; moreover these parts would vary at every instant, and the problem therefore become very intricate: were we however to suppose the initial state so chosen, that the total force to move any particle p within A, arising from its electric state and exterior actions, was then just equal to the coercive force β; also, that the alteration in the exterior forces should always be such, that if the electric fluid remained at rest during the next instant, this total force should no where be less than β; the problem would become more easy, and still possess a great degree of generality. For in this case, when the fluid is moveable, the whole force tending to move any particle p within A, will, at every instant, be exactly equal to the coercive force. If therefore x, y, z, represent the co-ordinates of p, and V the value of the total potential function at any instant of time t, arising from the electric state of the body and exterior forces, we shall have the equation

$$\beta^2 = \left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2, \quad . \quad . \quad . \quad . \quad (a)$$

whose general integral may be thus constructed:

Take the value of V arbitrarily over any surface whatever S, plane or curved, and suppose three rectangular co-ordinates w, w', w'', whose origin is at a point P on S: the axis of w being a normal to S, and those of w' w'', in its plane tangent. Then the values of $\frac{dV}{dw'}$ and $\frac{dV}{dw''}$ are known at the point P, and the value of $\frac{dV}{dw}$ will be determined by the equation

$$\left(\frac{dV}{dw}\right)^2 + \left(\frac{dV}{dw'}\right) + \left(\frac{dV}{dw''}\right)^2 = \beta^2,$$

which is merely a transformation of the above.

Take now another point $P_{\prime}$, whose co-ordinates referred to these axes are $\frac{dV}{dw}$, $\frac{dV}{dw'}$ and $\frac{dV}{dw''}$, and draw a right line L through the points $P, P_{\prime}$, then will the value of V at any point p, on L, be expressed by

$$V_0 + \beta\lambda;$$

λ being the distance Pp, measured along the line L, considered as increasing in the direction $PP_{\prime}$, and V_0, the given value of V at P. For it is very easy to see that the value of V furnished by this construction, satisfies the partial differential equation *(a)*, and is its general integral, moreover the system of lines L, L, L'', *etc.* belonging to the points P, P', P'', *etc.* on S, are evidently those along which the electric fluid tends to move, and will move during the following instant.

Let now $V + DV$ represent what V becomes at the end of the time $t + dt$; substituting this for V in *(a)* we obtain

$$0 = \frac{dV}{dx} \cdot \frac{dDV}{dx} + \frac{dV}{dy} \cdot \frac{dDV}{dy} + \frac{dV}{dz} \cdot \frac{dDV}{dz}. \quad . \quad . \quad . \quad . \quad (b)$$

Then, if we designate by $D'V$, the augmentation of the potential function, arising from the change which takes place in the exterior forces during the element of time dt,

$$DV - D'V$$

will be the increment of the potential function, due to the corresponding alterations $D\rho$ and $D\rho'$ in the densities of the electric fluid at the surface of A and within it, which may be determined from $DV - D'V$ by art. 7. But, by the known theory of partial differential equations, the most general value of DV satisfying *(b)*, will be constant along every one of the lines L, L', L'', *etc.*, and may vary arbitrarily in passing from one of them to another:

as it is also along these lines the electric fluid moves during the instant dt, it is clear the total quantity of fluid in any infinitely thin needle, formed by them, and terminating in the opposite surfaces of A, will undergo no alteration during this instant. Hence therefore

$$0=\int D\rho' dv + D\rho d\sigma + D\rho_{,} d\sigma_{,}; \quad . \quad . \quad . \quad . \quad . \quad (c)$$

dv being an element of the volume of the needle, and $d\sigma$, $d\sigma_{,}$, the two elements of A's surface by which it is terminated. This condition, combined with the equation (b), will completely determine the value of DV, and we shall thus have the value of the potential function $V+DV$, at the instant of time $t+dt$, when its value V, at the time t, is known.

As an application of this general solution; suppose the body A is a solid of revolution, whose axis is that of the co-ordinate z, and let the two other axes X, Y, situate in its equator, be fixed in space. If now the exterior electric forces are such that they may be reduced to two, one equal to c, acting parallel to z, the other equal to b, directed parallel to a line in the plane (xy), making the variable angle ϕ with X; the value of the potential function arising from the exterior forces, will be

$$-zc - xb\cos\phi - yb\sin\phi;$$

where b and c are constant quantities, and ϕ varies with the time so as to be constantly increasing. When the time is equal to t, suppose the value of V to be

$$V=\beta(x\cos\varpi + y\sin\varpi).$$

then the system of lines L, L', L'' will make the angle ϖ with the plane (xz), and be perpendicular to another plane whose equation is

$$0 = x\cos\varpi + y\sin\varpi.$$

If during the instant of time dt, ϕ becomes $\phi + D\phi$, the augmentation of the potential function due to the elementary change in the exterior forces, will be

$$D'V = (x\sin\phi - y\cos\phi)\, bD\phi;$$

moreover the equation (b) becomes

$$0 = \cos\varpi \cdot \frac{dDV}{dx} + \sin\varpi \cdot \frac{dDV}{dy}, \quad . \quad . \quad . \quad . \quad (b')$$

and therefore the general value of DV is

$$DV = DF\{y\cos\varpi - x\sin\varpi;\; z\};$$

DF being the characteristic of an infinitely small arbitrary function. But, it has been before remarked that the value of DV will be completely determined, by satisfying the equation (b) and the condition (c). Let us then assume

$$DF\{y\cos\varpi - x\sin\varpi;\; z\} = hD\phi(y\cos\varpi - x\sin\varpi);$$

h being a quantity independent of x, y, z, and see if it be possible to determine h so as to satisfy the condition (c). Now on this supposition

$$DV - D'V = hD\phi(y\cos\varpi - x\sin\varpi) - (x\sin\phi - y\cos\phi)\, bd\phi = D\phi\left[y(h\cos\varpi + b\cos\phi) - x(h\sin\varpi + b\cos\phi)\right]$$

The value of $D\rho'$ corresponding to this potential function is (art. 7)

$$D\rho' = 0,$$

and on account of the parallelism of the lines L, L', *etc.* to each other, and to A's equator $d\sigma = d\sigma_{,}$. The condition (c) thus becomes

$$0 = D\rho + D\rho_{,} : \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c')$$

$D\rho$ and $D\rho_{,}$ being the elementary densities on A's surface at opposite ends of any of the lines L, L', *etc.* corresponding to the potential function $DV - D'V$. But it is easy to see from the form of this function, that these elementary densities at opposite ends of any line perpendicular to a plane whose equation is

$$0 = y\,(h\cos\varpi + b\cos\phi) - x\,(h\sin\varpi + b\sin\phi),$$

are equal and of contrary signs, and therefore the condition (c) will be satisfied by making this plane coincide with that perpendicular to L, L', *etc.*, whose equation, as before remarked, is

$$0 = x\cos\varpi + y\sin\varpi;$$

that is the condition (c) will be satisfied, if h be determined by the equation

$$\frac{h\cos\varpi + b\cos\phi}{\sin\varpi} = -\frac{h\sin\varpi + b\sin\phi}{\cos\varpi},$$

which by reduction becomes

$$0 = h + b\cos(\phi - \varpi),$$

and consequently

$$V + DV = \beta\,(x\cos\varpi + y\sin\varpi) + hD\phi\,(y\cos\varpi - x\sin\varpi)$$

$$= \beta x\left(\cos\varpi + \frac{b}{\beta}\sin\varpi\cos(\phi - \varpi)\,D\phi\right) + \beta y\left(\sin\varpi - \frac{b}{\beta}\cos\varpi\cos(\phi - \varpi)D\phi\right)$$

$$= \beta x\cos\left(\varpi - \frac{b}{\beta}\cos(\phi - \varpi)\,D\phi\right) + \beta y\sin\left(\varpi - \frac{b}{\beta}\cos(\phi - \varpi)\,D\phi\right).$$

When therefore ϕ is augmented by the infinitely small angle $D\phi$, ϖ receives the corresponding increment $-\frac{b}{\beta}\cos(\phi - \varpi)\,D\phi$, and the form of V remains unaltered; the preceding reasoning is consequently applicable to every instant, and the general relation between ϕ and ϖ expressed by

$$0 = D\varpi + \frac{b}{\beta}\cos(\phi - \varpi)\,D\phi :$$

a common differential equation, which by integration gives

$$H . e^{\phi\,\mathrm{cotan.}\gamma} = \frac{\sin\left(\frac{3\pi}{4} - \frac{\gamma}{2} + \frac{\varpi}{2} - \frac{\phi}{2}\right)}{\sin\left(\frac{\pi}{4} + \frac{\gamma}{2} + \frac{\varpi}{2} - \frac{\phi}{2}\right)};$$

H being an arbitrary constant, and γ, as in the former part of this article, the smallest root of

$$0 = b\sin\gamma - \beta.$$

Let ϖ_0 and φ_0 be the initial values of ϖ and φ; then the total potential function at the next instant, if the electric fluid remained fixed, would be

$$V_{,}=\beta\,(x\cos\varpi_0+y\sin\varpi_0)+(x\sin\varphi_0-y\cos\varphi_0)\,bd\varphi,$$

and the whole force to move a particle p, whose co-ordinates are x, y, z,

$$\sqrt{\left(\frac{dV_{,}}{dx}\right)^2+\left(\frac{dV_{,}}{dy}\right)^2+\left(\frac{dV_{,}}{dz}\right)^2}=\beta+d\phi.b\sin(\phi_0-\varpi_0),$$

which, in order that our solution may be applicable, must not be less than β, and consequently the angle $\varphi_0-\varpi_0$ must be between 0 and π: when this is the case, ϖ is immediately determined from φ by what has preceded. In fact, by finding the value of H from the initial values ϖ_0 and φ_0, and making $\zeta=\frac{\pi}{4}+\frac{\gamma}{2}+\frac{\varpi}{2}-\frac{\varphi}{2}$, we obtain

$$\tan\zeta=\frac{\tan\zeta_0}{e^{(\varphi-\varphi_0)\cot\gamma}+\tan\gamma\tan\zeta_0\,(e^{(\varphi-\varphi_0)\cot\gamma}-1)};$$

ζ_0 being the initial value of ζ.

We have, in the latter part of this article, considered the body A at rest, and the line X', parallel to the direction of b, as revolving round it: but if, as in the former, we now suppose this line immovable and the body to turn the contrary way, so that the relative motion of X' to X may remain unaltered, the electric state of the body referred to the axes X,Y,Z, evidently depending on this relative motion only, will consequently remain the same as before. In order to determine it on the supposition just made, let X' be the axis of x', one of the co-ordinates of p, referred to the rectangular axes X',Y',Z, also y',z, the other two; the direction $X'Y'$, being that in which A revolves. Then, if ϖ' be the angle the system of lines L,L', *etc.* forms with the plane (x',z), we shall have

$$\varpi+\varpi'=\varphi;$$

φ, as before stated, being the angle included by the axes X,X'. Moreover the general values of V and ζ will be

$$V=\beta\,(x'\cos\varpi'+y'\sin\varpi')\quad\text{and}\quad\zeta=\frac{\pi}{4}+\frac{\gamma}{2}-\frac{\varpi'}{2},$$

and the initial condition, in order that our solution may be applicable, will evidently become $\varphi_0-\varpi_0=\varpi'_0=$ a quantity betwixt 0 and π.

As an example, let $\tan\gamma=\frac{1}{10}$, since we know by experiment that γ is generally very small; then taking the most unfavorable case, viz. where $\varpi'_0=0$, and supposing the body to make one revolution only, the value of ζ, determined from its initial one, $\zeta_0=\frac{\pi}{4}+\frac{\gamma}{2}-\frac{\varpi'_0}{2}$, will be found extremely small and only equal to a unit in the 27th decimal place. We thus see with what rapidity ζ decreases, and consequently, the body approaches to a permanent state, defined by the equation

$$0=\zeta=\frac{\pi}{4}+\frac{\gamma}{2}-\frac{\varpi'}{2}.$$

Hence, the polarity induced by the rotation is ultimately directed along a line, making an angle equal to $\frac{\pi}{2}+\gamma$ with the axis X', which agrees with what was shown in the former part of this article.

The value of V at the body's surface being thus known at any instant whatever, that of the potential function at a point p' exterior to the body, together with the forces acting there, will be immediately determined as before.

APPLICATION OF THE PRELIMINARY RESULTS TO THE THEORY OF MAGNETISM.

(14.) THE electric fluid appears to pass freely from one part of a continuous conductor to another, but this is by no means the case with the magnetic fluid, even with respect to those bodies which, from their instantly returning to a natural state the moment the forces inducing a magnetic one are removed, must be considered, in a certain sense, as perfect conductors of magnetism. COULOMB, I believe, was the first who proposed to consider these as formed of an infinite number of particles, each of which conducts the magnetic fluid in its interior with perfect freedom, but which are so constituted that it is impossible there shall be any communication of it from one particle to the next. This hypothesis is now generally adopted by philosophers, and its consequences, as far as they have hitherto been developed, are found to agree with observation; we will therefore admit it in what follows, and endeavour thence to deduce, mathematically, the laws of the distribution of magnetism in bodies of any shape whatever.

Firstly, let us endeavour to determine the value of the potential function, arising from the magnetic state induced in a very small body A, by the action of constant forces directed parallel to a given right line; the body being composed of an infinite number of particles, all perfect conductors of magnetism and originally in a natural state. In order to deduce this more immediately from art. 6, we will conceive these forces to arise from an infinite quantity Q of magnetic fluid, concentrated in a point p on this line, at an infinite distance from A. Then the origin O of the rectangular co-ordinates being any where within A, if x, y, z, be those of the point p, and x', y', z', those of any other exterior point p', to which the potential function V arising from A belongs, we shall have (vide Mec. Cel. Liv. 3)

$$V=\frac{U^{(0)}}{r'}+\frac{U^{(1)}}{r'^2}+\frac{U^{(2)}}{r'^3}+\text{etc.};$$

$r'=\sqrt{x'^2+y'^2+z'^2}$ being the distance Op'.

Moreover, since the total quantity of magnetic fluid in A is equal to *zero*, $U^{(0)}=0$. Supposing now r' very great compared with the dimensions of the body, all the terms after $\frac{U^{(1)}}{r'^2}$ in the expression just given will be exceedingly small compared with this, by neglecting them, therefore, and substituting for $U^{(1)}$ its most general value, we obtain

$$V=\frac{U^{(1)}}{r'^2}=\frac{Ax'+By'+Cz'}{r'^3};$$

A, B, C, being quantities independent of x', y', z', but which may contain x, y, z.

Now (art. 6) the value of V will remain unaltered, when we change x, y, z, into x', y', z', and reciprocally. Therefore

$$V=\frac{Ax'+By'+Cz'}{r'^3}=\frac{A'x+B'y+C'z}{r^3};$$

A', B', C', being the same functions of x', y', z', as A, B, C, are of x, y, z. Hence it is easy to see that V must be of the form

$$V=\frac{a''xx'+b''yy'+c''zz'+e''(xy'+yx')+f''(xz'+zx')+g''(yz'+zy')}{r^3\,r'^3};$$

$a'', b'', c'', e'', f'', g''$, being constant quantities.

If X, Y, Z, represent the forces arising from the magnetism concentrated in p, in the diretions of x, y, z, positive, we shall have

$$X=\frac{-Qx}{r^3};\quad Y=\frac{-Qy}{r^3};\quad Z=\frac{-Qz}{r^3};$$

and therefore V is of the form

$$V=\frac{a'Xx'+b'Yy'+c'Zz'+e'(Xy'+Yx')+f'(Xz'+Zx')+g'(Yz'+Zy')}{r'^3};$$

a', b', *etc.* being other constant quantities. But it will always be possible to determine the situation of three rectangular axes, so that e, f, and g may each be equal to *zero*, and consequently V be reduced to the following simple form

$$V=\frac{aXx'+bYy'+cZz'}{r'^3}; \quad \ldots\ldots \quad (a)$$

a, b, and c being three constant quantities.

When A is a sphere, and its magnetic particles are either spherical, or, like the integrant particles of non-crystalized bodies, arranged in a confused manner; it is evident the constant quantities a', b', c', *etc.* in the general value of V, must be the same for every system of rectangular co-ordinates, and consequently we must have $a'=b'=c'$, $e'=o$, $f'=o$, and $g'=o$, therefore in this case

$$V=\frac{a'(Xx'+Yy'+Zz')}{r'^3}; \quad \ldots\ldots \quad (b)$$

a' being a constant quantity dependant on the magnitude and nature of A.

The formula *(a)* will give the value of the forces acting on any point p', arising from a mass A of soft iron or other similar matter, whose magnetic state is induced by the influence of the earth's action; supposing the distance Ap' to be great compared with the dimensions of A, and if it be a solid of revolution, one of the rectangular axes, say X, must coincide with the axis of revolution, and the value of V reduce itself to

$$V=\frac{a'Xx'+b'(Yy'+Zz')}{r'^3};$$

a' and b' being two constant quantities dependant on the form and nature of the body. Moreover the forces acting in the directions of x', y', z', positive, are expressed by

$$-\left(\frac{dV}{dx'}\right), -\left(\frac{dV}{dy'}\right), -\left(\frac{dV}{dz'}\right).$$

We have thus the means of comparing theory with experiment, but these are details into which our limits will not permit us to enter.

The formula *(b)*, which is strictly correct for an infinitely small sphere, on the supposition of its magnetic particles being arranged in a confused manner, will, in fact, form the basis of our theory, and although the preceding analysis seems sufficiently general and rigorous, it may not be amiss to give a simpler proof of this particular case. Let, therefore, the origin O of the rectangular co-ordinates be placed at the centre of the infinitely small sphere A, and OB be the direction of the parallel forces acting upon it; then, since the total quantity of magnetic fluid in A is equal to *zero*, the value of the potential function V, at the point p', arising from A, must evidently be of the form

$$V=\frac{k\cos\theta}{r'^2};$$

r' representing as before the distance Op', and θ the angle formed between the line Op', and another line OD fixed in A. If now f be the magnitude of the force directed along OB, the constant k will evidently be of the form $k=a'f$; a' being a constant quantity. The value of V, just given, holds good for any arrangement, regular or irregular, of the magnetic particles composing A, but on the latter supposition, the value of V would evidently remain unchanged, provided the sphere, and consequently the line OD, revolved round OB as an axis, which could not be the case unless OB and OD coincided. Hence $\theta=$angle BOp' and

$$V=\frac{a'f\cos\theta}{r'^2}.$$

Let now α, β, γ, be the angles that the line $Op'=r'$ makes with the axes of x, y, z, and α', β', γ', those which OB makes with the same axes; then substituting for $\cos\theta$ its value $\cos\alpha\cos\alpha'+\cos\beta\cos\beta'+\cos\gamma\cos\gamma'$, we have, since $f\cos\alpha=X, f\cos\beta=Y, f\cos\gamma=Z$,

$$V=\frac{a'(X\cos\alpha+Y\cos\beta+Z\cos\gamma)}{r'^2}, \quad \ldots\ldots \quad (b')$$

Which agrees with the equation *(b)*, seeing that $\cos\alpha=\frac{x'}{r'}$, $\cos\beta=\frac{y'}{r'}$, $\cos\gamma=\frac{z'}{r'}$.

(15.) Conceive now, a body A, of any form, to have a magnetic state induced in its particles by the influence of exterior forces, it is clear that if dv be an element of its volume, the value of the potential function arising from this element, at any point p' whose co-ordinates are x', y', z', must, since the total quantity of magnetic fluid in dv is equal to *zero*, be of the form

$$\frac{dv\,[X(x'-x)+Y(y'-y)+Z(z'-z)]}{r^3}; \quad . \quad . \quad . \quad . \quad (a)$$

x, y, z, being the co-ordinates of dv, r the distance p', dv and X, Y, Z, three quantities dependant on the magnetic state induced in dv, and serving to define this state. If therefore dv' be an infinitely small volume within the body A and inclosing the point p', the potential function arising from the whole of A exterior to dv', will be expressed by

$$\int dx\,dy\,dz\,\frac{X(x'-x)+Y(y'-y)+Z(z'-z)}{r^3};$$

the integral extending over the whole volume of A exterior to dv'.

It is easy to show from this expression that, in general, although dv' be infinitely small, the forces acting in its interior vary in magnitude and direction by passing from one part of it to another; but, when dv' is spherical, these forces are sensibly constant in magnitude and direction, and consequently, in this case, the value of the potential function induced in dv' by their action, may be immediately deduced from the preceding article.

Let ψ' represent the value of the integral just given, when dv' is an infinitely small sphere. The force acting on p' arising from the mass exterior to dv', tending to increase x', will be

$$-\left(\frac{d\overline{\psi'}}{dx'}\right);$$

the line above the differential co-efficient indicating that it is to be obtained by supposing the radius of dv' to vanish after differentiation, and this may differ from the one obtained by first making the radius vanish, and afterwards differentiating the resulting function of x', y', z', which last being represented as usual by $\frac{d\psi'}{dx'}$, we have

$$\frac{d\overline{\psi'}}{dx'}=\frac{d}{dx'}\int dx\,dy\,dz\,\frac{X(x'-x)+Y(y'-y)+Z(z'-z)}{r^3}$$

$$\frac{d\psi'}{dx'}=\frac{d}{dx'}\int dx\,dy\,dz\,\frac{X(x'-x)+Y(y'-y)+Z(z'-z)}{r^3};$$

the first integral being taken over the whole volume of A exterior to dv', and the second over the whole of A including dv'. Hence

$$\frac{d\psi'}{dx'}-\frac{d\overline{\psi'}}{dx'}=\frac{d}{dx'}\int dx\,dy\,dz\,\frac{X(x'-x)+Y(y'-y)+Z(z'-z)}{r^3};$$

the last integral comprehending the volume of the spherical particle dv' only, whose radius a is supposed to vanish after differentiation. In order to effect the integration here indicated, we may remaik that X, Y and Z are sensibly constant within dv', and may therefore be replaced by $X_{,}$, $Y_{,}$ and $Z_{,}$, their values at the centre of the sphere dv', whose co-ordinates are $x_{,}, y_{,}, z_{,}$; the required integral will thus become

$$\int dxdydz\,\frac{X_{,}(x'-x)+Y_{,}(y'-y)+Z_{,}(z'-z)}{r^3}.$$

Making for a moment $E=X_{,}x+Y_{,}y+Z_{,}z$, we shall have $X_{,}=\frac{dE}{dx}$, $Y_{,}=\frac{dE}{dy}$, $Z_{,}=\frac{dE}{dz}$, and as also $\frac{x'-x}{r^3}=\frac{d\frac{1}{r}}{dx}$; $\frac{y'-y}{r^3}=\frac{d\frac{1}{r}}{dy}$; $\frac{z'-z}{r^3}=\frac{d\frac{1}{r}}{dz}$, this integral may be written

$$\int dxdydz\left\{\frac{dE}{dx}\cdot\frac{d\frac{1}{r}}{dx}+\frac{dE}{dy}\cdot\frac{d\frac{1}{r}}{dy}+\frac{dE}{dz}\cdot\frac{d\frac{1}{r}}{dz}\right\},$$

which since $\delta E=0$, and $\delta\frac{1}{r}=0$, reduces itself by what is proved in art. 3, to

$$-\int\frac{d\sigma}{r}\left(\frac{dE}{dw}\right)=\text{(because } dw=-da\text{)}\int\frac{d\sigma}{r}\frac{dE}{da};$$

the integral extending over the whole surface of the sphere dv', of which $d\sigma$ is an element; r being the distance p', $d\sigma$, and dw measured from the surface towards the interior of dv'. Now $\int\frac{d\sigma}{r}\frac{dE}{da}$ expresses the value of the potential function for a point p', within the sphere, supposing its surface every where covered with electricity whose density is $\frac{dE}{da}$, and may very easily be obtained by No. 13, Liv. 3, Mec. Celeste. In fact, using for a moment the notation there employed, supposing the origin of the polar co-ordinates at the centre of the sphere, we have

$$E=E_{,}+a\left[X_{,}\cos\theta+Y_{,}\sin\theta\cos\varpi+Z_{,}\sin\theta\sin\varpi\right];$$

$E_{,}$ being the value of E at the centre of the sphere. Hence

$$\frac{dE}{da}=X_{,}\cos\theta+Y_{,}\sin\theta\cos\varpi+Z_{,}\sin\theta\sin\varpi,$$

and as this is of the form $U^{(1)}$ (Vide Mec. Celeste Liv. 3.), we immediately obtain

$$\int\frac{d\sigma}{r}\frac{dE}{da}=\frac{4\pi r'}{3}\left\{X_{,}\cos\theta'+Y_{,}\sin\theta'\cos\varpi'+Z_{,}\sin\theta'\sin\varpi'\right\}$$

where r', θ', ϖ', are the polar co-ordinates of p'. Or by restoring x', y', and z'

$$\int\frac{d\sigma}{r}\frac{dE}{da}=\frac{4\pi}{3}\left\{X_{,}(x'-x_{,})+Y_{,}(y'-y_{,})+Z_{,}(z'-z_{,})\right\}.$$

Hence we deduce successively

$$\frac{d\psi'}{dx'}-\frac{d\overline{\psi'}}{dx'}=\frac{d}{dx'}\int dxdydz\,\frac{X(x'-x)+Y(y'-y)+Z(z'-z)}{r^3}$$

$$=\frac{d}{dx'}\int\frac{d\sigma}{r}\frac{dE}{da}=\frac{d}{dx'}\frac{4\pi}{3}\left\{X_{,}(x'-x_{,})+Y_{,}(y'-y_{,})+Z_{,}(z'-z_{,})\right\}=\frac{4\pi X_{,}}{3}.$$

If now we make the radius a vanish, $X_{\prime}$ must become equal to X', the value of X at the point p', and there will result

$$\frac{d\psi'}{dx'} - \frac{d\overline{\psi'}}{dx'} = \frac{4\pi X'}{3} \quad i.\,e. \quad \frac{d\overline{\psi'}}{dx'} = \frac{d\psi'}{dx'} - \frac{4\pi X'}{3}.$$

But $-\frac{d\overline{\psi'}}{dx'}$ expresses the value of the force acting in the direction of x positive, on a point p' within the infinitely small sphere dv', arising from the whole of A exterior to dv'; substituting now for $\frac{d\overline{\psi'}}{dx'}$ its value just found, the expression of this force becomes

$$\frac{4\pi}{3} X' - \frac{d\psi'}{dx'}.$$

Supposing V' to represent the value of the potential function at p', arising from the exterior bodies which induce the magnetic state of A, the force due to them acting in the same direction, is

$$-\frac{dV'}{dx'},$$

and therefore the total force in the direction of x' positive, tending to induce a magnetic state in the spherical element dv', is

$$\frac{4\pi}{3} X' - \frac{d\psi'}{dx'} - \frac{dV'}{dx'} = \overline{X}.$$

In the same way, the total forces in the directions of y' and z' positive, acting upon dv', are shown to be

$$\frac{4\pi}{3} Y' - \frac{d\psi'}{dy'} - \frac{dV'}{dy'} = \overline{Y}, \text{ and, } \frac{4\pi}{3} Z' - \frac{d\psi'}{dz'} - \frac{dV'}{dz'} = \overline{Z}.$$

By the equation (b') of the preceding article, we see that when dv' is a perfect conductor of magnetism, and its particles are not regularly arranged, the value of the potential function at any point p'', arising from the magnetic state induced in dv' by the action of the forces $\overline{X}, \overline{Y}, \overline{Z}$, is of the form

$$\frac{a'(\overline{X}\cos\alpha + \overline{Y}\cos\beta + \overline{Z}\cos\gamma)}{r'^2};$$

r' being the distance p'', dv', and α, β, γ, the angles which r' forms with the axes of the rectangular co-ordinates. If then x'', y'', z'', be the co-ordinates of p'', this becomes, by observing that here $a' = kdv'$,

$$\frac{kdv'\left[\overline{X}(x''-x') + \overline{Y}(y''-y') + \overline{Z}(z''-z')\right]}{r'^3}$$

k being a constant quantity dependant on the nature of the body. The same potential function will evidently be obtained from the expression (a) of this article, by changing dv, p', and their co-ordinates, into dv', p'', and their co-ordinates; thus we have

$$\frac{dv'\left[X'(x''-x')+Y'(y''-y')+Z'(z''-z')\right]}{r'^3}.$$

Equating these two forms of the same quantity, there results the three following equations:

$$X'=k\overline{X}=\frac{4\pi k}{3}X'-k\frac{d\psi'}{dx'}-k\frac{dV'}{dx'}$$

$$Y'=k\overline{Y}=\frac{4\pi k}{3}Y'-k\frac{d\psi'}{dy'}-k\frac{dV'}{dy'}$$

$$Z'=k\overline{Z}=\frac{4\pi k}{3}Z'-k\frac{d\psi'}{dz'}-k\frac{dV'}{dz'},$$

since the quantities x'', y'', z'', are perfectly arbitrary. Multiplying the first of these equations by dx', the second by dy', the third by dz', and taking their sum, we obtain

$$0=\left(1-\frac{4\pi k}{3}\right)(X'dx'+Y'dy'+Z'dz')+kd\psi'+kdV'.$$

But $d\psi'$ and dV' being perfect differentials, $X'dx'+Y'dy'+Z'dz'$ must be so likewise, making therefore

$$d\phi'=X'dx'+Y'dy'+Z'dz',$$

the above, by integration, becomes

$$\text{const}=\left(1-\frac{4\pi k}{3}\right)\phi'+k\psi'+kV'.$$

Although the value of k depends wholly on the nature of the body under consideration, and is to be determined for each by experiment, we may yet assign the limits between which it must fall. For we have, in this theory, supposed the body composed of conducting particles, separated by intervals absolutely impervious to the magnetic fluid; it is therefore clear the magnetic state induced in the infinitely small sphere dv', cannot be greater than that which would be induced, supposing it one continuous conducting mass, but may be made less in any proportion, at will, by augmenting the non-conducting intervals.

When dv' is a continuous conductor, it is easy to see the value of the potential function at the point p'', arising from the magnetic state induced in it by the action of the forces $\overline{X}, \overline{Y}, \overline{Z}$, will be

$$\frac{3dv}{4\pi}\cdot\frac{\overline{X}(x''-x')+\overline{Y}(y''-y')+\overline{Z}(z''-z)}{r'^3},$$

seeing that $\frac{3dv}{4\pi}=a^3$; a representing, as before, the radius of the sphere dv'. By comparing this expression with that before found, when dv' was not a continuous conductor, it is evident k must be between the limits 0 and $\frac{3}{4\pi}$, or, which is the same thing, $k=\frac{3g}{4\pi}$; g being any positive quantity less than 1.

The value of k, just found, being substituted in the equation serving to determine ϕ', there arises

$$\text{const} = (1-g)\,\phi' + \frac{3g}{4\pi}(\psi' + V').$$

Moreover

$$\psi' = \int dx\,dy\,dz\,\frac{X(x'-x) + Y(y'-y) + Z(z'-z)}{r^3} = \int dx\,dy\,dz \left\{ \frac{d\phi}{dx}\cdot\frac{d\frac{1}{r}}{dx} + \frac{d\phi}{dy}\cdot\frac{d\frac{1}{r}}{dy} + \frac{d\phi}{dz}\cdot\frac{d\frac{1}{r}}{dz} \right\}$$

$$= 4\pi\phi' - \int d\sigma\,\overline{\phi}\left(\frac{\overline{d\frac{1}{r}}}{dw}\right) \text{ (art. 3);}$$

the triple integrals extending over the whole volume of A, and that relative to $d\sigma$ over its surface, of which $d\sigma$ is an element; the quantities $\overline{\phi}$ and $\frac{\overline{d\frac{1}{r}}}{dw}$ belonging to this element. We have, therefore, by substitution

$$\text{const.} = (1+2g)\,\phi' + \frac{3g}{4\pi}\left(V' - \int d\sigma\,\overline{\phi}\left(\frac{\overline{d\frac{1}{r}}}{dw}\right)\right).$$

Now $\delta' V' = 0$, and $\delta'\int d\sigma\,\overline{\phi}\left(\frac{d\frac{1}{r}}{dw}\right) = 0$, and consequently $\delta'\phi' = 0$; the symbol δ' referring to x', y', z', the co-ordinates of p'; or, since x', y' and z' are arbitrary, by making them equal to x, y, and z, respectively, there results

$$0 = \delta\phi,$$

in virtue of which, the value of ψ', by article 3, becomes

$$\psi' = -\int \frac{d\sigma}{r}\left(\frac{\overline{d\phi}}{dw}\right); \quad . \quad . \quad . \quad . \quad (b)$$

r being the distance p', $d\sigma$, and $\left(\frac{\overline{d\phi}}{dw}\right)$ belonging to $d\sigma$. The former equation serving to determine ϕ' gives, by changing x', y', z', into x, y, z,

$$\text{const} = (1-g)\,\phi + \frac{3g}{4\pi}(\psi + V); \quad . \quad . \quad . \quad . \quad (c)$$

ϕ, ψ and V belonging to a point p, within the body, whose co-ordinates are x, y, z. It is moreover evident from what precedes that, the functions ϕ, ψ and V satisfy the equations $0 = \delta\phi$, $0 = \delta\psi$ and $0 = \delta V$ and have no singular values in the interior of A.

The equations *(b)* and *(c)* serve to determine ϕ and ψ, completely, when the value of V arising from the exterior bodies is known, and therefore they enable us to assign the magnetic state of every part of the body A, seeing that it depends on X, Y, Z, the differential co-efficients of ϕ. It is also evident that ψ', when calculated for any point p', not contained within the body A, is the value of the potential function at this point arising from the magnetic state induced in A, and therefore this function is always given by the equation *(b)*.

The constant quantity g, which enters into our formulæ, depends on the nature of the body solely, and, in a subsequent article, its value is determined for a cylindric wire used by Coulomb. This value differs very little from unity: supposing therefore $g = 1$, the equations *(b)* and *(c)* become

$$\overline{\psi}' = -\int \frac{d\sigma}{r} \left(\frac{d\overline{\phi}}{dw}\right) \quad . \quad . \quad . \quad . \quad (b')$$

$$\text{const} = \overline{\psi} + V, \quad . \quad . \quad . \quad . \quad (c')$$

evidently the same, in effect, as would be obtained by considering the magnetic fluid at liberty to move from one part of the conducting body to another; the density ρ being here replaced by $-\left(\frac{d\overline{\phi}}{dw}\right)$, and since the value of the potential function for any point exterior to the body is, on either supposition, given by the formula *(b)*, the exterior actions will be precisely the same in both cases. Hence, when we employ iron, nickel, or similar bodies, in which the value of g is nearly equal to 1, the observed phenomena will differ little from those produced on the latter hypothesis, except when one of their dimensions is very small compared with the others, in which case the results of the two hypotheses differ widely, as will be seen in some of the applications which follow.

If the magnetic particles composing the body were not perfect conductors, but indued with a coercive force, it is clear there might always be equilibrium, provided the magnetic state of the element dv' was such as would be induced by the forces $\frac{d\overline{\psi}'}{dx'} + \frac{dV'}{dx'} + A'$, $\frac{d\overline{\psi}'}{dy'} + \frac{dV'}{dy'} + B'$ and $\frac{d\overline{\psi}'}{dz'} + \frac{dV'}{dz'} + C'$, instead of $\frac{d\overline{\psi}'}{dx'} + \frac{dV'}{dx'}$, $\frac{d\overline{\psi}'}{dy'} + \frac{dV'}{dy'}$ and $\frac{d\overline{\psi}'}{dz'} + \frac{dV'}{dz'}$; supposing the resultant of the forces A', B', C', no where exceeds a quantity β, serving to measure the coercive force. This is expressed by the condition

$$A'^2 + B'^2 + C'^2 \angle \beta^2.$$

The equation *(c)* would then be replaced by

$$0 = (1-g)\, d\overline{\phi} + \frac{3g}{4\pi}\,(d\overline{\psi} + dV + A dx + B dy + C dz)\,; \quad . \quad . \quad . \quad . \quad (c')$$

A, B, C, being any functions of x, y, z, as A', B', C', are of x', y', z', subject only to the condition just given.

It would be extremely easy so to modify the preceding theory, as to adapt it to a body whose magnetic particles are regularly arranged, by using the equation *(a)* in the place of the equation *(b)* of the preceding article; but, as observation has not yet offered any thing which would indicate a regular arrangement of magnetic particles, in any body hitherto examined, it seems superfluous to introduce this degree of generality, more particularly as the omission may be so easily supplied.

(16.) As an application of the general theory contained in the preceding article, suppose the body A to be a hollow spherical shell of uniform thickness, the radius of whose inner surface is a, and that of its outer one $a_{,}$; and let the forces inducing a magnetic state in A, arise from any bodies whatever, situate at will, within or without the shell. Then since in the interior of A's mass $0=\delta\phi$, and $0=\delta V$, we shall have (Mec. Cel. Liv. 3)

$$\phi = \Sigma \phi^{(i)} r^i + \Sigma \phi_{,}^{(i)} r^{-i-1} \quad \text{and} \quad V = \Sigma U^{(i)} r^i + \Sigma U_{,}^{(i)} r^{-i-1};$$

r being the distance of the point p, to which ϕ and V belong, from the shell's centre,

$\varphi^{(0)}, \varphi^{(1)}$, *etc.*—$U^{(0)}, U^{(1)}$, *etc.* functions of θ and ϖ, the two other polar co-ordinates of p, whose nature has been fully explained by LAPLACE in the work just cited; the finite integrals extending from $i=0$ to $i=\infty$.

If now, to prevent ambiguity, we enclose the r of equation *(b)* art. 15 in a parenthesis. it will become

$$\psi=\int\frac{d\sigma}{(r)}\left(\frac{\overline{d\varphi}}{dw}\right);$$

(r) representing the distance p, $d\sigma$, and the integral extending over both surfaces of the shell. At the inner surface we have $\frac{\overline{d\varphi}}{dw}=\frac{\overline{d\varphi}}{dr}$ and $r=a$: hence the part of ψ due to this surface is

$$-\int\frac{d\sigma}{(r)}\frac{\overline{d\varphi}}{dr}=-\int\frac{d\sigma}{(r)}\Sigma i\varphi^{(i)}a^{i-1}+\int\frac{d\sigma}{(r)}\Sigma(i+1)\varphi_{,}^{(i)}a^{-i-2};$$

the integrals extending over the whole of the inner surface, and $d\sigma$ being one of its elements. Effecting the integrations by the formulæ of LAPLACE (Mec. Celeste, Liv. 3), we immediately obtain the part of ψ, due to the inner surface, viz.

$$\frac{4\pi a^2}{r}\Sigma\frac{a^i}{(2i+1)r^i}\left(-ia^{i-1}\varphi^{(i)}+(i+1)\varphi_{,}^{(i)}a^{-i-2}\right).$$

In the same way the part of ψ due to the outer surface, by observing that for it $\frac{\overline{d\varphi}}{dw}=-\frac{\overline{d\varphi}}{dr}$ and $r=a_{,}$, is found to be

$$4\pi a_{,}\Sigma\frac{r^i}{(2i+1)a_{,}^i}\left(ia_{,}^{i-1}\varphi^{(i)}-(i+1)\varphi^{(i)}a_{,}^{-i-2}\right).$$

The sum of these two expressions is the complete value of ψ, which, together with the values of φ and V before given, being substituted in the equation *(c)* art. 15, we obtain

$$\text{const}=(1-g)\Sigma\varphi_{,}^{(i)}r^{-i-1}+(1-g)\Sigma\varphi^{(i)}r^i+\frac{3g}{4\pi}\Sigma U_{,}^{(i)}r^{-i-1}+\frac{3g}{4\pi}\Sigma U^{(i)}r^i$$
$$+\frac{3ga^2}{r}\Sigma\frac{a^i}{(2i+1)r^i}(-ia^{i-1}\varphi^{(i)}+(i+1)\varphi_{,}^{(i)}a^{-i-2})+3ga_{,}\Sigma\frac{r^i}{(2i+1)a_{,}^i}(ia_{,}^{i-1}\varphi^{(i)}-(i+1)\varphi_{,}^{(i)}a_{,}^{-i-2}).$$

Equating the co-efficients of like powers of the variable r, we have generally, whatever i may be,

$$0=(1-g)\varphi_{,}^{(i)}+\frac{3ga^{i+2}}{2i+1}\left(-ia^{i-1}\varphi^{(i)}+(i+1)\varphi_{,}^{(i)}a^{-i-2}\right)+\frac{3g}{4\pi}U_{,}^{(i)}$$

$$0=(1-g)\varphi^{(i)}+\frac{3g}{(2i+1)a_{,}^{i-1}}\left(ia_{,}^{i-1}\varphi^{(i)}-(i+1)\varphi_{,}^{(i)}a_{,}^{-i-2}\right)+\frac{3g}{4\pi}U^{(i)};$$

neglecting the constant on the right side of the equation in r as superfluous, since it may always be made to enter into $\varphi^{(0)}$. If now, for abridgment, we make

$$D=(2i+1)^2(1+g)+(i-1)(i+2)g^2-9g^2i(i+1)\left(\frac{a}{a_{,}}\right)^{2i+1},$$

we shall obtain by elimination

$$\varphi^{(i)} = -\frac{3g}{4\pi} U^{(i)} \frac{(2i+1)(2i+1+(i+2)g)}{D} - \frac{3g}{4\pi} U_{,}^{(i)} \frac{3g(i+1)(2i+1)a_{,}^{-2i-1}}{D}$$

$$\varphi_{,}^{(i)} = -\frac{3g}{4\pi} U^{(i)} \frac{3gi(2i+1)a^{2i+1}}{D} - \frac{3g}{4\pi} U_{,}^{(i)} \frac{(2i+1)(2i+1+(i-1)g)}{D}.$$

These values substituted in the expression

$$\varphi = \Sigma \varphi^{(i)} r^{i} + \Sigma \varphi_{,}^{(i)} r^{-i-1},$$

give the general value of φ in a series of the powers of r, when the potential function due to the bodies inducing a magnetic state in the shell is known, and thence we may determine the value of the potential function ψ arising from the shell itself, for any point whatever, either within or without it.

When all the bodies are situate in the space exterior to the shell, we may obtain the total actions exerted on a magnetic particle in its interior, by the following simple method, applicable to hollow shells of any shape and thickness.

The equation *(c)* art. 15 becomes, by neglecting the superfluous constant,

$$0 = (1-g)\varphi + \frac{3g}{4\pi}(\psi + V).$$

If now (φ) represent the value of the potential function, corresponding to $\bar{\varphi}$ the value of φ at the inner surface of the shell, each of the functions (φ), ψ and V, will satisfy the equations $0 = \delta(\varphi)$, $0 = \delta\psi$ and $0 = \delta V$, and moreover, have no singular values in the space within the shell; the same may therefore be said of the function

$$(1-g)(\varphi) + \frac{3g}{4\pi}(\psi + V),$$

and as this function is equal to *zero* at the inner surface, it follows (art. 5) that it is so for any point p of the interior space. Hence

$$0 = (1-g)(\varphi) + \frac{3g}{4\pi}(\psi + V).$$

But $\psi + V$ is the value of the total potential function at the point p, arising from the exterior bodies and shell itself: this function will therefore be expressed by

$$-\frac{4\pi(1-g)}{3g}(\varphi).$$

In precisely the same way, the value of the total potential function at any point p', exterior to the shell, when the inducing bodies are all within it, is shown to be

$$-\frac{4\pi(1-g)}{3g}.(\varphi');$$

(φ') being the potential function corresponding to the value of φ at the exterior surface of the shell. Having thus the total potential functions, the total action exerted on a magnetic particle in any direction, is immediately given by differentiation.

To apply this general solution to our spherical shell, the inducing bodies being all exterior to it, we must first determine $\bar{\phi}$, the value of φ at its inner surface, making $0 = \Sigma U_{\prime}^{(i)} r^{-i-1}$ since there are no interior bodies, and thence deduce the value of (φ). Substituting for $\varphi^{(i)}$ and $\varphi_{\prime}^{(i)}$ their values before given, making $U_{\prime}^{(i)} = 0$ and $r = a$, we obtain

$$\bar{\phi} = \frac{-3g}{4\pi}(1+2g)\sum U^{(i)}\frac{(2i+1)^2 a^i}{D},$$

and the corresponding value of (φ) is (Mec. Cel. Liv. 3)

$$(\phi) = \frac{-3g}{4\pi}(1+2g)\sum U^{(i)}\frac{(2i+1)^2 r^i}{D}.$$

The value of the total potential function at any point p within the shell, whose polar co-ordinates are r, θ, ϖ, is

$$-\frac{4\pi}{3g}(1-g)(\varphi) = (1-g)(1+2g)\sum U^{(i)}\frac{(2i+1)^2 r^i}{D}.$$

In a similar way, the value of the same function at a point p' exterior to the shell, all the inducing bodies being within it, is found to be

$$(1-g)(1+2g)\sum U_{\prime}^{(i)}\frac{(2i+1)^2}{D.r^{i+1}};$$

r, θ and ϖ in this expression representing the polar co-ordinates of p'.

To give a very simple example of the use of the first of these formulæ, suppose it were required to determine the total action exerted in the interior of a hollow spherical shell, by the magnetic influence of the earth; then making the axis of x to coincide with the direction of the dipping needle, and designating by f, the constant force tending to impel a particle of positive fluid in the direction of x positive, the potential function V, due to the exterior bodies, will here become

$$V = -f.x = -f\cos\theta.r = U^{(1)}.r.$$

The finite integrals expressing the value of V reduce themselves therefore, in this case, to a single term, in which $i=1$, and the corresponding value of D being $9\left(1+g-2g^2\frac{a^3}{a_{\prime}^3}\right)$, the total potential function within the shell is

$$-(1-g)(1+2g)U^{(1)}\frac{r}{1+g-2g^2\frac{a^3}{a_{\prime}^3}} = -\frac{1+g-2g^2}{1+g-2g^2\frac{a^3}{a_{\prime}^3}}f.x.$$

We therefore see that the effect produced by the intervening shell, is to reduce the directive force which would act on a very small magnetic needle,

$$\text{from } f, \text{ to } \frac{1+g-2g^2}{1+g-2g^2\frac{a^3}{a_{\prime}^3}}f.$$

In iron and other similar bodies, g is very nearly equal to 1, and therefore the directive force in the interior of a hollow spherical shell is greatly diminished, except when its thickness is very small compared with its radius, in which case, as is evident from the formula, it approaches towards the original value f, and becomes equal to it when this thickness is infinitely small.

To give an example of the use of the second formula, let it be proposed to determine the total action upon a point p, situate on one side of an infinitely extended plate of uniform thickness, when another point P, containing a unit of positive fluid, is placed on the other side of the same plate considering it as a perfect conductor of magnetism. For this, let fall the perpendicular PQ upon the side of the plate next P, on PQ prolonged, demit the perpendicular $p\,q$, and make $PQ=b$, $Pq=u$, $pq=v$, and $t=$the thickness of the plate; then, since its action is evidently equal to that of an infinite sphere of the same thickness, whose centre is upon the line QP at an infinite distance from P, we shall have the required value of the total potential function at p by supposing $a_{,}=a+t$, a infinite, and the line PQ prolonged to be the axis from which the angle θ is measured. Now in the present case

$$V=\frac{1}{Pp}=\frac{1}{\sqrt{r^2-2r(a-b)\cos\theta+(a-b)^2}}=\Sigma U_{,}^{(i)}r^{-i-1},$$

and the value of the potential function, as before determined; is

$$(1-g)\,(1+2g)\sum\frac{(2i+1)^2}{D}\,U_{,}^{(i)}r^{-i-1}.$$

From the first expression we see that the general term $U_{,}^{(i)}r^{-i-1}$ is a quantity of the order $(a-b)^i r^{-i-1}$. Moreover, by substituting for r its value in u

$$(a-b)^i r^{-i-1}=(a-b)^i(a-b+u)^{-i-1}=\frac{1}{a}\,e^{-\frac{iu}{a}};$$

neglecting such quantities as are of the order $\frac{1}{a}$ compared with those retained. The general term $U_{,}^{(i)}r^{-i-1}$, and consequently $U_{,}^{(i)}$, ought therefore to be considered as functions of $\frac{i}{a}=\gamma$. In the finite integrals just given, the increment of i is 1, and the corresponding increment of γ is $\frac{1}{a}=d\gamma$ (because a is infinite), the finite integrals thus change themselves into ordinary integrals or fluents. In fact (Mec. Cel. Liv. 3), $U_{,}^{(i)}$ always satisfies the equation

$$\frac{d^2U_{,}^{(i)}}{d\theta^2}+\frac{\cos\theta}{\sin\theta}\,\frac{dU_{,}^{(i)}}{d\theta}+i(i+1)U_{,}^{(i)}=0,$$

and as θ is infinitely small whenever V has a sensible value, we may eliminate it from the above by means of the equation $a\theta=v$, and we obtain by neglecting infinitessimals of higher orders than those retained, since $\frac{i}{a}=\gamma$,

$$0=\frac{d^2U_{,}^{(i)}}{dv^2}+\frac{dU_{,}^{(i)}}{vdv}+\gamma^2U_{,}^{(i)}.$$

Hence the value $U_{,}^{(i)}$ is of the form

$$U_i^{(i)} = A\int_0^1 \frac{d\beta}{\sqrt{1-\beta^2}} \cos(\beta\gamma v);$$

seeing that the remaining part of the general integral becomes infinite when v vanishes, and ought therefore to be rejected. It now only remains to determine the value of the arbitrary constant A. Making, for this purpose, $\theta=0$, i. e. $v=0$, we have

$$U_i^{(i)}=(a-b)^i \text{ and} \int_0^1 \frac{d\beta}{\sqrt{1-\beta^2}} = \frac{\pi}{2}: \text{ hence } (a-b)^i = \frac{A\pi}{2} \quad i.\ e.\quad A = \frac{2}{\pi}(a-b)^i.$$

By substituting for A and r their values, there results

$$U_i^{(i)} r^{-i-1} = \frac{2}{\pi}(a-b)^i (a-b+u)^{-i-1} \int_0^1 \frac{d\beta}{\sqrt{1-\beta^2}} \cos(\beta\gamma v) = \frac{2d\gamma}{\pi} e^{-\gamma u} \int_0^1 \frac{d\beta}{\sqrt{1-\beta^2}} \cos(\beta\gamma v);$$

because $\frac{i}{a}=\gamma$ and $\frac{1}{a}=d\gamma$. Writing now in the place of i its value $a\gamma$, and neglecting infinitessimal quantities, we have

$$\frac{(2i+1)^2}{D} = \frac{4}{4+4g+g^2-9g^2 e^{-2\gamma t}}.$$

Hence the value of the total potential function becomes

$$\frac{8}{\pi}(1-g)(1+2g)\int_0^\infty \frac{d\gamma . e^{-\gamma u}}{4+4g+g^2-9g^2 e^{-2\gamma t}} \int_0^1 \frac{d\beta}{\sqrt{1-\beta^2}} \cos(\beta\gamma v);$$

where the integral relative to γ is taken from $\gamma=0$ to $\gamma=\infty$, to correspond with the limits 0 and ∞ of i, seeing that $i=a\gamma$.

The preceding solution is immediately applicable to the imaginary case only, in which the inducing bodies reduce themselves to a single point P, but by the following simple artifice we may give it a much greater degree of generality:

Conceive another point P', on the line PQ, at an arbitrary distance c from P, and suppose the unit of positive fluid concentrated in P' instead of P; then if we make $r'=Pp$, and $\theta'=\angle pPQ$, we shall have $u=r'\cos\theta'$, $v=r'\sin\theta'$, and the value of the potential function arising from P' will be

$$\frac{1}{P'p} = \frac{1}{\sqrt{r'^2-2r'c\cos\theta'+c^2}} = Q^{(0)}\frac{1}{r'} + Q^{(1)}\frac{c}{r'^2} + Q^{(2)}\frac{c^2}{r'^3} + etc.$$

Moreover, the value of the total potential function at p due to this, arising from P' and the plate itself, will evidently be obtained by changing u into $u-c$ in that before given, and is therefore

$$\frac{8}{\pi}(1-g)(1+2g)\int_0^\infty \frac{e^{\gamma c} d\gamma e^{-\gamma u}}{(2+g)^2-9g^2 e^{-2\gamma t}} \int_0^1 \frac{d\beta}{\sqrt{1-\beta^2}} \cos(\beta\gamma v).$$

Expanding this function in an ascending series of the powers of c, the term multiplied by c^i is

$$\frac{8}{\pi}(1-g)(1+2g)\int_0^\infty \frac{\frac{\gamma^i c^i}{1.2.3\ldots n} d\gamma e^{-\gamma u}}{(2+g)^2-9g^2 e^{-2\gamma t}} \int_0^1 \frac{d\beta}{\sqrt{1-\beta^2}} \cos(\beta\gamma v),$$

which, as c is perfectly arbitrary, must be the part due to the term $Q^{(i)}\frac{c^i}{r^{i+1}}$ in the potential function arising from the inducing bodies. If then this function had been

$$Q^{(0)}\frac{k_0}{r'} + Q^{(1)}\frac{k_1}{r'^2} + Q^{(2)}\frac{k_2}{r'^3} + Q^{(3)}\frac{k_3}{r'^4} + etc.;$$

where the successive powers c^0, c^1, c^2 *etc.* of c are replaced by the arbitrary constant quantities k_0, k_1, k_2, *etc.*, the corresponding value of the total potential function will be given by making a like change in that due to P'. Hence if, for abridgment, we make

$$\varphi(\gamma) = k_0 + \frac{k_1}{1}\gamma + \frac{k_2}{1\cdot 2}\gamma^2 + \frac{k_3}{1.2\cdot 3}\gamma^3 + etc.,$$

the value of this function at the point p will be

$$\frac{8}{\pi}(1-g)(1+2g)\int_0^\infty \frac{\varphi(\gamma)d\gamma\, e^{-\gamma u}}{(2+g)^2-9g^2e^{-2\gamma t}}\int_0^1 \frac{d\beta}{\sqrt{1-\beta^2}}\cos(\beta\gamma v).$$

Now, if the original one due to the point P be called F, it is clear the expression just given may be written

$$\varphi\left(\frac{-d}{du}\right)\cdot F;$$

where the symbols of operation are separated from those of quantity, according to ARBOGAST's method; thus all the difficulty is reduced to the determination of F.

Resuming therefore the original supposition of the plate's magnetic state being induced by a particle of positive fluid concentrated in P, the value of the total potential function at p will be

$$F = \frac{8}{\pi}(1-g)(1+2g)\int_0^\infty \frac{d\gamma\, e^{-u\gamma}}{(2+g)^2-9g^2e^{-2\gamma t}}\int_0^1 \frac{d\beta}{\sqrt{1-\beta^2}}\cos(\beta\gamma v),$$

as was before shown. Writing now $e^{-v\beta\gamma\sqrt{-1}}$ in the place of $\cos(\beta\gamma v)$, we obtain

$$F = \frac{8}{\pi}(1-g)(1+2g)\int_0^1 \frac{d\beta}{\sqrt{1-\beta^2}}\int_0^\infty \frac{d\gamma\, e^{-\gamma(u+\beta v\sqrt{-1})}}{(2+g)^2-9g^2e^{-2\gamma t}},$$

provided we reject the imaginary quantities which may arise. In order to transform this double integral let $z = \frac{3g}{2+g}e^{-\gamma t}$, and we shall have

$$F = \frac{8(1-g)(1+2g)}{9\pi g^2 t}\left(\frac{2+g}{3g}\right)^{\frac{u}{t}-2}\int_0^1 \frac{d\beta}{\sqrt{1-\beta^2}}\left(\frac{2+g}{3g}\right)^{\frac{\beta v\sqrt{-1}}{t}}\int \frac{dz.z^{\frac{u}{t}-1+\frac{\beta v\sqrt{-1}}{t}}}{1-z^2};$$

the integral relative to z being taken from $z=0$ to $z=\frac{3g}{2+g}$.

The value of $1-g$, for iron and other similar bodies, is very small, neglecting therefore quantities which are of the order $(1-g)$ compared with those retained, there results

$$F=\frac{8(1-g)}{3\pi t}\int_0^1\frac{d\beta}{\sqrt{1-\beta^2}}\int_0^1\frac{dz}{1-z^2}\,z^{\frac{u}{t}-1+\frac{\beta v}{t}\sqrt{-1}}\,;\quad\ldots\quad(a)$$

where u and v may have any values whatever provided they are not very great and of the order $\frac{t}{1-g}$. If F_1 represents what F becomes by changing u into $u+2t$, we have

$$F_1=\frac{8(1-g)}{3\pi t}\int_0^1\frac{d\beta}{\sqrt{1-\beta^2}}\int_0^1\frac{z^2dz}{1-z^2}\,z^{\frac{u}{t}-1+\frac{\beta v}{t}\sqrt{-1}},$$

and consequently

$$F-F_1=\frac{8(1-g)}{3\pi t}\int_0^1\frac{d\beta}{\sqrt{1-\beta^2}}\int_0^1 dz\cdot z^{\frac{u}{t}-1+\frac{\beta v}{t}\sqrt{-1}}$$

which, by effecting the integrations and rejecting the imaginary quantities, becomes

$$F-F_1=\frac{4(1-g)}{3\sqrt{u^2+v^2}}=\frac{4(1-g)}{3r'}.$$

Suppose now pO is a perpendicular falling from the point p upon the surface of the plate, and on this line, indefinitely extended in the direction Op, take the points p_1, p_2, p_3, *etc.*, at the distances $2t$, $4t$, $6t$, *etc.* from p; then F_1, F_2, F_3, *etc.* being the values of F, calculated for the points p_1, p_2, p_3, *etc.* by the formula *(a)* of this article, and r'_1, r'_2, r'_3, *etc.* the corresponding values of r', we shall equally have

$$F_1-F_2=\frac{4(1-g)}{3r'_1};\quad F_2-F_3=\frac{4(1-g)}{3r'_2},\ etc.;$$

and consequently

$$F=\frac{4(1-g)}{3}\left\{\frac{1}{r'}+\frac{1}{r'_1}+\frac{1}{r'_2}+\text{etc. in infinitum}\right\};\ \text{seeing that } F_\infty=0.$$

From this value of F, it is evident the total action exerted upon the point p, in any given direction pn, is equal to the sum of the actions which would be exerted without the interposition of the plate, on each of the points p, p_1, p_2, *etc.* in infinitum, in the directions pn, p_1n_1, p_2n_2, *etc.* multiplied by the constant factor $\frac{4(1-g)}{3}$: the lines pn, p_1n_1, p_2n_2, *etc.* being all parallel. Moreover, as this is the case wherever the inducing point P may be situate, the same will hold good when, instead of P, we substitute a body of any figure whatever magnetized at will. The only condition to be observed, is, that the distance between p and every part of the inducing body be not a very great quantity of the order $\frac{t}{1-g}$.

On the contrary, when the distance between p and the inducing body is great enough to render $\frac{(1-g)r'}{t}$ a very considerable quantity, it will be easy to show, by expanding F in a descending series of the powers of r', that the actions exerted upon p are very nearly the same as if no plate were interposed.

We have before remarked (art. 15,) that when the dimensions of a body are all quantities of the same order, the results of the true theory differ little from those, which would be obtained by supposing the magnetic like the electric fluid, at liberty to move from one part of a conducting body to another; but when, as in the present example, one of the dimensions is very small compared with the others, the case is widely different; for if we make g rigorously equal to 1 in the preceding formulæ, they will belong to the latter supposition (art. 15), and as F will then vanish, the interposing plate will exactly neutralize the action of any magnetic bodies however they may be situate, provided they are on the side opposite the attracted point. This differs completely from what has been deduced above by employing the correct theory. A like difference between the results of the two suppositions takes place, when we consider the action exerted by the earth on a magnetic particle, placed in the interior of a hollow spherical shell, provided its thickness is very small compared with its radius, as will be evident by making $g=1$ in the formulæ belonging to this case, which are given in a preceding part of the present article.

(17.) Since COULOMB's experiments on cylindric wires magnetized to saturation are numerous and very accurate, it was thought this little work could not be better terminated, than by directly deducing from theory such consequences as would admit of an immediate comparison with them, and in order to effect this, we will, in the first place, suppose a cylindric wire whose radius is a and length 2λ, is exposed to the action of a constant force, equal to f, and directed parallel to the axis of the wire, and then endeavour to determine the magnetic state which will thus be induced in it. For this, let r be a perpendicular falling from a point p within the wire upon its axis, and x, the distance of the foot of this perpendicular from the middle of the axis; then f being directed along x positive, we shall have for the value of the potential function due to the exterior forces

$$V=-fx,$$

and the equations (b), (c) (art. 15) become, by omitting the superfluous constant,

$$\psi'=-\int\frac{d\sigma}{(r)}\left(\overline{\frac{d\phi}{dw}}\right) \quad . \quad . \quad . \quad . \quad . \quad (b)$$

$$0=(1-g)\phi+\frac{3g}{4\pi}\psi-\frac{3gf}{4\pi}x: \quad . \quad . \quad . \quad . \quad (c)$$

(r), the distance $p', d\sigma$ being inclosed in a parenthesis to prevent ambiguity, and p' being the point to which ψ' belongs. By the same article we have $0=\delta\phi$ and $0=\delta\psi$, and as ϕ and ψ evidently depend on x and r only, these equations being written at length are

$$0=r^2\frac{d^2\phi}{dx^2}+\frac{rd}{dr}\left(\frac{rd\phi}{dr}\right)$$

$$0=r^2\frac{d^2\psi}{dx^2}+\frac{rd}{dr}\left(\frac{rd\psi}{dr}\right).$$

Since r is always very small compared with the length of the wire, we may expand ϕ in an ascending series of the powers of r, and thus

$$\phi=X+X_1r+X_2r^2+ etc.;$$

X, X_1, X_2, *etc.* being functions of x only. By substituting this value in the equation just given, and comparing the co-efficients of like powers of r, we obtain

$$\varphi = X - \frac{d^2X}{dx^2}\frac{r^2}{2^2} + \frac{d^4X}{dx^4}\frac{r^4}{2^2.4^2} + etc.$$

In precisely the same way the value of ψ is found to be

$$\psi = Y - \frac{d^2Y}{dx^2}\frac{r^2}{2^2} + \frac{d^4Y}{dx^4}\frac{r^4}{2^2.4^2} - etc:$$

It now only remains to find the values of X and Y in functions of x. By supposing p' placed on the axis of the wire, the equation *(c)* becomes

$$Y = -\int \frac{d\sigma}{(r)}\left(\frac{\overline{d\varphi}}{dw}\right);$$

the integral being extended over the whole surface of the wire: Y' belonging to the point p', whose co-ordinates will be marked with an accent.

The part of Y' due to the circular plane at the end of the cylinder, where $x = -\lambda$, is

$$-\frac{dX''}{dx}\int_0^a \frac{2\pi r dr}{(r)} = -2\pi\frac{dX''}{dx}\left\{\sqrt{(\lambda + x')^2 + a^2} - \lambda - x'\right\},$$

since here $d\sigma = 2\pi r dr$ and $\frac{\overline{d\varphi}}{dw} = \frac{dX''}{dx}$, by neglecting quantities of the order a^2 on account of their smallness; X'' representing the value of X when $x = -\lambda$.

At the other end where $x = +\lambda$ we have $d\sigma = 2\pi r dr$, $\frac{\overline{d\varphi}}{dw} = -\frac{dX'''}{dx}$, and consequently the part due to it is

$$\frac{dX'''}{dx}\int_0^a \frac{2\pi r dr}{(r)} = 2\pi\frac{dX'''}{dx}\left\{\sqrt{(\lambda - x')^2 + a^2} - \lambda + x'\right\};$$

X''' designating the value of X when $x = +\lambda$.

At the curve surface of the cylinder

$$d\sigma = 2\pi a dx \text{ and } \frac{\overline{d\varphi}}{dw} = -\frac{\overline{d\varphi}}{dr} = \frac{a}{2}\frac{d^2X}{dx^2}$$

provided we omit quantities of the order a^2 compared with those retained. Hence the remaining part due to this surface is

$$-\pi a^2 \int \frac{dx}{(r)}\frac{d^2X}{dx^2};$$

the integral being taken from $x = -\lambda$ to $x = +\lambda$. The total value of Y' is therefore

$$Y' = 2\pi\frac{dX'''}{dx}\left\{\sqrt{(\lambda - x')^2 + a^2} + \lambda - x'\right\} - 2\pi\frac{dX''}{dx}\left\{\sqrt{(\lambda + x')^2 + a^2} - \lambda - x'\right\} - \pi a^2 \int \frac{dx}{(r)}\frac{d^2X}{dx^2};$$

the limits of the integral being the same as before. If now we substitute for *(r)* its value $\sqrt{(x - x')^2 + a^2}$ we shall have

$$-\pi a^2\int\frac{dx}{(r)}\,\frac{d^2X}{dx^2}=-\pi a^2\int\frac{dx}{\sqrt{(x-x')^2+a^2}}\,\frac{d^2X}{dx^2};$$

both integrals extending from $x=-\lambda$ to $x=+\lambda$.

On account of the smallness of a, the elements of the last integral where x is nearly equal to x' are very great compared with the others, and therefore the approximate value of the expression just given, will be

$$-\pi a^2 A\frac{d^2X'}{dx'^2}\text{ where }A=\int\frac{dx}{\sqrt{(x-x')^2+a^2}}=2\log\frac{2\mu}{a}\text{ very nearly};$$

the two limits of the integral being $-\mu$ and $+\mu$ and μ so chosen that when p' is situate any where on the wire's axis, except in the immediate vicinity of either end, the approximate shall differ very little from the true value, which may in every case be done without difficulty. Having thus, by substitution, a value of Y' free from the sign of integration, the value of Y is given by merely changing x' into x and X' into X; in this way

$$Y=2\pi\frac{dX'''}{dx}\left\{\sqrt{(\lambda-x)^2+a^2}-\lambda+x\right\}-2\pi\frac{dX''}{dx}\left\{\sqrt{(\lambda+x)^2+a^2}-\lambda-x\right\}-\pi a^2 A\frac{d^2X}{dx^2}.$$

The equation *(c)*, by making $r=0$, becomes

$$0=(1-g)X+\frac{3g}{4\pi}Y-\frac{3gf}{4\pi}x,$$

or by substituting for Y

$$0=(1-g)X-\frac{3ga^2A}{4}\,\frac{d^2X}{dx^2}-\frac{3gf}{4\pi}x$$

$$+\frac{3g}{2}\,\frac{dX'''}{dx}\left\{\sqrt{(\lambda-x)^2+a^2}-\lambda+x\right\}-\frac{3g}{2}\,\frac{dX''}{dx}\left\{\sqrt{(\lambda+x)^2+a^2}-\lambda-x\right\};$$

an equation which ought to hold good, for every value of x, from $x=-\lambda$ to $x=+\lambda$.

In those cases to which our theory will be applied, $1-g$ is a small quantity of the same order as a^2A, and thus the three terms of the first line of our equation will be of the order a^2AX; making now $x=+\lambda$, $\frac{3g}{2}\,\frac{dX'''}{dx}\,a$ is shown to be of the order a^2AX''', and therefore $\frac{dX'''}{dx}\div X'''$ is a small quantity of the order aA; but for any other value of x the function multiplying $\frac{dX'''}{dx}$ becomes of the order a^2, and therefore we may without sensible error neglect the term containing it, and likewise suppose

$$\frac{dX'''}{dx}\div X'''=0.$$

In the same way by making $x=-\lambda$, it may be shown that the term containing $\frac{dX''}{dx}$ is negligible, and

$$\frac{dX''}{dx}\div X''=0.$$

Thus our equation reduces itself to

$$0=(1-g)X-\frac{3ga^2A}{4}\frac{d^2X}{dx^2}-\frac{3gf}{4\pi}x,$$

of which the general integral is

$$X=\frac{3gfx}{4\pi(1-g)}+Be^{-\beta x}+Ce^{+\beta x};$$

where $\beta^2=\frac{4(1-g)}{3ga^2A}$: B and C being two arbitrary constants. Determining these by the conditions $0=\frac{dX'''}{dx}\div X'''$ and $0=\frac{dX''}{dx}\div X''$, we ultimately obtain

$$X=\frac{3gf}{4\pi(1-g)}\left\{x-\frac{e^{\beta x}-e^{-\beta x}}{\beta\left(e^{\beta\lambda}+e^{-\beta\lambda}\right)}\right\}.$$

But the density of the fluid at the surface of the wire, which would produce the same effect as the magnetized wire itself, is

$$-\frac{d\overline{\phi}}{dw}=\frac{d\overline{\phi}}{dr}=-\frac{a}{2}\frac{d^2X}{dx^2} \text{ very nearly,}$$

and therefore the total quantity in an infinitely thin section whose breadth is dx, will be

$$-\pi a^2\frac{d^2X}{dx^2}dx=\frac{3gf\beta a^2}{4(1-g)}\cdot\frac{e^{\beta x}-e^{-\beta x}}{e^{\beta\lambda}+e^{-\beta\lambda}}dx.$$

As the constant quantity f may represent the coercive force of steel or other similar matter, provided we are allowed to supposse this force the same for every particle of the mass, it is clear that when a wire is magnetized to saturation, the effort it makes to return to a natural state must, in every part, be just equal to f, and therefore, on account of its elongated form, the degree of magnetism retained by it will be equal to that which would be induced in a conducting wire of the same form by the force f, directed along lines parallel to its axis. Hence the preceding formulæ are applicable to magnetized steel wires. But it has been shown by M. Biot (Traite de Phy. Tom. 3, Chap. 6,) from Coulomb's experiments, that the apparent quantity of free fluid in any infinitely thin section is represented by

$$A'\left(\mu'^{-x}-\mu'^{+x}\right)dx.$$

This expression agrees precisely with the one before deduced from theory, and gives, for the determination of the constants A' and μ', the equations

$$\beta=-\log\mu'; \; A'=\frac{3gf\beta a^2}{4(1-g)\left(e^{\beta\lambda}+e^{-\beta\lambda}\right)}.$$

The chapter in which these experiments are related, contains also a number of results, relative to the forces with which magnetized wires tend to turn towards the meridian, when retained at a given angle from it, and it is easy to prove that this force for a fine wire, whose variable section is s, will be proportional to the quantity

$$\int s dx \frac{d\phi}{dx},$$

where the wire is magnetized in any way either to saturation or otherwise, the integral extending over its whole length. But in a cylindric wire magnetized to saturation, we have, by neglecting quantities of the order a^2,

$$\frac{d\phi}{dx} = \frac{dX}{dx} = \frac{3gf}{4\pi(1-g)} \left\{ 1 - \frac{e^{\beta x} + e^{-\beta x}}{e^{\beta\lambda} + e^{-\beta\lambda}} \right\} \text{ and } s = \pi a^2,$$

and therefore for this wire the force in question is proportional to

$$\frac{3gfa^2}{4(1-g)} \left\{ 2\lambda - \frac{2\left(e^{\beta\lambda} - e^{-\beta\lambda}\right)}{\beta\left(e^{\beta\lambda} + e^{-\beta\lambda}\right)} \right\}.$$

The value of g, dependant on the nature of the substance of which the needles are formed, being supposed given as it ought to be, we have only to determine β in order to compare this result with observation. But β depends upon $A = 2 \log \frac{\mu}{a}$, and on account of the smallness of a, A undergoes but little alteration for very considerable variations in μ, so that we shall be able in every case to judge with sufficient accuracy what value of μ ought to be employed: nevertheless, as it is always desirable to avoid every thing at all vague, it will be better to determine A by the condition, that the sum of the squares of the errors committed by employing, as we have done, $A \frac{d^2X'}{dx'^2}$ for the approximate value of $\int_{-\lambda}^{+\lambda} \frac{dx'}{\sqrt{(x-x')^2+a^2}}$, shall be a minimum for the whole length of the wire. In this way I find when λ is so great that quantities of the order $\frac{1}{\beta\lambda}$ may be neglected

$$A = ,231863 - 2 \log a\beta + 2a\beta;$$

where ,231863 *etc.* $= 2 \log 2 - 2(A)$; (A) being the quantity represented by A in Lacroix' Traite du Cal. Diff. Tome 3, p. 521. Substituting the value of A just found in the equation $\beta^2 = \frac{4(1-g)}{3ga^2A}$ before given, we obtain

$$\frac{4(1-g)}{3g \cdot a^2\beta^2} = ,231863 - 2 \log a\beta + 2a\beta \quad . \quad . \quad . \quad . \quad (a)$$

We hence see that when the nature of the substance of which the wires are formed remains unchanged, the quantity $a\beta$ is constant, and therefore β varies in the inverse ratio of a. This agrees with what M. BIOT has found by experiment in the chapter before cited, as will be evident by recollecting that $\beta = -\log \mu'$.

From an experiment made with extreme care by COULOMB, on a magnetized wire whose radius was $\frac{1}{12}$ inch, M. BIOT has found the value of μ' to be ,517948 (Traite de Phy. Tome 3, p. 78). Hence we have in this case

$$a\beta = \frac{-1}{12} \log \mu' = ,0548235,$$

which, according to a remark just made, ought to serve for all steel wires. Substituting this value in the equation *(a)* of the present article, we obtain

$$g = ,986636.$$

With this value of g we may calculate the forces with which different lengths of a steel wire whose radius is $\frac{1}{12}$ inch, tend to turn towards the meridian, in order to compare the results with the table of COULOMB's observations, given by M. BIOT (Traite de Phy. Tome 3, p. 84). Now we have before proved that this force for any wire may be represented by

$$K\left(\beta\lambda - \frac{e^{\beta\lambda} - e^{-\beta\lambda}}{e^{\beta\lambda} + e^{-\beta\lambda}}\right) = K\left(\beta\lambda - \frac{1 - e^{-2\beta\lambda}}{1 + e^{-2\beta\lambda}}\right);$$

where, for abridgment, we have supposed

$$K = \frac{3gfa^2}{2\beta(1-g)}.$$

It has also been shown that for any steel wire

$$a\beta = ,0548235:$$

the French inch being the unit of space, and as in the present case $a = \frac{1}{12}$ there results $\beta = ,657882$. It only remains therefore to determine K from one observation, the first for example, from which we obtain $K = 58°,5$ very nearly; the forces being measured by their equivalent torsions. With this value of K we have calculated the last column of the following table:—

Length 2λ.	*Observed Torsion.*	*Calculated Torsion.*
18 in.	288°	287°9
12	172	172,1
9	115	115,3
6	59	59,3
4,5	34	33,9
3	13	13,5

The three last observations have been purposely omitted, because the approximate equation *(a)* does not hold good for very short wires.

The very small difference existing between the observed and calculated results will appear the more remarkable, if we reflect that the value of β was determined from an experiment of quite a different kind to any of the present series, and that only one of these has been employed for the determination of the constant quantity K, which depends on f, the measure of the coercive force.

The table page 87 of the volume just cited, contains another set of observed torsions, for different lengths of a much finer wire whose radius $a=\frac{1}{12}\sqrt{\frac{38}{865}}$: hence we find the corresponding value of $\beta=3{,}13880$, and the first observation in the table gives $K={}^{\circ}{,}6448$. With these values the last column of the following table has been calculated as before.

Length 2λ.	*Observed Torsion.*	*Calculated Torsion.*
12 in.	11°50	11°50
9	8,50	8,46
6	5,30	5,43
3	2,30	2,39
2	1,30	1,38
1	,35	,42
,5	,07	,084
,25	,02	,012

Here also the differences between the observed and calculated values are extremely small, and as the wire is a very fine one, our formula is applicable to much shorter pieces than in the former case. In general, when the length of the wire exceeds 10 or 15 times its diameter, we may employ it without hesitation.

THE END.

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