THE DIFFERENTIAL ANALYZER. A NEW MACHINE FOR SOLVING DIFFERENTIAL EQUATIONS.

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1. This paper will describe a new machine for the solution of ordinary differential equations recently placed in service at the Massachusetts Institute of Technology. First, there will be outlined some of the objectives in view, and the general philosophy underlying the development of this type of analyzing device. Various serious mechanical problems have of necessity been met in the effort to produce a machine which is versatile and accurate, and the means by which these have been attacked will be treated. The general method of placing an equation on the machine will be presented and illustrated. Finally, a brief discussion will be given of a method of successive approximations for obtaining solutions of high accuracy, a method for avoiding difficulties due to singularities in coefficients, and a treatment of the problem of obtaining critical values of a parameter in an equation.

2. The handling of the processes of arithmetic by mechanical computation has recently revolutionized business accounting methods. The use in mechanical analysis of more advanced mathematical processes will ultimately be of comparable importance in scientific research and its applications in engineering. This idea is far from original, for Leibnitz envisaged it comprehensively over two hundred years ago. The far-reaching project of utilizing complex mechanical interrelationships as substitutes for intricate processes of reasoning owes its inception to an inventor of the calculus itself.[1] Leibnitz lived in an age when any comprehensive

[1] See "La Logique de Leibnitz," L. Couturat, Chap. IV, p. 115. Leibnitz invented the calculus as one step in a general program of symbolic reasoning. As steps in the program of the mechanization of reasoning he invented a machine

realization of his plan was prevented by lack of the devices and technique by which we are now surrounded. Had he been able to utilize thermionic tubes, photoelectric cells, accurate machine tools, and new alloys, he would certainly have built him many a strange device; for he was a genius and an inventor in the highest sense of the term, and he was fired with the idea of relegating to the machine those parts of the processes of thought which are inherently mechanical and repetitive.

The status of physics and engineering at the present time is peculiarly favorable to a development of this sort. Electrical engineering, for example, having dealt with substantially linear networks throughout the greater part of its history, is now rapidly introducing into these networks elements the non-linearity of which is their salient feature, and is baffled by the mathematics thus presented and requiring solution. Mathematical physicists are continually being hampered by the complexity rather than the profundity of the equations they employ; and here also even a numerical solution or two would often be a relief.

Not any one machine, nor even any one program of development can meet these needs. It was a long hard road from the adding machine of Pascal to the perforated card accounting machines of the present day. There must be much of labor and many struggles before the full ideal of Leibnitz can be consummated.

In the Department of Electrical Engineering of the Massachusetts Institute of Technology the development has followed three lines. First is the process of solving complicated simultaneous algebraic equations as they occur for example in the treatment of modern power networks, by means of alternating-current measurements made on an electrical replica of the power system. The flexible congeries of coils, resistances, and condensers by which this is accomplished is called a network analyzer.[2] Second is the attack

for performing arithmetical processes (see the "Source Book in Mathematics," McGraw-Hill) and an algebraic machine for solving equations. (Various references in Couturat.)

[2] Hazen, H. L., Schurig, O. R., and Gardner, M. F., "The M. I. T. Network Analyzer, Design and Application to Power System Problems," *Trans. A. I. E. E.*, October, 1930. M. I. T. E. E. Research Bulletin No. 66.

on the integral with a variable parameter by an optical method [3] first suggested by Wiener. This gives an approach to the integral equation and to certain processes of statistical analysis. The device itself is called a photoelectric integraph. The third deals with the ordinary differential equation, and provides solutions in the form of plotted curves for specified boundary conditions. The machine treated in this present paper is the latest step [4] along this third line. It is called a differential analyzer.[5]

3. The machine for solving second-order ordinary differential equations which was described in this journal in 1928 has, in spite of its limitations, been successfully used in a large number of problems, some of which have been described in the literature.[6] It has proved very definitely that such a machine, if flexible, rugged, and precise, could be of much

[3] Gould, K. E., "A New Machine for Integrating a Functional Product," *Jour. Math. & Physics*, **17**, 305–316, 1929. Gray, T. S., "A Photoelectric Integraph," JOUR. FRANK. INST., July, 1931.

[4] Bush, V., Gage, F. D., and Stewart, H. R., "A Continuous Integraph," JOUR. FRANK. INST., 63–84, 1927. M. I. T. E. E. Research Bulletin No. 56. Bush, V., and Hazen, H. L., "Integraph Solution of Differential Equations," JOUR. FRANK. INST., 575–615, 1927. M. I. T. E. E. Research Bulletin No. 59.

[5] This name was suggested by Prof. W. V. Lyon.

[6] Bush, V., and Gould, K. E., "Temperature Distribution along an Emitting Filament," *Physical Review*, 337–345, February, 1927. Bush, V., "Mechanical Solution of Engineering Problems," *The Tech. Engineering News*, **IX**, No. 2, 52, March, 1928. Bush, V., "Transmission Line Transients," M. I. T. E. E. Research Bulletin No. 50. Teplow, L., "Stability of Synchronous Motors under Variable Load," *General Electric Review*, 356, June, 1928. Summers, I. H., and McClure, J. B., "Progress in the Study of System Stability," Presented at the Pacific Coast Convention of the A. I. E. E., Santa Monica, Calif., September 3–6, 1929. Gardner, M. F., "The M. I. T. Integraph, Its Operation and Use," Report of the Colloquium on Power-Circuit Analysis, 91–100, June, 1929. Spencer, H. H., "The Solution of a Typical Stability Problem by Means of the Integraph," Report of the Colloquium on Power-Circuit Analysis, 102–112, June, 1929. Lyon, W. V., and Edgerton, H. E., "Transient Torque-Angle Characteristics of Synchronous Machines," Presented at the Mid-Winter Convention of the A. I. E. E., New York, January 31, 1930. Edgerton, H. E., and Zak, F. J., "The Pulling into Step of a Synchronous Induction Motor," *Jour. I. E. E.* (London), **68**, No. 405, 1205, September, 1930. Sears, F. W., "A Study of Electron Orbits in the Barkhausen-Kurz Effect," JOUR. FRANK. INST., **209**, No. 4, 459, April, 1930. Edgerton, H. E., and Fourmarier, D., "The Pulling into Step of a Salient-Pole Synchronous Motor," Presented at the Mid-Winter Convention of the A. I. E. E., January, 1931.

service in the problems of engineering and physics. It was because of a conviction that such devices are destined to fill an increasing need that the present development was undertaken.

There has been an enormous change in technique since the time when Sir William Thomson first suggested, more than fifty years ago,[7] that the integrators developed by his brother could be connected together and thus forced to produce solutions of differential equations. The idea could then hardly be carried out, for one reason because an integrator, which is simply a variable-speed drive, could not then be built both accurate and capable of carrying sufficient load to move numerous mechanical parts.

The present device incorporates the same basic idea of interconnection of integrating units as did the machine previously described. In detail, however, there is little resemblance to the earlier model. The addition of torque amplifiers has rendered the integrating units capable of carrying a considerable load. A very flexible system of "bus" shafts has been provided by means of which these units can be interconnected or "back coupled" at will. Large size has been preserved in order to obtain accuracy of plotting of the variable coefficients and of the result. Various auxiliary units, such as multipliers, have been provided. The aim has been to produce extreme flexibility, sufficient ruggedness, and reasonable precision. A precision of one part in one thousand under ordinary conditions of use has been arrived at in individual units, with the intention of achieving a somewhat less overall precision except in extraordinary circumstances. The arrangement is still based on the use of integrators because of their inherent average precision, due to the important balancing of momentary errors which occurs in their use. Such effects as are due, for example, to failure of an operator to exactly follow a plotted coefficient are integrated out to an extent which is really surprising, unless one has experienced the balancing of deviations when using a planimeter or steering a ship. A

[7] Thomson, Sir William, "Mechanical Integration of the Linear Differential Equations of the Second Order with Variable Coefficients," *Proceedings of the Royal Society*, **24**, 269, 1876.

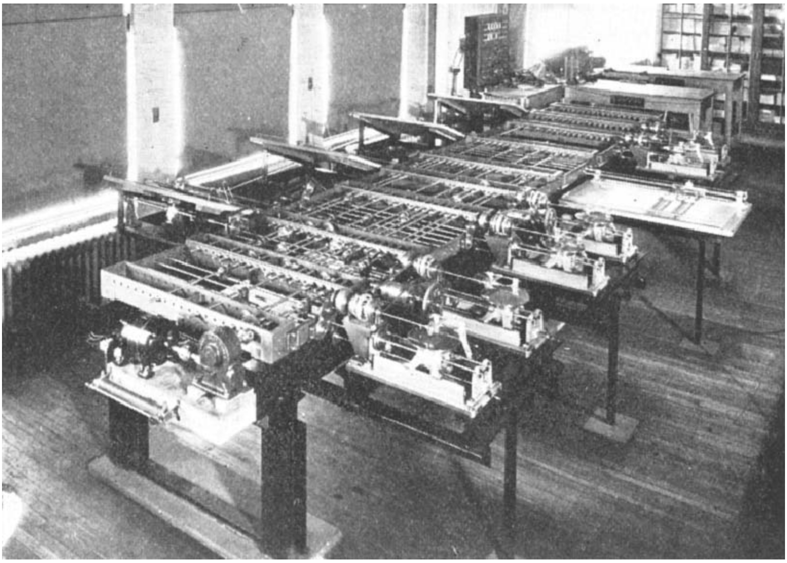
precise differentiator lending itself to automatic operation has never been produced. Electrical integration by watthour meter, very useful in the previous device on account of the possibility of simultaneously multiplying functions, had to be abandoned when precision was extended to the next decimal place in the present machine. All main operations are now mechanical, with electrical devices used only for drives and controls.

The machine as constructed is intended for the solution of ordinary differential equations of any order up to the sixth, and with any amount of complexity within reason. The frequency with which problems are encountered involving two or three simultaneous second-order equations made this range desirable, although it could readily be extended. It is readily possible, when plots have been made and a schematic diagram giving scales and connections prepared, to set up the machine for a given problem in a few hours. The time necessary for solutions varies with the complexity of the problem and the precision desired, and in representative cases is about ten minutes for each solution corresponding to a given set of boundary conditions. Experience is necessary, of course, in order to use the device effectively. This is actually one of the most attractive aspects of the machine; one acquires an entirely new appreciation of the innate nature of a differential equation as that experience is gained.

The attainment of precision in a device of this sort, granted sound mechanical design and accurate construction, is largely a struggle with the problems of backlash and integrator slip, and these will hence receive most attention below. First, however, it will be well to consider briefly the general nature of the units of the machine and the manner in which these various units may be interconnected to solve an equation, assuming that each unit performs exactly as intended.

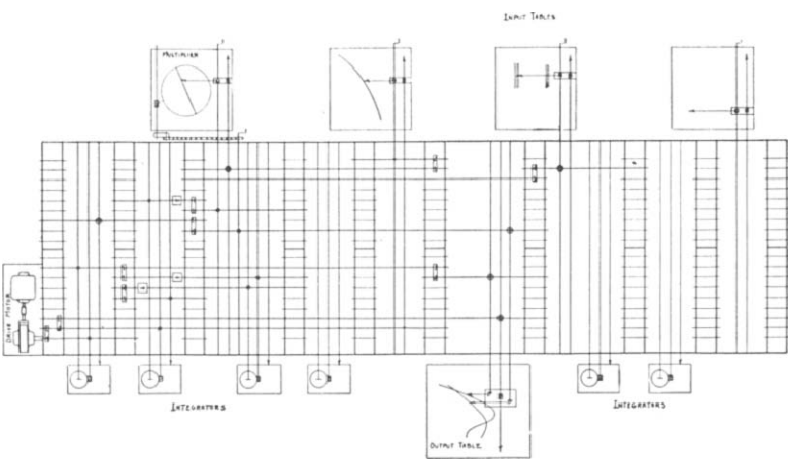
4. A photograph of the machine is shown in Fig. 1, and a schematic diagram of the layout in Fig. 2. It will be noted that there are provided eighteen longitudinal or bus shafts, and that these can be readily uncoupled at many points. Along the sides of the device are ranged the main units: the integrators, input tables, multipliers and output table,

FIG. 1.



The differential analyzer.

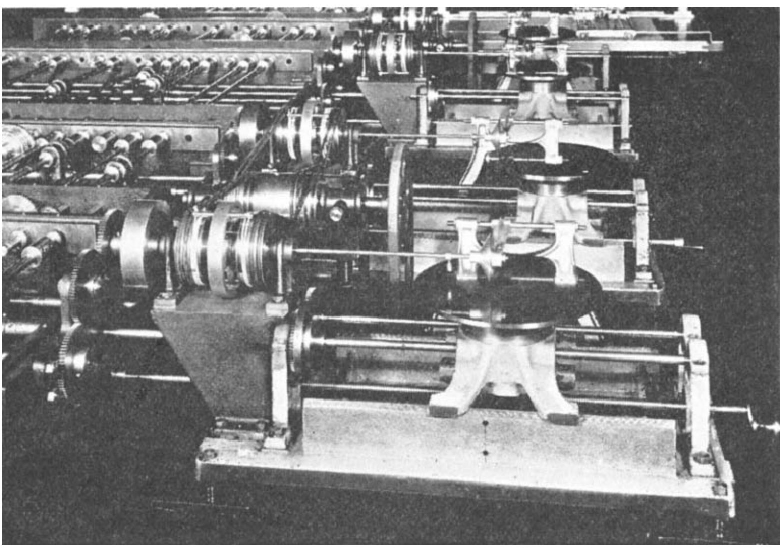
FIG. 2.



Schematic diagram of the analyzer.

each connected to cross shafts. An integrator, Fig. 3, may be considered as a unit with three shafts, the angular movements of which are u, v, and w, so connected that at all instants $u = K\int w dv$. An input table, Fig. 4, has two shafts with revolutions p and q, one of which moves a pointer horizontally in the direction of abscissas and the other vertically in the direction of ordinates across a plot of a function. One shaft is controlled manually to keep the pointer on the plot, thus giving at all times $p = f(q)$. A

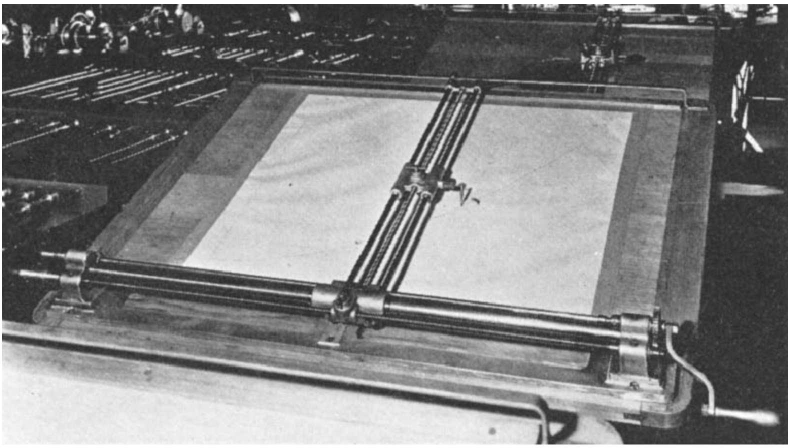
FIG. 3.



An integrator.

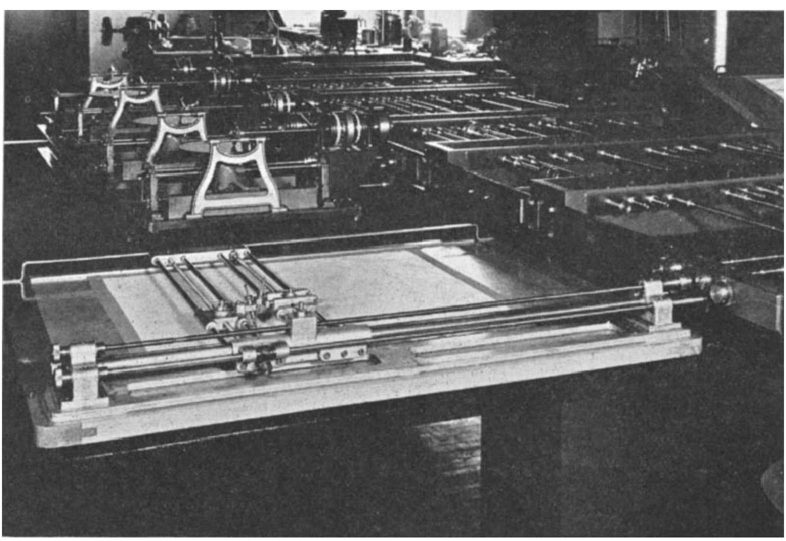
multiplier is obtained by using an attachment on an input table. There are then three shafts, and the revolutions or total angle turned through by one is, at any instant, equal to the product of the revolutions of the other two. An output table, Fig. 5, has three shafts, one of which moves a carriage horizontally, and the other two of which move recording pencils carried by this carriage vertically over the recording paper. Thus two quantities may be simultaneously recorded as a function of any chosen variable. Any cross shaft can be

FIG. 4.



An input table.

FIG. 5.



The output table.

readily connected to any bus shaft by inserting a spiral gear box between them, and one will then drive the other. Right- and left-hand boxes are supplied in order to secure correctly related directions of rotation. One bus shaft, assigned permanently to the independent variable, is driven by a variable-speed motor. There are provided also sets of spur gears which can be connected between adjacent shafts so that one will drive the other with a chosen ratio of speed, and differential gears or "adders" which can be connected to three shafts so that the revolutions of one will be the sum of the revolutions of the other two. These last units are inserted as needed in the body of the table.

It is interesting to contrast the integrators, multipliers, and adders. Each has three shafts. If the angles turned through by these shafts are θ_1, θ_2, θ_3, and their speeds are $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\theta}_3$, then at any instant for the integrator

$$\theta_1 = \int \theta_2 d\theta_3 \quad \text{or} \quad \dot{\theta}_1 = \theta_2 \dot{\theta}_3,$$

for the multiplier

$$\theta_1 = \theta_2 \theta_3 \quad \text{or} \quad \dot{\theta}_1 = \theta_2 \dot{\theta}_3 + \theta_3 \dot{\theta}_2,$$

for the adder

$$\theta_1 = \theta_2 + \theta_3 \quad \text{or} \quad \dot{\theta}_1 = \dot{\theta}_2 + \dot{\theta}_3.$$

It is convenient to have standard symbols for all these various units, and these are shown in Fig. 6.

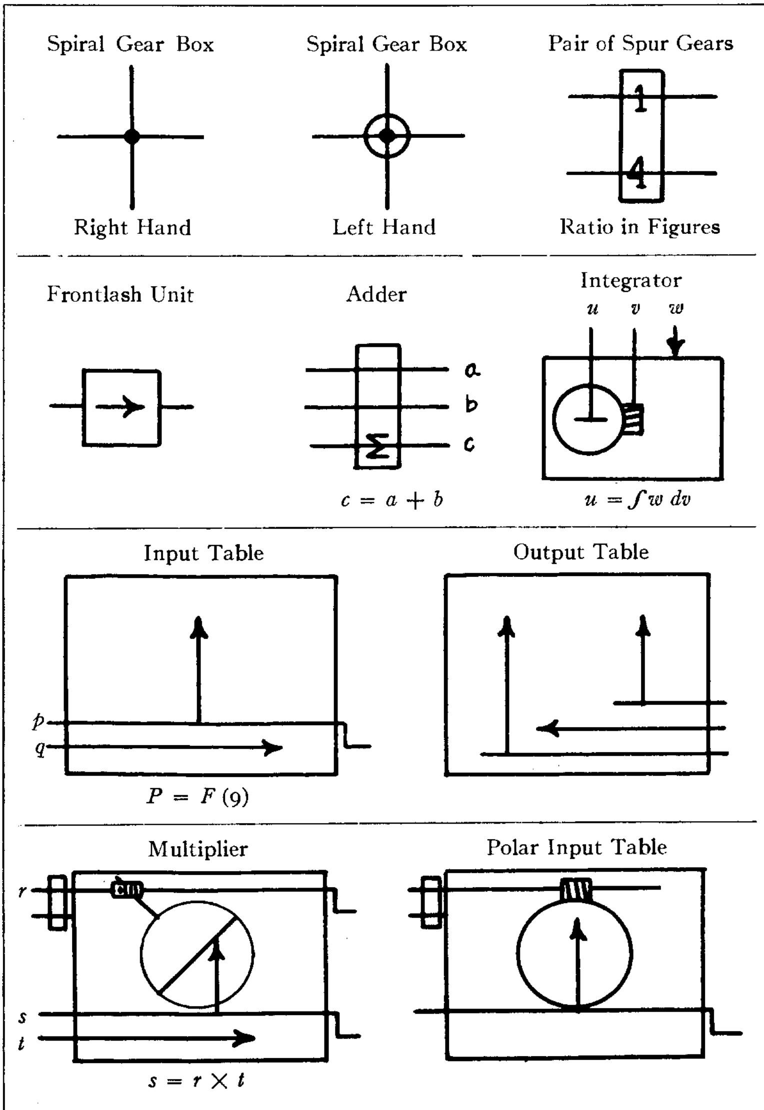
5. Let us now examine somewhat in detail the method of placing a simple equation on the machine. As an example consider the equation

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + g = 0$$

of a body projected vertically in a constant gravitational field, the resistance being assumed proportional to the first power of the speed. Disregarding scales and directions of rotation for the present we may place this on the machine as shown in Fig. 7. The connections will be more apparent if the equation is written in the form:

$$\frac{dx}{dt} = -\int \left(k\frac{dx}{dt} + g \right) dt.$$

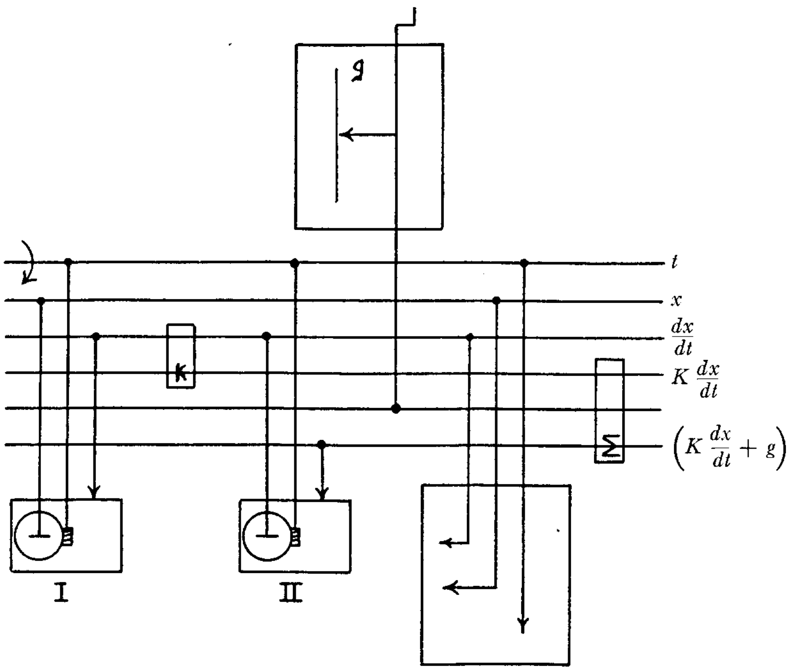
FIG. 6.



Standard symbols for units.

The symbols entered at the end of each bus shaft indicate the equivalent of unit angular rotation of the shaft. We have assumed that k is exactly a certain gear ratio. The procedure, if it is not, will be apparent below. The value of g is, for convenience, placed on an index on an input table where it may be readily adjusted to any desired value. In this problem, of constant coefficients, there are no manual opera-

FIG. 7.



Analyzer connections for simple problem of falling body.

$$\frac{d^2x}{dt^2}+k\frac{dx}{dt}+g=0.$$

tions. When the time shaft is started, by motor drive, all other shafts are driven. The values of x and dx/dt will be recorded on the output table as functions of the time. The initial values of x and of dx/dt, that is the boundary conditions for the problem, are assigned by specifying the initial displacements of the carriages of the two integrators. For example, suppose we are to start at $x = 0$, $t = 0$, with

$dx/dt = V$. We initially displace integrator I by an amount V, and integrator II by an amount $(kV + g)$ from their mid positions.

Now let us pass to the problem which arises when the gravitational force varies with the distance x, and where the resistance is a complicated but known function of the speed: $f(dx/dt)$. The equation is now

$$\frac{d^2x}{dt^2} + f\left(\frac{dx}{dt}\right) + g(x) = 0$$

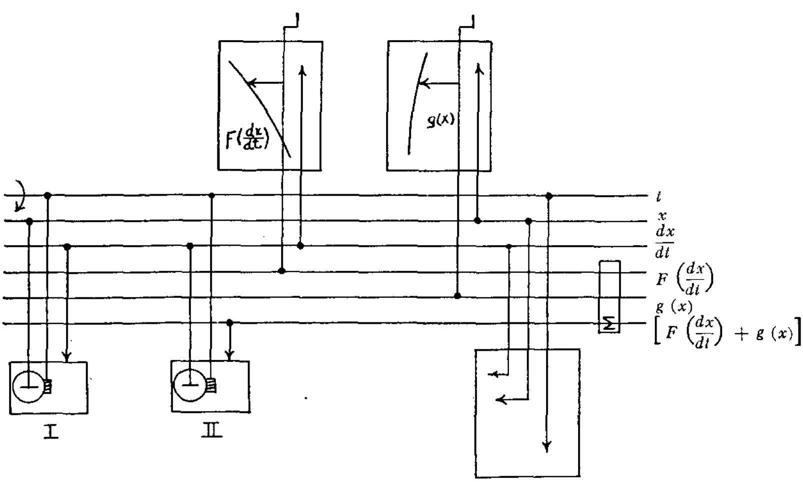
or

$$\frac{dx}{dt} = -\int\left[f\left(\frac{dx}{dt}\right) + g(x)\right]dt.$$

It should be emphasized that, from the standpoint of the machine, the problem is now but little more difficult than before; although formally it may be very troublesome.

The scheme of connections is shown in Fig. 8. Note that

FIG. 8.



Falling-body problem, somewhat generalized.

$$\frac{d^2x}{dt^2} + f\left(\frac{dx}{dt}\right) + g(x) = 0.$$

we have simply entered the variable coefficients by the use of input tables. An operator is stationed at each of these points, and, as the solution proceeds, he turns a crank keeping an index on a plotted curve. Otherwise nothing is altered.

The machine, as before, causes the distance and speed to be recorded on the output table as functions of the time.

In each of these problems it will be noted that the procedure of placing the equation on the machine is somewhat as follows: A bus shaft is assigned to each significant quantity appearing in the equation. The several relations existing between these are then set up by means of connections to the operating units: a functional relation by connecting the two corresponding shafts to an input table, a sum by placing an adder in position, an integral relationship by an integrator, and so on. When all the relationships which are involved have been thus represented a final connection is made which represents the equality expressed in the equation. In the example above this connection is through an integrating unit, for it is desired to represent the fact that the integral of a certain function is equal to the first derivative of the dependent variable. When this has been done the machine is locked, and the rotation of the independent-variable shaft will drive everything else, thus forcing the machine to move in accordance with the expressed relationship of the equation. The speed of this independent shaft, within mechanical limits, makes no difference. It is, in fact, driven for convenience by a variable-speed motor. This enables everything to be run slowly when an operator is called upon to follow a difficult part of a plot.

The scheme of connecting the machine for a specific problem which has been illustrated is quite general; more so in fact than might at first appear. It has certain features in common with the "plugging" of a desired circuit on a switchboard, and the resulting diagrams have something of an electrical atmosphere about them. Evidently the complexity of the equations which can be handled is limited only by the number of units available.

There will be discussed below more complex situations in connection with matters of discontinuities in coefficients, and successive approximations; and the ways in which precision may be attained in difficult situations will then be apparent. In order that this matter may be clear, however, the design of various individual units will first be treated, with particular attention to precision and reliability of operation.

6. There are six integrating units in the present machine, which form the central feature of the device. Their design constituted a major problem in the development. The task of making a satisfactory integrating unit is that of building a variable-speed drive capable of substantial mechanical power output, accurate in ratio at all speeds and loads, and with this ratio accurately adjustable. Meeting these requirements resulted in a compact rigid mechanical construction, with the backlash in the drives carefully removed, and with a high-ratio torque amplifier added to relieve the friction drive from the necessity of supplying more than a minute torque.

A complete integrating unit is shown in Fig. 3. A massive carriage is moved horizontally on ways by means of an accurate lead screw. This carries a disc in a horizontal plane which can be revolved, independently of the carriage position, by means of splined shafts. Resting on this disc, and pivoted in accurate bearings, is a wheel or roller with its axis parallel to the ways and lying in a vertical plane through the center of the disc. Disc and roller are of hardened steel, ground and lapped. The edge of the roller is given a radius of about 0.002 inch. The outer bearing of the roller shaft is carried by a hinged carriage from the base. This bearing is jewelled and is located directly over the point of contact. End play in roller-shaft and carriage pivots is removable by fine screw adjustment. There are also provided adjustments by which the roller shaft can be brought accurately to its correct position.

Backlash in the lead-screw drive is almost completely removed by using two nuts on the screw, with spring-backed wedges between them forcing the nuts apart, the wedge supports being rigidly fastened to the carriage. By proper choice of the wedge angle this scheme gives a positive drive in either direction, while allowing the slight irregularities of the screw to be taken up without binding. This device, called a "lashlock," was developed by Mr. C. W. Nieman of the Bethlehem Steel Corporation, and, as it has been described elsewhere,[8] will not be discussed in detail here.

The same idea is utilized in the disc drive. On the disc shaft is a spiral gear meshing with two spiral pinions located

[8] *American Machinist*, 66, 921–924, June 2, 1927.

in the carriage and driven by two symmetrically-disposed horizontal splined shafts. This is illustrated diagrammatically in Fig. 9. These shafts are coupled by a pair of spur gears, and at this point is introduced the lashlock. The lashlock-wedge tends to separate the parts against which it presses in such manner as to bring gear teeth into contact all the way around the double drive to the disc. The stud on which the wedge moves is driven by the main disc-drive shaft which projects from the base. Thus backlash in this drive is also eliminated. More care was taken in removing backlash from the integrating units than was perhaps entirely necessary, as will be seen below. It is very convenient,

FIG. 9.

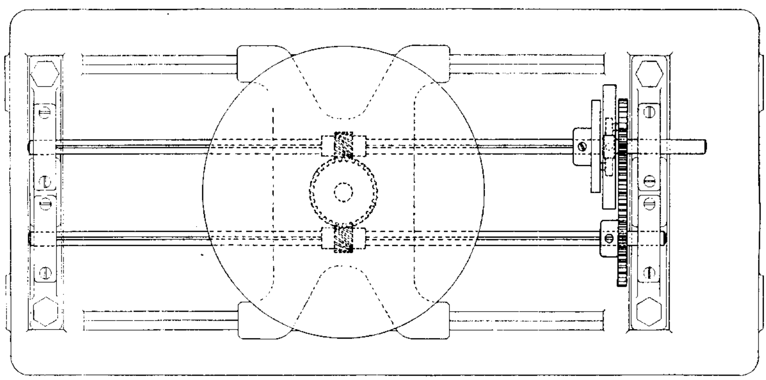


Diagram of integrator drive.

however, to have these integrators precise self-contained units free from substantial errors due to this cause, in order more readily to isolate other possible sources of error.

When the roller is entirely free in its bearings, and when all construction has been made accurately, free from play or backlash, and rigid, the integrator becomes a very precise unit. It takes only an extremely small torque opposing the rotation of the roller entirely to remove this precision. Offhand it might be thought that there would be a threshold value of torque which would cause the roller to slip, and below which precision would be obtained. Actually, however, there is an error caused by any load torque however small; and tests show that the departure from the correct ratio of disc to roller speeds varies continuously with the torque and is nearly proportional thereto in the range of small values. This effect is undoubtedly caused by the existence of a finite area of contact between disc and roller, at only one point of which is there at any instant zero relative velocity between the surfaces in contact. Such a point will shift its position in the area slightly for any small load increment, and hence cause a small change in drive ratio. It is interesting to note that the error thus caused was found to be substantially the same whether the surfaces were washed free from grease and dried, or thoroughly lubricated with a light oil.

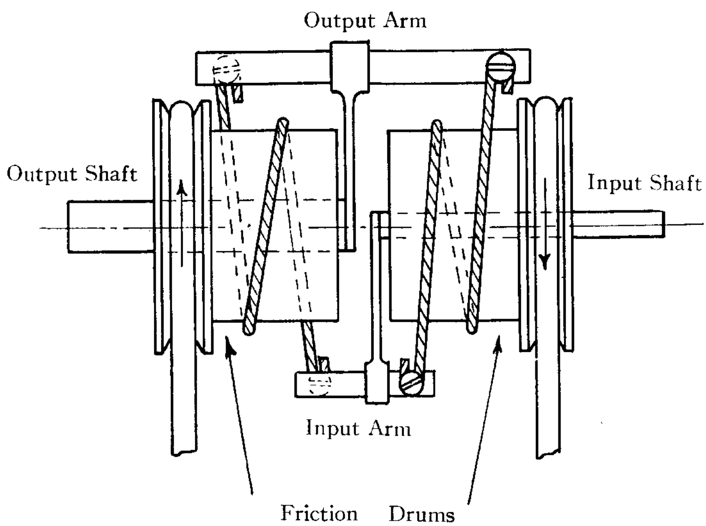
In the previous machine this fundamentally important difficulty was met by the use of servo-mechanisms, the integrating units being then called upon only to move very light contacts, while a motor controlled thereby did the actual work. Unfortunately this almost necessarily involves an oscillatory output drive which is very inconvenient and hardly compatible with extreme precision when complicated interconnection is involved. Thanks to the much-appreciated assistance of Mr. Nieman a much better solution was available for the present machine in the form of the torque amplifier which has been recently developed by the Bethlehem Steel Corporation.[9] This is, in brief, a device having an input and an output shaft, and so arranged that, when the input shaft is turned, the output shaft will turn an equal amount, but with a greatly increased torque, so that a small torque applied to the input shaft is magnified and applied to a load. The energy is supplied by an independent and constantly-running motor which drives a pair of drums in opposite directions. The central idea involved is the use of friction bands wrapped around the rotating drums. A small force applied at one end of such a band produces a much amplified force at the other end, the ratio of these forces varying exponentially with the angle of wrap. The two drums

[9] *American Machinist*, **66**, 895–7, May 26, 1927. *Automotive Ind.*, **56**, 896–7, June 11, 1927. *Chem. & Met. Eng.*, **34**, 443–4, July, 1927. *Elec. Ry. Jour.*, **70**, 18, July 2, 1927. *Mech. Eng.*, **49**, 1017–18, September, 1927. *Power*, **66**, 413, Sept. 13, 1927. *Railway Age*, **83**, 1310–11, Sec. 2, Dec. 24, 1927. *Power Plant Eng.*, **33**, 772–3, July 1, 1929.

provide for rotation in both directions. Fig. 10 shows the basic idea somewhat diagrammatically.

A study showed that about a pound-foot of output torque from integrators would probably be ample, and that, in order that errors due to load might be negligible within the limits of precision aimed at, the torque supplied by the roller could not well be more than a ten-thousandth part of this. It was therefore necessary to develop the torque amplifier along the

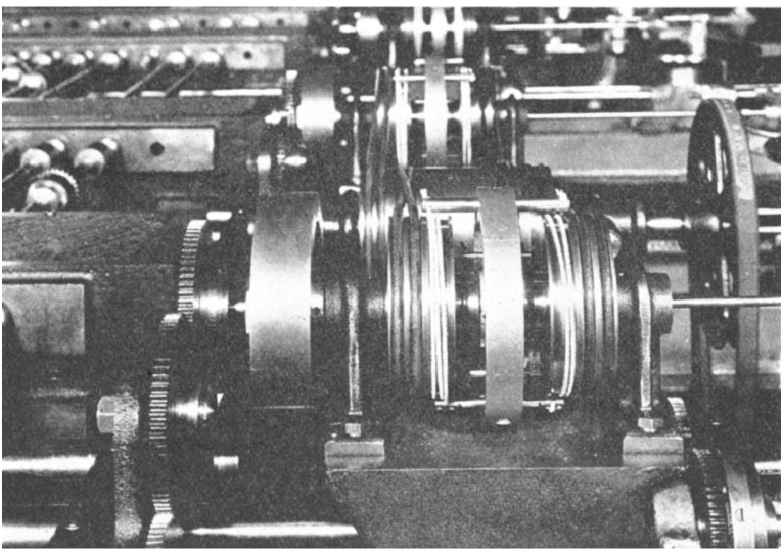
FIG. 10.



Simplified diagram of torque amplifier.

lines of very high torque-ratio and very low input-torque. A two-stage device was found necessary in order that the bands on the first stage might be exceedingly flexible and thus properly controlled by small forces, while the bands of the second stage might be sufficiently rugged to carry the load. The output of the first stage simply moves a pivoted lever which in turn supplies the input to the second stage. The final model is shown in Fig. 11. The rotating drums are belt-driven in opposite directions, and are stepped to furnish two small-diameter friction surfaces for the first-stage bands, and larger ones for the second-stage bands. There are two concentric shafts. The internal shaft is the roller shaft, held in a light bearing mounted in the outer shaft, and furnished with an arm which projects through a hole in the latter. This arm carries the input ends of the first-stage bands, which are actually pieces of silk-braided fishline, the output ends of these bands being connected to the pivoted lever mentioned above. The second-stage bands are heavy cord, with their input ends connected to this same lever, and their output ends attached to a ring carried rigidly on the output shaft. One band in each stage is wound clockwise and the

FIG. 11.



A torque amplifier.

other counterclockwise. Every band is given two and one-half turns wrap. There is a very slight initial tension in all the bands, and the effect of a movement of the input shaft is to loosen one pair and tighten the other. The relative angular motion between input and output shafts is only a very few degrees even when carrying full load. This angle has been made very small indeed in certain designs produced by Nieman, but in the design for use with the integrator, where the problem of extremely small input torque is paramount, it has been found advisable to allow it to be several degrees. The error which might be caused by this small play is avoided by an arrangement described below. The input shaft may, of course, revolve at any speed whatever, and exert complete control over the output, provided the fixed drum speed is not reached.

Now such a torque amplifier is quite analogous to a two-stage thermionic-tube amplifier, and it has many of the properties of the latter, including the possibility of self-oscillation. It was soon found, in fact, that when the amplification of such a low-input unit was raised to around 10,000 it was very prone to go into a condition of violent oscillation ending usually in disaster. This was presumably caused by a small part of the output being fed back in one way or another into the input. This problem caused quite a struggle. It was finally solved by the use of a vibration damper, which is analogous to the introduction of resistance by transformer coupling into the corresponding thermionic-tube circuit. In Fig. 11 it will be noted that there is a relatively massive flywheel on the output shaft. This is loose on the shaft and coupled thereto only through a slight friction introduced by pressure on a felt washer. During uniform rotation the flywheel simply rotates with the shaft. If there are oscillations it slips and introduces a damping force. As in the tube circuit, its presence does not simply limit the amplitude of oscillations; rather it prevents them from starting at all. This scheme, long used for limiting the torsional vibration of machinery, here performs a new function in preventing the initiation of self-oscillations. The performance of the torque amplifiers is now smooth and reliable, and their ratio is ample. It is still necessary to exercise some care in their use, for a spot of dirt on a first-stage friction drum, for example, can cause rough operation. With proper care they give no trouble, and the life of the friction bands is long.

A series of forty test runs was made on a completed integrating unit with various loads of from zero to one pound-foot output torque, at various displacements, and for both directions of disc rotation. The calibration constant was evaluated by averaging all these runs, and the departure for individual runs then computed. The mean departure was 0.032 per cent., and the maximum departure in any single run 0.12 per cent. This is as good precision as is necessary at this point for all ordinary purposes, although it could now undoubtedly be improved, as the result of experience, if improvement should become advisable.

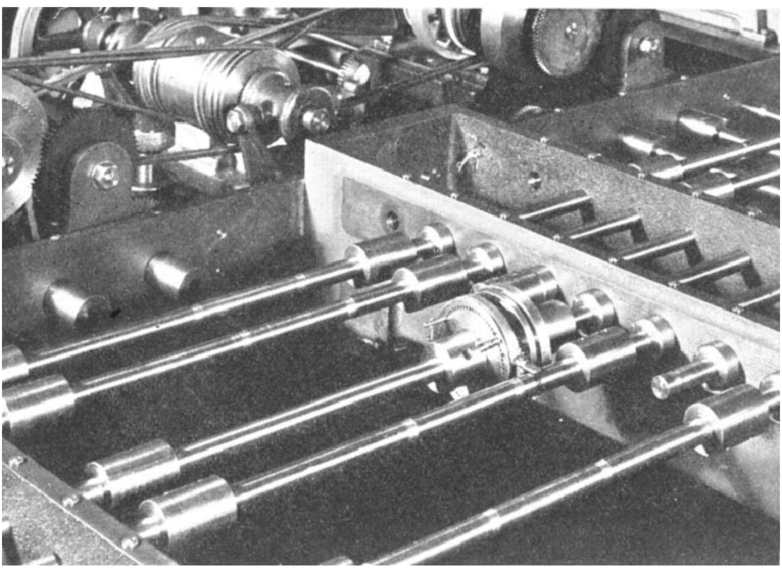
7. A second major problem in the development has been backlash. A serious matter in any precision device, it becomes especially important in this, because the interconnection of units often renders cumulative the error caused by its presence. The problem is acute when it is desired to produce complete flexibility, for the way in which shafts are geared together is then continually altered.

The general scheme of attack has been as follows: Within the drives of integrating units themselves backlash has been substantially eliminated by the use of lashlocks. Elsewhere it has been held down to a small amount by careful construction, although a certain amount of clearance is imperative for the free running of any extensive system of shafts and gearing. In any specific problem the effect of a given backlash angle in a particular drive can then be reduced by specifying scales so that the shafts of that drive will make a large total number of revolutions in the course of a problem. In many situations this is sufficient. Finally, for important drives where the effect of backlash might be especially serious, there has been developed a unit which can be conveniently inserted in any such drive and which reduces the influence of backlash at that point to a second-order effect. This is, in reality, a unit having a negative backlash which can be adjusted to compensate for the positive backlash present in the drive into which it is inserted. It has been aptly termed a "frontlash" unit.

The underlying idea is as follows: The unit, when connected into a line of shafting ordinarily furnishes simply a rigid driving connection from the in-going to the out-going shaft. When the direction of rotation changes, however, it immediately steps the outgoing shaft ahead in the new direction of rotation by an adjustable amount. This it does during part of the first revolution in the new direction. Hence for the balance of the revolutions up to the next reversal the net backlash in the total drive is brought to zero. This action is repeated at each reversal, so that there is always zero backlash in the shaft, except during the brief periods at each reversal of direction. The effect of backlash itself being small, the residual effect due to this transition period is negligible.

The unit is shown in position in Fig. 12, and disassembled in Fig. 13. A drum is loosely mounted on the in-going shaft,

FIG. 12.

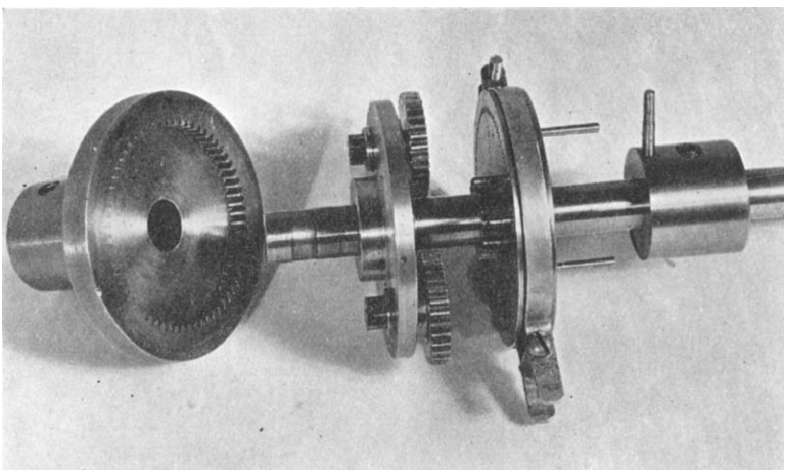


A frontlash unit.

but limited in its rotation with respect thereto by two stops which are readily adjustable. A light friction band, engaging for convenience with an adjacent shaft, rests on the drum and holds it stationary with respect to the base, except when it is caused to rotate in spite of this friction by coming up against one of the stops. A planetary gear train is carried by a mounting fixed on the in-going shaft. One end of this train engages with an internal gear fixed on the out-going shaft. The other end terminates in the drum.

When the drum is up against one of the stops, the whole mechanism rotates as a unit, slipping the friction band. The drive is then direct and none of the gears are rotating with respect to one another, all of course having a common rotation about the main axis. If the direction of rotation of the in-going shaft is now reversed, there is a period before the opposite stop is engaged during which the drum is held stationary by the friction band. During this period the out-going shaft is being driven through the planetary train in such manner that the out-going shaft revolves about ten per

FIG. 13.



Disassembled view of frontlash unit.

cent. faster than the in-going shaft. This continues until the opposite stop engages, when the drive is again one to one. Evidently the actual angular separation of the stops will be about ten times the angular backlash which it is desired to cancel, and hence can readily be set with precision. The friction torque on the drum need be but little more than ten per cent. of the transmitted torque. Hence the load imposed on the drive is not seriously increased by the presence of the frontlash unit.

The usual manner of adjusting a frontlash unit is as follows: It is inserted in a drive, or singly-connected system of shafts and gearing, in which it is important that backlash should be removed, and usually near the output end of the drive. The drive is marked at each end so that angles may be accurately noted. The input end is then turned through several revolutions in one direction, brought to an accurate mark and the output position noted. This is repeated in the opposite directino. The stops are then adjusted until there is no detectable difference in the position of the output end when a definite position of the input end is thus approached from the two directions of rotation. Gear ratios, if present in the drive, can readily be taken into account. Usually the input end of the drive will be at the roller of an integrating unit. The frontlash unit is then set to cancel the backlash of the torque amplifier as well as that of the balance of the drive. It is of course not always necessary to cancel backlash in this manner. Where it occurs in a drive connected to an input table, for example, it may if necessary be compensated for by adjustment of the plot. Six frontlash units are available, and this appears to be a sufficient number for the present.

A brief study was made of the effect of backlash and its cancellation. For this purpose the machine was set up to solve the simple equation:

$$\frac{d^2z}{dx^2} = -z$$

and the output table was connected to record dz/dx as a function of z, so that the plotted result should be a circle. Since the equation has constant coefficients there were no manual controls involved, and the test was hence unaffected by personal errors. The backlash in the important drive which interconnected the integrators was now measured; and, by what amounted to a perturbation method, its effect on the actual solution calculated. It was found that with this backlash present the machine could be expected to record, in place of a circle, a close spiral with an increment per cycle of about one per cent. in radius. On operating the machine this was indeed found to be the case. A frontlash unit was then inserted and set to cancel this amount of backlash. Again operating the machine it was found to draw accurate circles, so that if allowed to repeat the record for several cycles the line drawn by a fine pencil point was not appreciably widened. A final test was as follows: The recording pencil was sharpened and allowed to draw one record circle about ten inches in diameter. It was then raised from the paper, but the machine was left running. At the end of an hour the pencil was again dropped on the paper and allowed to draw a second circle. The two circles were accurately concentric, and differed in radius by about one per cent. Even this result could undoubtedly be bettered now, with the better technique in setting frontlash units which has since been developed.

8. There are four input tables, one of which is shown in Fig. 4. They are built to carry plots 18 by 24 inches in size, which allows for considerable precision in plotting. It is possible to increase the precision at this point, as is sometimes necessary, by plotting a curve in several sections, each occupying substantially the whole extent of the table. Successive sections are then brought into play by stopping the machine during the course of the problem and transferring the pointer from one section of the curve to another.

On each table there is a lead screw which is connected with one of the cross shafts and which moves a carriage in the direction of abscissas. This carriage extends perpendicularly across the table in the direction of ordinates, and on it moves a slider carrying an adjustable pointer. A second lead screw located on this carriage moves this slider, and is driven through a pair of spiral gears by another cross shaft of the device. This second drive is also controlled by a crank. Thus, if the first cross shaft is connected to the machine so as to revolve proportionately to a certain variable, and if the crank is turned so that the pointer always registers with a curve placed on the table, the second cross shaft may be connected to pass out into the machine a function represented by this curve and with the variable as argument.

The output table, Fig. 5, is very similar except that there are on the carriage two sliders, each driven by its own cross shaft, and each carrying a recording pencil. The cross shaft which traverses the carriage is ordinarily connected to revolve proportionately to the independent variable, but may be connected otherwise. Any two chosen quantities may be simultaneously recorded by properly connecting the cross shafts. The usual procedure is to record the dependent variable and its derivative. With simultaneous equations two dependent variables may conveniently be recorded. Sometimes, for example in problems with periodic coefficients, it is expedient to use only one slider and record the dependent variable against its derivative.

For convenience the two pencils are so placed that they record in the same vertical line. This allows both records to have the same abscissas. It requires, however, that provision be made to allow one pencil to pass the other without interference. This is accomplished by mounting one pencil on a spring-controlled plunger attached to the slider, and providing this plunger with a projection which will engage with a similar projection on the opposite slider. Thus, when the pencils come close together, the plunger is caused to recede, and one pencil describes a small arc about the other. The presence of this small arc in the record gives little difficulty.

The lead screws of these tables are interrupted near the ends so that overtravel can not cause a wreck. The connection between lead screws and carriages or sliders is made by a half nut, forced into place by a spring to reduce backlash, and supplied with an arrangement so that it can readily be thrown out of engagement to allow rapid adjustment to a desired position. Final close adjustment is made by moving the pointers in their mountings or by uncoupling and turning cross shafts.

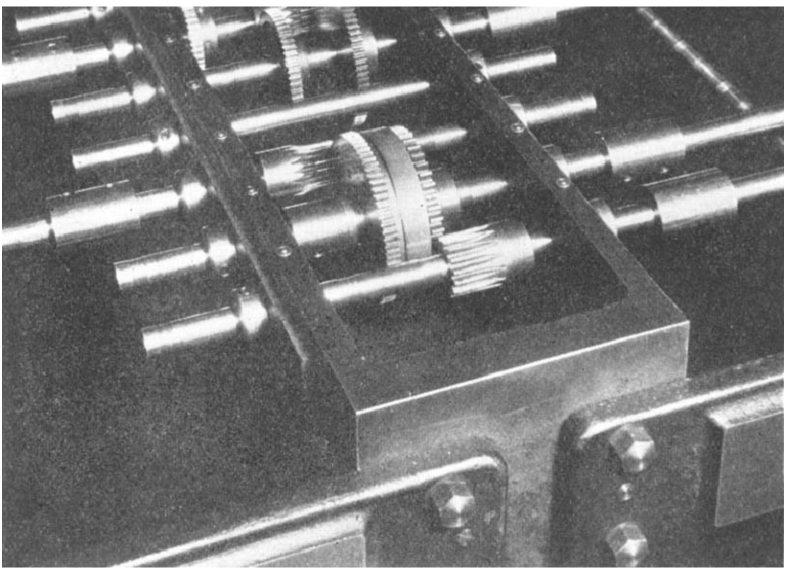
There are two forms of differential gears or adders, and little description of these is necessary. The one shown in Fig. 14 is arranged so that it can be placed in position engaging any three adjacent bus shafts. It is of the planetary type. Gear ratios are so chosen that the revolutions of one shaft are caused to be the sum of the revolutions of the other two without the introduction of multiplying factors.

The binary system has been adhered to in nearly all gear ratios, so that the spur gears which may be introduced to connect adjacent shafts are supplied in ratios 1 : 1, 1 : 2, and 1 : 4. By successive steps higher ratios can also be obtained. Steps as close together as this are necessary for convenience in scales. In order to be consistent with this scheme the factor of the integrating units is made to be exactly 32, so that we then have

$$32u = \int v dw,$$

where u, v, and w are the revolutions of the three cross shafts connected to the unit.

FIG. 14.



A differential gear or adder.

9. An important addition which greatly increases the convenience and flexibility of the machine is a combined polar input table and multiplier. The polar table consists essentially of a large circular plate which can be placed in position on the platen of an input table. This plate can be turned by means of an additional cross shaft by a worm and gear, so that its angular movement is proportional to the revolutions of the cross shaft. The connection as a multiplier will be treated below. There is a handle which is geared to this third cross shaft so that it may, when desired, be turned manually.

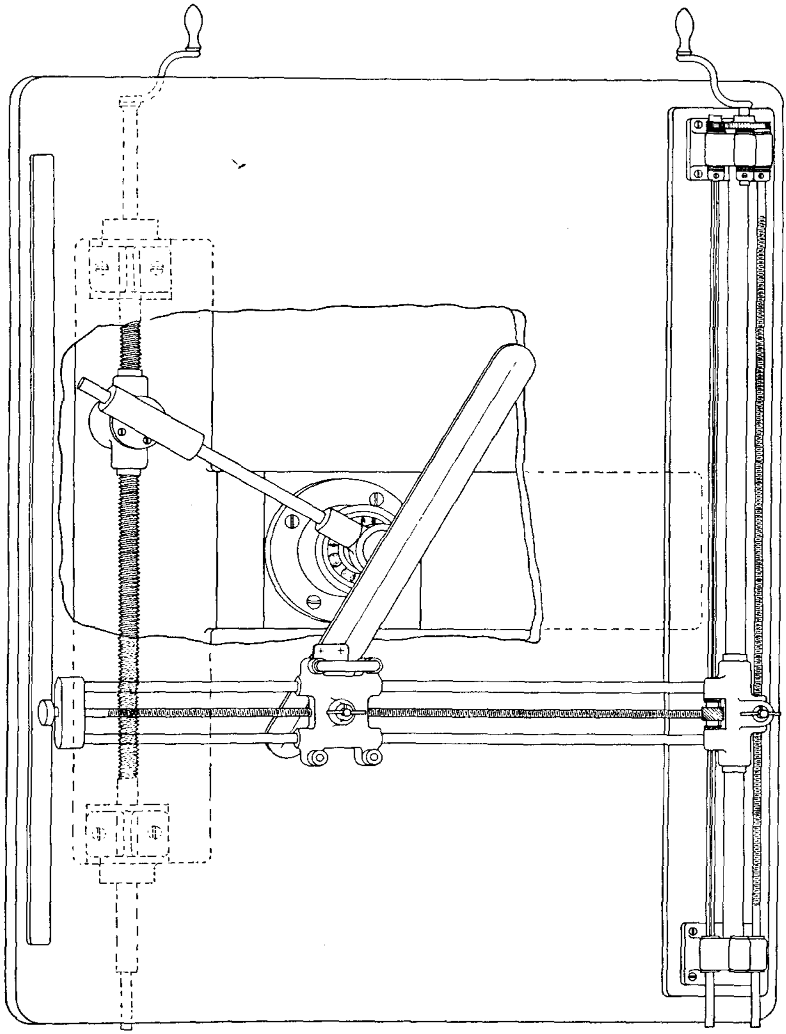
The polar input table has several uses. One use is in connection with periodic coefficients appearing in an equation. Often, in such cases, the operation of the machine will need to be carried on for many cycles of the coefficient; for example, when it is desired to find a stable steady-state solution and to determine a value of a parameter or a set of initial conditions which will produce such a result. The machine can then be left running continuously with the output table set to record a periodic result, such as a plot of dependent variable against its derivative. The parameter is then adjusted until the output is found to exactly repeat cycles, whereupon it may be convenient to alter the connection and record against the independent variable.

For a condition such as this the circular plate is driven from the machine, the input-table carriage is permanently placed accurately in mid-position so that the pointer moves in a vertical line through the center of the plate, and the vertical motion of the pointer is manually controlled so that it is caused to follow a polar plot of the coefficient carried by the plate. The corresponding cross shaft passes this coefficient out into the machine. Gear changes in the plate drive are provided. By the use of these, by the use of plots which repeat several times during a plate revolution, and by proper choice of scales in the balance of the machine, a coefficient of any period may be treated.

A very similar use appears when it is desired to plot a coefficient in much extended manner for purposes of precision. It is plotted in polar coördinates on the plate and winds many times about the center. An inverse procedure is to use a polar input table as an output table, replacing the pointer by a recording pencil, so that a result may be recorded directly in polar coördinates. This again is useful in problems with periodic solutions.

The worm gear drive may be disconnected and the second form of drive mentioned above used in its stead. The circular plate is now replaced by a bar having on its face an accurately scribed line passing through the center. A diagrammatic view of this arrangement is shown in Fig. 15, and a photograph in Fig. 16. In this arrangement there is a lead screw placed parallel to the axis of abscissas and driven

Fig. 15.

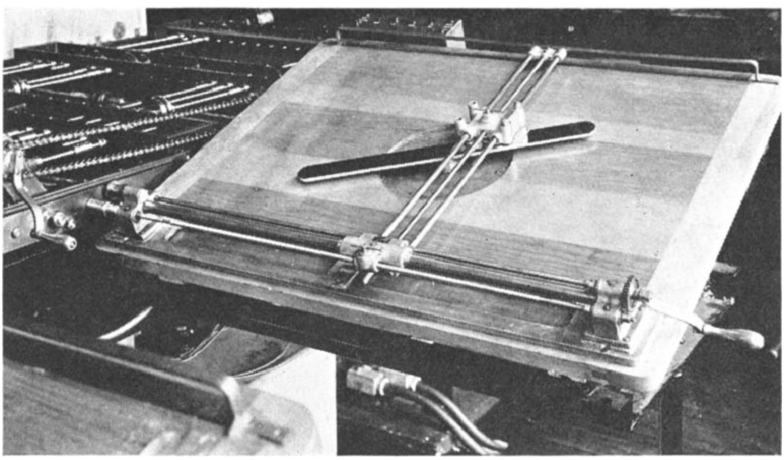


Diagrammatic view of multiplier.

by the cross shaft previously connected to the worm and gear. On this lead screw travels a carriage with a swivelled bearing, and through this bearing passes a rod firmly attached to an extension to the shaft carrying the bar, and located perpendicular to the axis about which the bar revolves. Evidently with this arrangement the revolutions of the third cross shaft will be proportional to the tangent of the angle turned through by the bar. Call the revolutions of the third cross shaft from the position in which the rod is perpendicular to the lead screw z, and the revolutions of the other cross shafts from the positions in which the pointer lies at the center of the plate x and y. If the manipulation is such that the pointer remains always on the diametrical straight line, we have then

$$y = xz$$

FIG. 16.



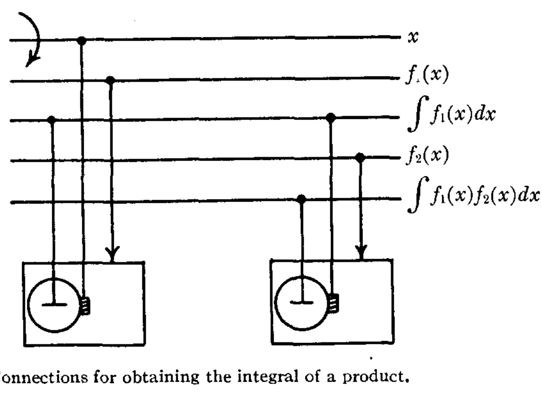
A multiplier.

with proper proportionality factors not now considered. There is thus available a multiplier. If x and z are driven from the machine, and y is controlled manually, a product of two variables is obtained and passed out to the machine. Either x or z can go through zero and take on negative values. If we control x and y by drive from the machine, and control z manually, a quotient can be obtained. Of course in this case x can not go through zero unless y does simultaneously. The choice of scales, and scale changes during a solution when necessary, will maintain the pointer normally at a considerable distance from the center of the plate. This matter of scales always needs to be considered in connection with the question of precision in the use of an input table or multiplier. The maximum angle with the axis of abscissas through which the bar can turn when the multiplier arrangement is in use is a little over 45 degrees. When this is inconvenient the bar may be unclamped from the rest of the mechanism, rotated through 90 degrees and then clamped again so that the straight line occupies a position perpendicular to the lead screw when $z = 0$. We have then an interchange of variables, so that

$$x = yz.$$

It is always possible to evaluate the integral of the product of two functions by a proper interconnection of two integrating units. The arrangement for accomplishing this is indicated diagrammatically in Fig. 17. In this diagram it is assumed

FIG. 17.



Connections for obtaining the integral of a product.

that $f_1(x)$ and $f_2(x)$ have been introduced into the machine by means of input tables, and integration against x is indicated. The essential point is the connection whereby the output of one integrator drives the disc of the other integrator. Of course any functions appearing in the machine, represented by the revolutions of bus shafts, may be thus multiplied and integrated, instead of these functions of the independent variable used here for illustration. Integration may also be accomplished with any desired variable as the variable of integration. This scheme for obtaining the integral of a product is very convenient, and can readily be extended to the case where there are more than two factors. Of course, when employed, it cuts down the number of integrators available for other purposes. In some cases it is desirable to multiply functions without immediately integrating the product, as for example when we meet an expression such as

$$\int [f_1(x)f_2(x) + f_3(x)]dx.$$

In such cases the multiplier is necessary, for there is no accurate automatic differentiator available to undo an integration when once it has been performed. The multiplier, moreover, serves a more vital need than this, as will appear in an illustrative example below.

The drive motor is driven by a push-button-controlled Ward-Leonard system. Automatic stops prevent overtravel of integrators and multipliers.

10. The detailed procedure to be followed in placing a problem on the machine belongs in an operating manual rather than in the present paper, but a general discussion of scales and limits is in order, for it is rather essential to a complete understanding of the uses to which the device may be put. There is an art and a technique in its application, just as in the formal treatment of equations, and the technique must be mastered by any one who would become proficient in the art.

The first step is always the preparation of a connection diagram similar to that of Fig. 7 or 8, and some to follow, which shows the essential interrelations only and disregards all scales, limits, and backlash. This is more than a diagram —it is a process of reasoning, and as such it is recommended to those who seek to impart to youth the meaning, as contrasted with the formalism, of the differential equation. An example to illustrate how alternatives arise will be given below.

From this is prepared a second diagram, exactly similar except that undetermined constant factors and gear ratios are introduced. Thus the independent-variable shaft may be labelled Ax, the shaft for the first derivative $B(dy/dx)$, and so on. When these two shafts are plugged to an integrator, the factor of the integrator is immediately introduced, and the shaft to which the integrator output is connected is labelled $(AB/32)y$. Sufficient gear changes are introduced to provide complete flexibility, with the expectation that many of them will later come out 1 : 1 and be eliminated. Thus we would now gear a new bus shaft to the shaft above and label it $n_1(AB/32)y$, where n_1 is some power of 2, thus available by the usual gear changes, and to be determined. On connecting to input or output tables we introduce the factors given by the lead screws and label scales accordingly. When a final connection which expresses the equality is made, there will be introduced a relationship between shafts already labelled, and we shall obtain an equality between undetermined factors which must be satisfied. There can also be then written down a set of inequalities expressing limits. These involve the maximum values of variables, which must needs be estimated, and are introduced in order that nothing can go out of range when carrying out the problem. For example, the maximum number of turns which can be given to the cross shaft which displaces an integrator carriage is 40. We should hence have, following out the above example,

$$B\left(\frac{dy}{dx}\right)_{\max} \leq 40$$

as one of the inequalities. We are now at liberty to assign all our factors arbitrarily, provided we keep within these inequalities. We can usually do so in such a way as to make most of the scales of plots come out to be exact. The assignment of A above determines the length of time for a solution, for the maximum speed at which the independent-variable shaft can be driven is known. Usually A will be picked to be as small as possible, to cut down the solution time, noting, however, that there should be a substantial number of revolutions of every bus shaft of the machine in order to preserve precision. Suppose the output table is connected to this independent-variable shaft through a gear reduction of 1 : 4, so that the label on the axis of abscissas is $Ax/4.20$, where the 20 appears from the lead-screw factor. This means that one inch of abscissa corresponds to $Ax/80$, just as one revolution of the bus shaft corresponds to Ax. If now we choose A to be 480, the scale of abscissas will be such that 6 inches represents unit value of x. The entire 24-inch range will then allow x to range from zero to four during the course of the problem. Moreover, since 480 revolutions per minute is approximately the maximum speed of drive, a single run through the problem will take about four minutes. The various factors are thus chosen, satisfying the equation between them, and keeping in mind the inequalities. Of course in this process, on a new problem, we do not know at the outset the maximum range of the variables involved, sometimes not even approximately. It is best then to choose factors conservatively so that there is a good probability that the inequalities will be satisfied with plenty to spare, run a first rough solution with the poor precision thus imposed in order merely to determine approximate excursions, and then on the basis of these reassign new factors which will allow good open scales for plots, provide for large numbers of revolutions of bus shafts, utilize integrators and multipliers over a large part of their range, and generally produce precision.

When the factors have been thus chosen a final diagram is prepared showing the actual location of units in the machine. On this are indicated the positions of frontlash units in every critical drive where the presence of backlash might be serious. The direction of rotation of each shaft is indicated, and made correct where necessary by the use of left-hand gear boxes. Plots of coefficients to the proper scales are prepared, and the problem is ready for the machine.

The procedure probably sounds complicated. With practice it can be carried out on problems of moderate complexity in an hour or two, exclusive of the time necessary for plotting. With the final diagram available a few hours more will suffice to connect up the machine, whereupon it is usually employed to obtain a considerable family of solutions before again altering the connections.

11. In this section, and the following, there will be discussed several interesting points which arise in the process of machine solution. Problems will not be treated from the standpoint of the interest in the solutions themselves, for this hardly comes within the scope of the present paper;

but rather it will be the object to illustrate the flexibility of arrangement which is available, and special schemes of avoiding difficulties and securing simplicity and precision. In order to illustrate most readily the Legendre equation,

$$(x^2 - 1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - \lambda y = 0$$

will be treated, for it brings out excellently several points of interest.

We know, of course, that there are solutions of this equation which are finite in the interval $-1 \leq x \leq +1$ provided λ has a value

$$\lambda = n(n + 1)$$

where n is zero or a positive integer. Let us discuss the manner of handling the equation, however, as though nothing were known concerning it, and we were interested in discovering critical values of λ for which solutions would remain finite over this range, and the corresponding solutions.

The straightforward way of treating the equation proves to be cumbersome, but this method will be treated first for purposes of later comparison. This involves solving for the highest-order derivative and writing

$$\frac{d^2y}{dx^2} = \frac{2x}{x^2 - 1}\frac{dy}{dx} - \frac{\lambda}{x^2 - 1}y$$

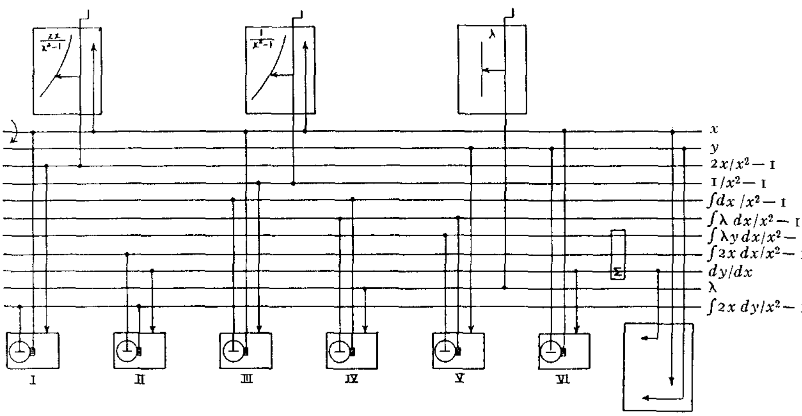
or, what is the same thing,

$$\frac{dy}{dx} = \int \left(\frac{dy}{dx}\right)\frac{2x}{x^2 - 1}dx - \int \frac{\lambda y}{x^2 - 1}dx,$$

and using this as a guide in placing the equation on the machine. This results in the diagram of Fig. 18. In preparing this layout no multiplying units were utilized, in order further to illustrate the use of integrators for multiplying purposes. Scale factors are omitted for simplicity, and no attention is paid to signs, limits of travel, or directions of rotation. The arrangement is thought to be clear without discussion except for one matter. The value of λ is placed on an input table, using a scale on this table simply as an index, and multiplication by λ is then obtained by use of an integrator, the carriage of which thus remains fixed at a certain displacement throughout a single solution. This allows λ to be readily adjustable without altering plotted curves.

Now this arrangement allows us to obtain solutions of the equation for any value of λ and for any given set of initial conditions, provided only that we do not need to examine matters exactly at $x = \pm 1$. We can approach as closely as we please by changing gear ratios and plots every time we

FIG. 18.



Connections for Legendre equation in form:

$$\frac{d^2y}{dx^2} = \frac{2x}{1 - x^2}\frac{dy}{dx} - \frac{\lambda}{1 - x^2}y.$$

get close enough so that the curves we are using go off scale, but this is likely to be laborious.

It will be well to digress a moment to consider initial conditions. Consider that we are examining solutions in the region from zero to nearly unity, and are starting at $x = 0$. At this point

$$\frac{d^2y}{dx^2} + \lambda y = 0,$$

so that we see there is one set of odd, and one set of even solutions. Let us, to be explicit, consider the second set. These start with zero first derivative, and the solution obtained will be proportional throughout to the initial value of y chosen, so that we may consider y to start at any convenient value. The initial displacement of integrators I, IV, and V will be zero, that of III and VI will be λ, and that of II will be the initial assigned value of y.

We can now start the x shaft, and run the solution close to $x = 1$. In the immediate neighborhood of this point we have the approximate relation

$$\frac{dy}{dx} = \frac{\lambda}{2x}y.$$

Having carried a solution close to $x = 1$, and recorded dy/dx and y, we now stop and compare the ratio of our final recorded values with $\lambda/2x$. If the ratio checks we know that the solution, if continued, would remain finite up to $x = 1$. Otherwise it would not. If, upon running curves for various values of λ we find two such values which differ by a small amount, one of which makes the final ratio too large, and the other too small, then evidently the critical value of λ which is sought lies between them.

This scheme, of working by the use of approximate solutions valid in the immediate vicinity of points of discontinuity of coefficients, is of general applicability in handling equations having such discontinuities, and is useful when critical values of parameters are sought. An alternative procedure is, however, possible and will now be discussed.

Let the equation be written in the form

$$\frac{d}{dx}\left[(1 - x^2)\frac{dy}{dx}\right] + \lambda y = 0$$

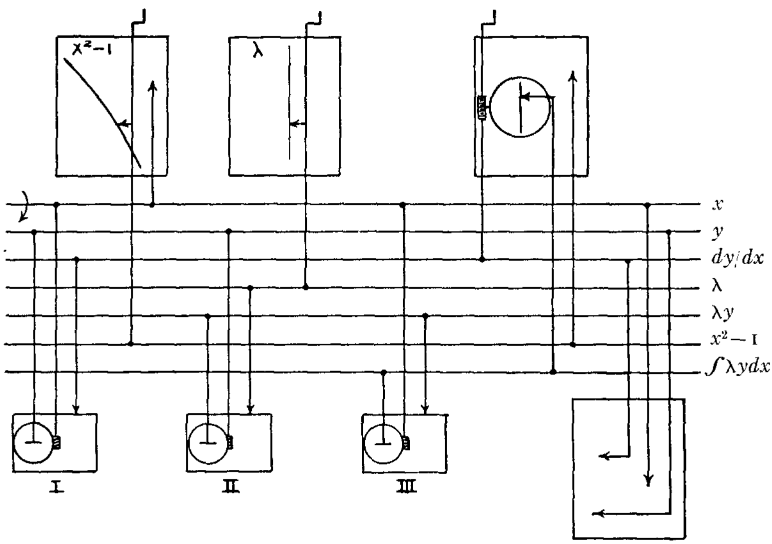
or

$$(x^2 - 1)\frac{dy}{dx} = \int \lambda y dx.$$

With this as a guide we have an arrangement of connections such as is shown in Fig. 19. A multiplier is now used, and a very considerable simplification has been produced. Moreover we have largely removed the difficulty with coefficients, for a zero point is involved rather than a pole.

If we start a solution at $x = 0$, and examine as before the set of even solutions, the initial displacement of integrator I will be zero, that of integrator II will be λ, and that of integrator III will be the assigned initial value of λy. As x approaches unity there is now no difficulty provided we have started with a value of λ for which the solution remains finite. As $x = 1$ is approached the pointer on the multiplier will approach the center of the plate, and will finally move in toward the center in a uniform manner so that the operator will have no difficulty in maintaining the line on the plate in register with the pointer even up to the end of the travel. On the other hand if λ is too large or too small the angle will

FIG. 19.

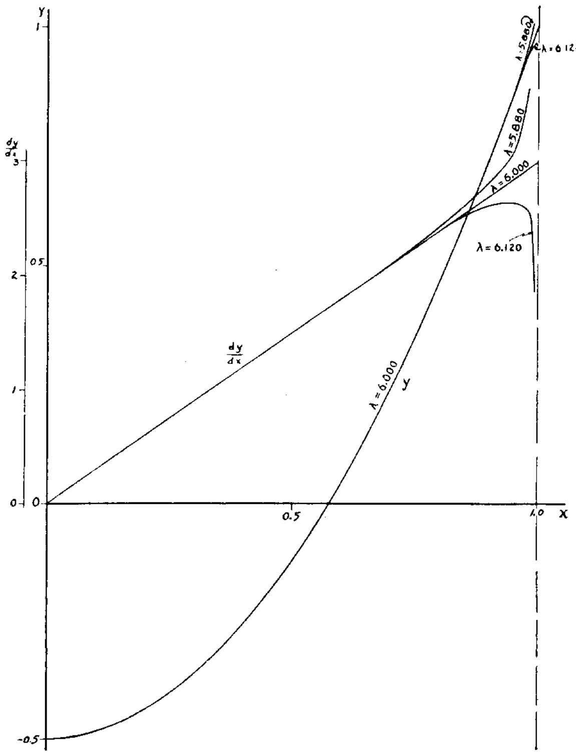


Connections for Legendre equation in form:

$$(x^2 - 1)\frac{dy}{dx} = \int \lambda y dx.$$

change rapidly as the final point is approached. Thus there is no difficulty in identifying critical values of λ within close limits. In Fig. 20 is reproduced a set of solutions obtained in this manner for values of λ in the vicinity of 6, and in Fig. 21 for values near 20. The solutions for the critical value of the parameter are given, and then solutions in which the parameter was altered by 2 per cent. The reproduced curves cannot give, of course, the detail of the original. In the figure no attempt has been made to reproduce the separate curves except near to $x = 1$, and even in that region the correct details cannot be made apparent at small scale. It was evident, however, from the behavior of the curves for dy/dx that a departure of λ from its critical value to this extent shows up unmistakably, and that critical values can be obtained directly in important cases within rather close

FIG. 20.

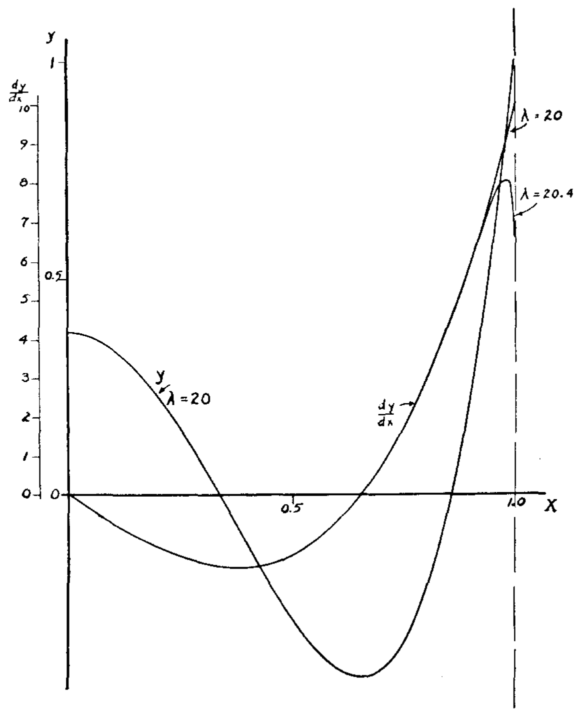


Solutions of Legendre's equation for λ near 6.

limits. This set of solutions was carried out primarily to test the multiplier, and the procedure for avoiding poles of coefficients, and showed that no difficulty is to be expected in approaching a point at which a coefficient goes through zero.

On the other hand we may, if we wish, actually start our solution at the end point. To do this we start at $x = 1$, and run the independent-variable shaft backward. Integrator II starts with a displacement of λ, integrator III with a value λy corresponding to an arbitrarily chosen initial value of y, and integrator I with a displacement corresponding to an initial value of dy/dx equal to $\lambda y/2$ in which y has this same value. The pointer of the multiplier starts at the center of the plate, and the manually-controlled shaft of the multiplier is initially set so as to produce an initial angle for the

FIG. 21.



Solutions of Legendre's equation for λ near 20.

line on the plate corresponding to this same initial $\lambda y/2$. Before starting, in other words, the handle on the multiplier is turned to the position in which the initial displacement of integrator I is brought to the correct computed initial value. Now the instant the machine starts the pointer of the multiplier will move out from the center along the line on the plate, and as soon as it departs from this line visibly correction is made by the operator, who manually controls the slope of the line in such a way as to maintain it registered with the pointer. There is nothing any more critical in adjustment involved here than at any other part of the solution. In this case, when x reaches zero, we note the recorded value of dy/dx. If it is zero we have so chosen λ as to produce a solution which is finite from $x = -1$ to $x = +1$. If λ deviates from such a value in one direction the recorded value of dy/dx will be positive, and if in the other direction it will be negative. In searching for a critical value of λ we may use interpolation as a guide. Of course the above is written as though we were handling an equation the critical values of the parameter of which were not known, although in this example they are of course well known.

Thus when coefficients appearing in an equation have singularities in the range under investigation it is often best to multiply the equation through in such a way as to produce zeros rather than singularities, whereupon the use of multipliers will ordinarily allow the equation to be treated with little difficulty.

12. As a final topic there will be treated a method of successive approximations which is applicable when an accuracy is necessary which is beyond the direct precision of the machine. It is expected that this will be especially expedient when using the machine to tabulate functions which can be defined by means of linear differential equations. The efficacy of the procedure depends upon the fact that complication of the formal expressions which appear as coefficients in an equation does not then render the machine solution appreciably more laborious or less precise.

The tactics employed are as follows: A first set of solutions is made as accurately as the precision of the machine will allow. Each of these graphical solutions is closely fitted by use of a formal expression which can be easily computed, and with a sufficient number of readily computed derivatives. Then a new equation is set up and solved to give the difference between the functions sought and the formal expressions which were adopted. Finally a numerical combination is made of the exact values obtainable from the formal expressions and the graphically-determined differences from the second machine solution. The percentage accuracy in evaluating the differences in the second solution will be limited by machine precision in the usual way, but a small error in these small differences will make only a second-order error in the final result.

Thus suppose we have made a first solution which is accurate within a few tenths of a per cent., and that we have fitted formally so that the maximum departure between these solutions and the formal expressions is about one per cent. The values attained by the dependent variable in the second solution will be only about one per cent. of those appearing in the first solution; but by proper choice of scales approximately the full range of plots and units will be utilized in this second equation as well as in the first. We can then expect to evaluate the differences from the second solution to within a few tenths of a per cent. While the situation will be altered in accordance with the nature of the problem it may be then expected that about four-figure accuracy can be obtained from the double solution. Of course the process may be continued another step if necessary.

The second equation is not ordinarily difficult to put on the machine. As an illustration suppose we wished to obtain a very accurate solution of the Legendre equation for a non-integral value of λ over some range within which the solution remains finite. A first solution would be made as described in the preceding section. Suppose this were approximated by a formal and readily computed function $F(x)$, say a polynomial. We then write

$$y_1 = y - F(x)$$

and substitute in the equation. The resulting equation to be solved for differences is

$$(x^2 - 1)\frac{d^2y_1}{dx^2} + 2x\frac{dy_1}{dx} - \lambda y_1 = \phi(x),$$

in which

$$\phi(x) = \lambda F(x) - 2xF'(x) - (x^2 - 1)F''(x).$$

This is but little more difficult to place on the machine than the original equation. All that is necessary is the addition of an input table for introducing $\phi(x)$ at the proper point.

It is not necessary that $F(x)$ be a single formal expression; the first solution may be split up into sections and each represented by a simple formal expression. It is necessary, however, that in each section $F(x)$ have continuous derivatives up to the derivative of an order one less than the order of the equation treated. Otherwise the procedure is not significantly altered from that described above. This method of successive approximations is of course somewhat long, and it will presumably be employed only where the extra precision is vital, but nevertheless it is a very potent method.

13. The machine is not yet completed; in fact it is questionable whether it will ever be complete, for it can always be extended by the addition of units to cover greater order or complexity of equations. It is capable at the present time, however, of handling a wide range of problems of extreme interest.

It has been thoroughly tested for precision. Its speed has been investigated in connection with a problem in which about 100 solutions of a moderately complex second-order equation were evaluated in five days' work. The procedure in locating critical values of a parameter has been somewhat explored. No great experience has as yet been had in work involving successive approximations.

This machine forms part of a departmental program of development. It is a pleasure to recall the effective and enthusiastic support of many of the staff. Prof. H. L. Hazen, who has been associated with me practically throughout the entire development, has consistently contributed in this latest work. Professor M. F. Gardner, and later Mr. S. H. Caldwell have been in direct charge of the laboratory where the work has been in progress, and have greatly helped. I also appreciate the able work of Mr. L. E. Frost, designing draftsman, and Mr. Maurice Forbes, expert machinist.